Functions

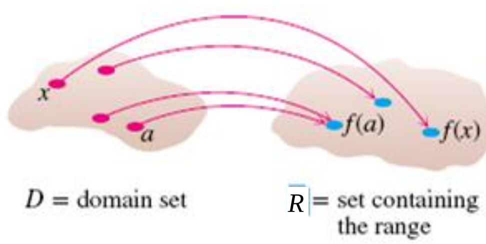
DEFINITION: Function

A **function** from a set D (domain) to a set R (range) is a rule that assigns to *unique* (single) element $f(x)\in R$ to each element $x\in D$

$f: x \longrightarrow f(x)$ it means that f sends x to $f(x)$

$f: x \longrightarrow \frac{1}{x^2}$ it means that f sends x to

$f(x) = y = \frac{1}{x^2}$



- The set of x is called the *"Domain"* of the function (D_f).
- The set of y is called the *"Range"* of the function (R_f).

❖ x & y are variables.

❖ x is independent variable.

❖ y is dependent variable.

Domain (D_f): is the set of all possible inputs (x-values).

Range (R_f): is the set of all possible outputs (y-values).

To find Domain (D_f) and the Range (R_f) the following points must be noticed:

1. The denominator in a function must not equal zero (never divide by zero).
2. The values under even roots must be positive.

Examples: Find the Domain (D_f) and Range (R_f) of the following functions:

1. $y = f(x) = \frac{1}{x}$

Sol: denominator must not equal zero $\Rightarrow x \neq 0 \quad \Rightarrow \; D_f=\{x: x \neq 0\}$.

To find R_f : we must convert the function from $y=f(x)$ into $x=f(y)$.

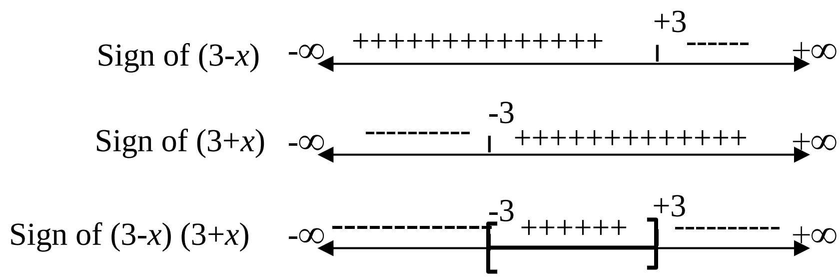
$\therefore x = \frac{1}{y} \quad \Rightarrow R_f=\{y: y \neq 0\}$.

2. $y = f(x) = \sqrt{9 - x^2}$

Sol: The values under even roots must be positive

$\Rightarrow$ $9\text{-}x^2 \geq 0 \Rightarrow (3\text{-}x)\ (3\text{+}x) \geq 0$

$\therefore$ $D_f = \{x: -3 \leq x \leq 3\}$.



To find R_f: we must convert the function from $y=f(x)$ into $x=f(y)$.

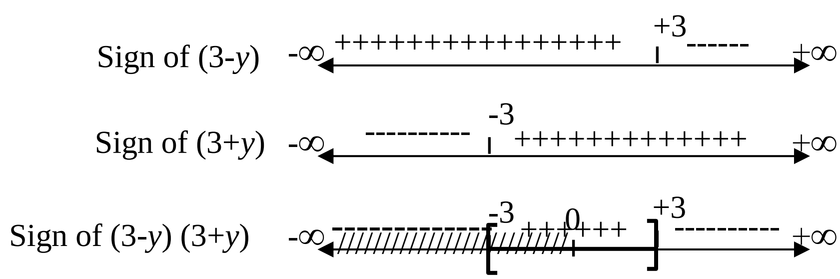
$y = \sqrt{9 - x^2} \Rightarrow y^2 = 9 - x^2$

$\Rightarrow x^2 = 9 - y^2 \Rightarrow x = \pm\sqrt{9 - y^2}$

So the values under even roots must be positive

$9\text{-}y^2 \geq 0 \Rightarrow (3\text{-}y)\ (3\text{+}y) \geq 0$

$\therefore$ $R_f = \{y: -3 \leq y \leq 3\}$.



But the values of y must be always positive, we must exclude negative values, $\Rightarrow$ $R_f = \{y: 0 \leq y \leq 3\}$.

3. $y = f(x) = \dfrac{1}{\sqrt{9 - x^2}}$

Sol: The values under even roots must be positive and the denominator must not equal zero, so:

$9\text{-}x^2 > 0 \Rightarrow (3\text{-}x)\ (3\text{+}x) > 0$

$\therefore$ $D_f = \{x: -3 < x < 3\}$.

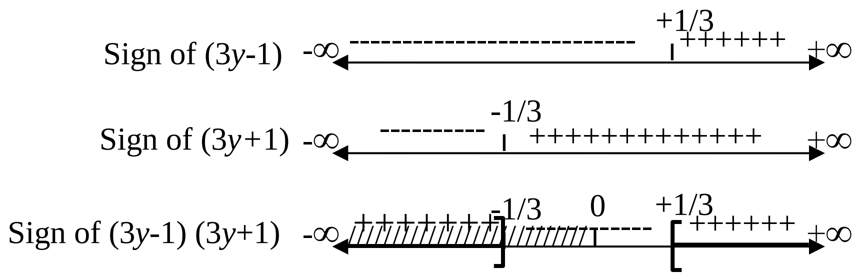
To find R_f : we must convert the function from $y=f(x)$ into $x=f(y)$.

$$y = \frac{1}{\sqrt{9 - x^2}} \quad \Rightarrow \quad y^2 = \frac{1}{9 - x^2} \quad \Rightarrow \quad 9 - x^2 = \frac{1}{y^2} \quad \Rightarrow \quad x^2 = 9 - \frac{1}{y^2}$$

$$\Rightarrow \quad x^2 = \frac{9y^2 - 1}{y^2} \quad \Rightarrow \quad x = \mp\sqrt{\frac{9y^2 - 1}{y^2}} \quad \Rightarrow \quad x = \mp\frac{\sqrt{9y^2 - 1}}{y}$$

The values under even roots must be positive and the denominator must not equal zero, so: $9y^2-1 \geq 0$

$\Rightarrow \quad (3y-1)(3y+1) \geq 0$



$\Rightarrow \quad R_f=\{y: -\infty \leq y \leq -1/3\} \cup \{y: 1/3 \leq y \leq \infty\}.$

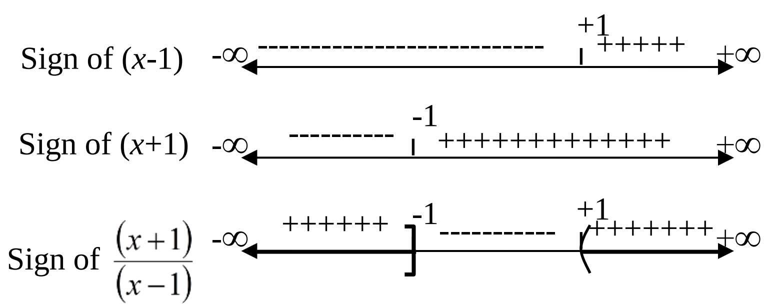
denominator must not equal zero $\Rightarrow y \neq 0$

But the values of y must be always positive; we must exclude negative values,

$\Rightarrow \quad R_f=\{y: 1/3 \leq y \leq \infty\}$

4. $y = f(x) = \sqrt{\dfrac{x+1}{x-1}}$

Sol: The values under even roots must be positive and the denominator must not equal zero, so:



$x-1 \neq 0 \quad \Rightarrow \quad x \neq 1$

and $\quad \dfrac{x+1}{x-1} \geq 0$

$D_f = (-\infty, -1] \cup (1, +\infty)$

To find R_f: we must convert the function from $y=f(x)$ into $x=f(y)$.

$$y = \sqrt{\frac{x+1}{x-1}} \quad \Rightarrow \quad y^2 = \frac{x+1}{x-1} \quad \Rightarrow \quad y^2(x-1) = x+1 \quad \Rightarrow \quad y^2x - y^2 = x+1$$

$$\Rightarrow \quad y^2x - x = y^2 + 1 \quad \Rightarrow \quad x(y^2-1) = y^2+1 \quad \Rightarrow \quad x = \frac{y^2+1}{y^2-1}$$

You can see that there is no root in the form $x = f(y)$, so only the denominator must not equal zero so:

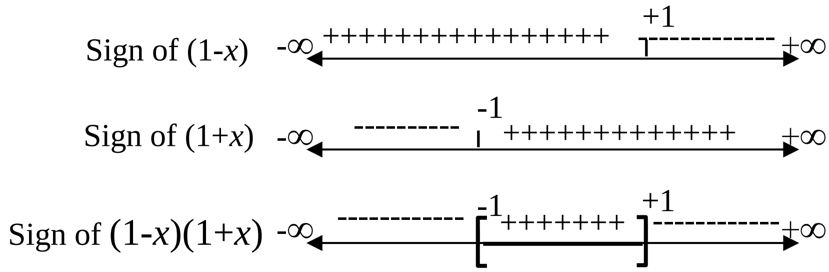
$y^2-1 \neq 0 \quad \Rightarrow \quad y^2 \neq 1 \quad \Rightarrow \quad y \neq \pm 1 \quad \Rightarrow \quad R_f = R\backslash\{-1,+1\}$

But the values of y must be always positive; we must exclude negative values,

$\Rightarrow \quad R_f=[0,+\infty)\backslash\{+1\}$

5. $y=f(x)=-\sqrt{1-x^2}$

Sol: The values under even roots must be positive:



$1-x^2 \geq 0 \quad \Rightarrow \quad (1-x)(1+x) \geq 0$

$\therefore \; D_f=[-1,+1]$

To find R_f: we must convert the function from $y=f(x)$ into $x=f(y)$.

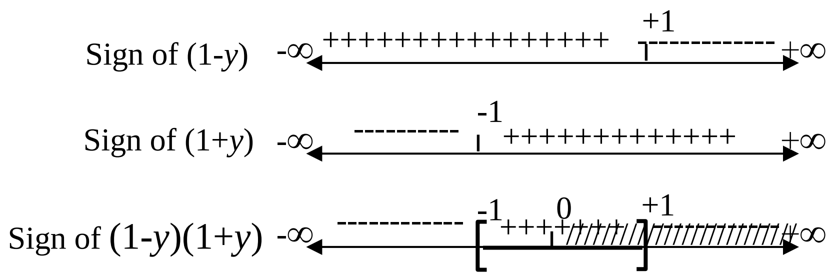
$$y=-\sqrt{1-x^2} \quad \Rightarrow \quad y^2=1-x^2 \quad \Rightarrow \quad x^2=1-y^2 \quad \Rightarrow \quad x=\mp\sqrt{1-y^2}$$

The values under even roots must be positive:

$1-y^2 \geq 0 \quad \Rightarrow \quad (1-y)(1+y) \geq 0$

$\Rightarrow \; R_f=[-1,+1]$

But the values of y must be always negative; we must exclude positive values,



$\Rightarrow \quad R_f=[-1,0]$

Homework: Find the domains and ranges of the following functions.

1. $y=\dfrac{x^2-1}{x^2+1}$
2. $y=\dfrac{2x}{(x-2)(x+1)}$
3. $y=\sqrt{\dfrac{2x}{(x-2)(x+1)}}$
4. $y=\sqrt{x^2+4}$
5. $y=\sqrt{x^2-4}$
6. $y=\sqrt{\dfrac{2x}{2-x}}$
7. $y=\sqrt{9-\sqrt{x}}$
8. $y=\dfrac{1}{3+\sqrt{x}}$
9. $x^2+xy+y^2-3=0$

Absolute Value Function: it is defined as:

$$y = |x| = \sqrt{x^2} = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values.
4. $|a+b| \leq |a| + |b|$ — The triangle inequality. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Absolute Values and Intervals

If a is any positive number, then

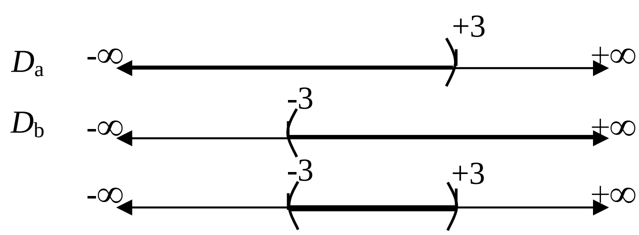
5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

Examples: Solve the following for x?

1. $|x| = 3$ **Sol.:** So $x=3$ & $x=-3$ $D_f=\{3,-3\}$

2. $|x| < 3$ **Sol.:** So a: $x < 3$

$\Rightarrow$ $D_a=(-\infty,3)$



and b: $-x < 3$ (multiply by -1) $\Rightarrow x > -3 \Rightarrow D_b = (-3,\infty)$

$\therefore D_f = D_a \cap D_b = (-3,3)$

3. $|x| \geq 3$

Sol.: *Note* (solution must consist the remaining part of the real numbers line of the previous example)

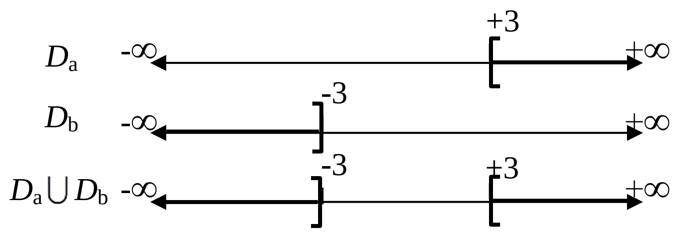
So a: either $x \geq 3 \quad \Rightarrow \quad D_a=[3,\infty)$

b: or $-x \geq 3$ (multiply by -1) $\Rightarrow x \leq -3$

$\Rightarrow \quad D_b=(-\infty,-3]$

$\therefore D_f = D_a \cup D_b = (-\infty,-3] \cup [3, \infty)$

$=R\backslash(-3,3)$



4. $|2x-3| \leq 7$

Sol.: $-7 \leq 2x-3 \leq 7$ (6^{th} property)

$-4 \leq 2x \leq 10 \quad \Rightarrow \quad -2 \leq x \leq 5$

$\therefore D_f = [-2,5]$

5. $|x-9| > 3$

Sol.: $x-9 > 3$ or $x-9 < -3$ (7^{th} property)

$\Rightarrow x > 12$ or $x < 6$

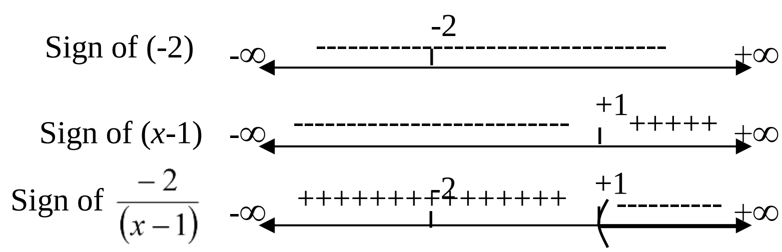
$\therefore D_f = (-\infty,6) \cup (12, \infty)$

6. $|x-3| < |x-1|$

Sol.: $\dfrac{|x-3|}{|x-1|} < 1$ (divided by $|x-1|$)

$\Rightarrow \quad \left|\dfrac{x-3}{x-1}\right| < 1$ (3^{rd} property)

$\Rightarrow \quad -1 < \dfrac{x-3}{x-1} < 1$ (6^{th} property)



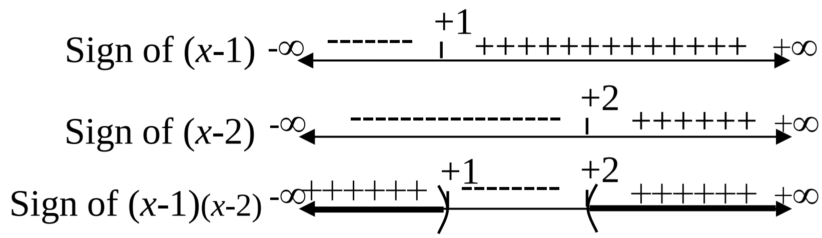
[syllabus](#)

a. $\frac{x-3}{x-1}<1 \quad \Rightarrow \quad \frac{x-3}{x-1}-1<0$

$\Rightarrow \frac{x-3-(x-1)}{x-1}<0$

$\Rightarrow \frac{x-3-x+1}{x-1}<0 \Rightarrow \frac{-2}{x-1}<0$

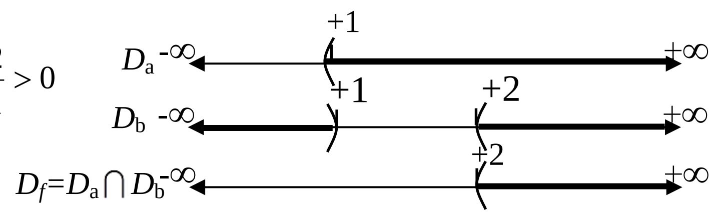
$\therefore D_a = (1, \infty)$



b. $\frac{x-3}{x-1}>-1 \Rightarrow \frac{x-3}{x-1}+1>0$

$\Rightarrow \frac{x-3+x-1}{x-1}>0$

$\Rightarrow \frac{2x-4}{x-1}>0 \Rightarrow \frac{2(x-2)}{x-1}>0 \Rightarrow \frac{x-2}{x-1}>0$



$\therefore D_b = (-\infty,1) \cup (2, \infty)$

$\therefore D_f = D_a \cap D_b = (2, \infty)$

7. $\left|\frac{x-4}{x-3}\right| \geq 2$

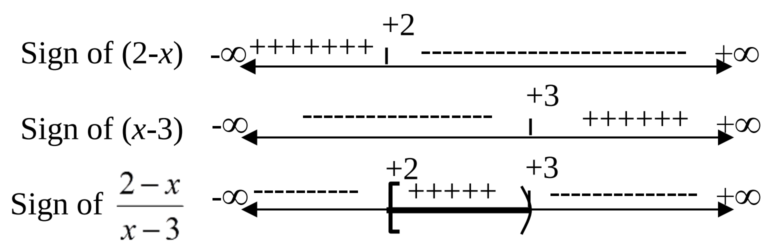
Sol.: either $\frac{x-4}{x-3} \geq 2$ or $\frac{x-4}{x-3} \leq -2$

a. $\frac{x-4}{x-3} \geq 2 \Rightarrow \frac{x-4}{x-3}-2 \geq 0$

$\Rightarrow \quad \frac{x-4-2(x-3)}{x-3} \geq 0 \quad \Rightarrow$

$\frac{x-4-2x+6}{x-3} \geq 0 \quad \Rightarrow$

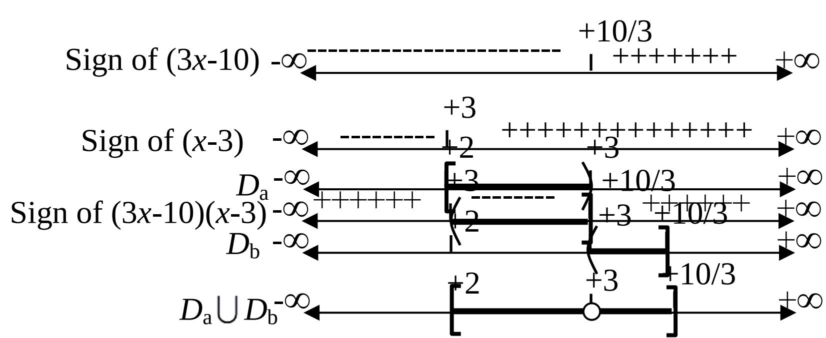
$\frac{2-x}{x-3} \geq 0$



$\therefore D_a = [2,3)$ ($x \neq 3$ because denominator should not equal zero.)

b. $\frac{x-4}{x-3} \leq -2 \quad \Rightarrow$

$\frac{x-4}{x-3}+2 \leq 0 \quad \Rightarrow$



$$\frac{x-4+2(x-3)}{x-3} \le 0 \quad \Rightarrow \quad \frac{x-4+2x-6}{x-3} \le 0 \quad \Rightarrow \quad \frac{3x-10}{x-3} \le 0$$

$\therefore D_b = (3, 10/3]$ $(x \neq 3)$

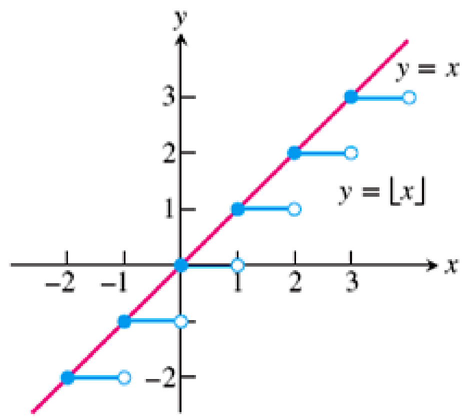
So $D_f = D_a \cup D_b = [2,10/3] \setminus \{3\}$

Homework: Find the values of x that satisfies the inequality.

1. $|x-5| < 9$
2. $\left|\frac{3x+1}{2}\right| < 1$
3. $\left|\frac{x}{2}-1\right| \le 1$
4. $|x-3| \le |2x-6|$
5. $\left|\frac{x-3}{6-5x}\right| \le 2$

The Greatest Integer Function (Stepped Function):

The function whose values at any number x is *the greatest integer less than or equal to x* is called **greatest integer function.** It is denoted $\lfloor x \rfloor$, or in some books **[x]** or **[[x]]** or **int x**



The greatest integer function:

$$[x] \le x < [x] + 1$$

Example 1: Find the integer of the following:

[2.4]=2, [1.9]=1, [0.1]=0, [0.0]=0,

[-1.2]=-2, [-0.3]=-1, [-2.0]=-2

Example2: Find the interval of the following functions:

1. $[x]=2$

Sol. $2 \le x < 3$

2. $[2x]=1$

Sol. $1 \le 2x < 2 \Rightarrow \frac{1}{2} \le x < 1$

3. $\left[\frac{1}{3}x\right] = 2$

Sol. $2 \le \frac{1}{3}x < 3 \Rightarrow 6 \le x < 9$

4. $[2x] = -1$

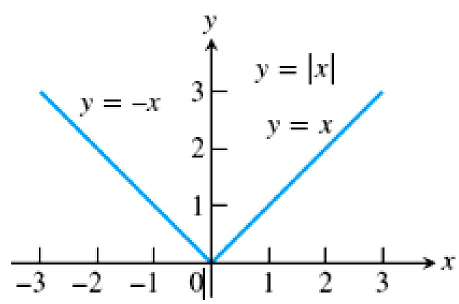
Sol. $-1 \le 2x < 0 \Rightarrow -\frac{1}{2} \le x < 0$

Properties of greatest integer value:

1. $[[[[x]]]] = [x]$
2. $[x+n] = [x] + n$ where n is integer
3. $[x-n] = [x] - n$ where n is integer
4. $-[x] \neq [-x]$

Function Defined in Pieces:

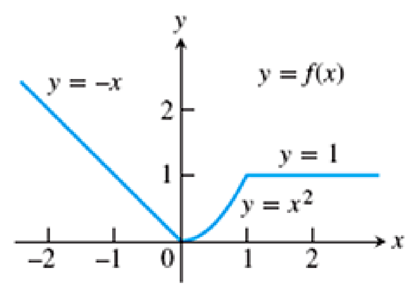
While some functions defined by single formulas, others are defined by applying different formulas to different parts of their domains. One example is the **absolute value function**



$$y = |x| = \sqrt{x^2} = \begin{cases} x & if \quad x \ge 0 \\ -x & if \quad x < 0 \end{cases}$$

Example: the values of the function

$$f(x) = \begin{cases} -x & if \quad x < 0 \\ x^2 & if \quad 0 \le x \le 1 \\ 1 & if \quad x > 1 \end{cases}$$



Example: Graph the function

$$y = f(x) = |x-3| + |x+2|$$

Sol. Recall the definition of absolute value:

$$y = |x| = \sqrt{x^2} = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

It follows that: $|x-3| = \begin{cases} (x-3) & if \quad (x-3) \geq 0 \\ -(x-3) & if \quad (x-3) < 0 \end{cases}$

$$= \begin{cases} x-3 & if \quad x \geq 3 \\ -x+3 & if \quad x < 3 \end{cases}$$

Similarly $|x+2| = \begin{cases} (x+2) & if \quad (x+2) \geq 0 \\ -(x+2) & if \quad (x+2) < 0 \end{cases}$

$$= \begin{cases} x+2 & if \quad x \geq -2 \\ -x-2 & if \quad x < -2 \end{cases}$$

These expressions show that we must consider three cases:

$x < -2$, $-2 \leq x < 3$ and $x \geq 3$

Case I: if $x < -2$ we have $f(x) = |x-3| + |x+2|$

$$f(x) = -x+3-x-2$$

$$f(x) = -2x+1$$

Case II: if $-2 \leq x < 3$ we have $f(x) = |x-3| + |x+2|$

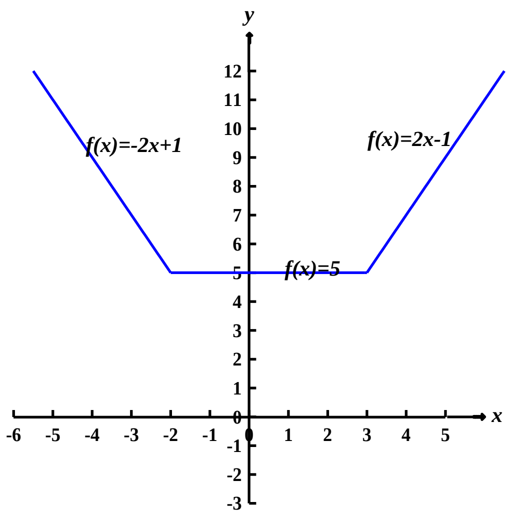
$$f(x) = -x+3+x+2$$

$$f(x) = 5$$

Case III: if $x \geq 3$ we have $f(x) = |x-3| + |x+2|$

$$f(x) = x-3+x+2$$

$$f(x) = 2x-1$$



Thus $f(x) = \begin{cases} -2x+1, & if \quad x < -2 \\ 5, & if \quad -2 \leq x < 3 \\ 2x-1, & if \quad x \geq 3 \end{cases}$

Sums, Difference, Product and Quotients of Functions:

Definition: If f and g are functions, then we define the functions

Sum $\Rightarrow (f+g)(x)= f(x)+g(x)$

Difference $\Rightarrow (f-g)(x)= f(x)-g(x)$ or $(g-f)(x)= g(x)-f(x)$

Product $\Rightarrow (f.g)(x)= f(x).g(x)$

Quotient $\Rightarrow (f/g)(x)= f(x)/g(x)$; where $g(x) \neq 0$ …(1)

or $(g/f)(x)= g(x)/f(x)$; where $f(x) \neq 0$ …(2)

are also functions of (x), defined for any value of x that lies in both D_f and D_g ($x \in D_f \cap D_g$), except the points which $g(x) =0$ in eq.(1) or $f(x) =0$ in eq.(2)

Example 1: **Combining Functions Algebraically**:

The function defined by the formulas

$f(x) = \sqrt{x} \qquad and \ g(x) = \sqrt{1-x}$

have domains $D(f) = [0, \infty)$ *and* $D(g) = (-\infty, 1]$. The points common to these domains are the points

$[0, \infty) \cap (-\infty, 1] = [0, 1]$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f * g$ for the product function fg.

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f*g$	$(f*g)(x) = f(x)g(x) = \sqrt{x(1-x)} = \sqrt{x-x^2}$	$[0,1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0,1)$ $(x=1 \ excluded)$

g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]\,(x=0\ \ excluded)$

Example 2: Give the domain of $f(x)$ and $g(x)$ and the corresponding domains of $f+g$, $f-g$, $g-f$, $f.g$, f/g and g/f where $f(x)=\sqrt{4-x^2}$ and $g(x) = 3x+1$

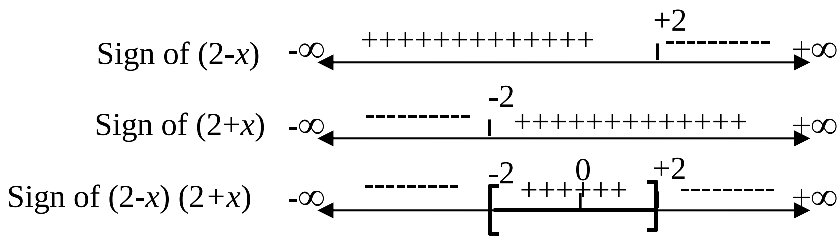
Sol. The domain of $f(x)$ (D_f) is:

$4 - x^2 \geq 0 \qquad \Rightarrow (2-x)\,(2+x) \geq 0$

$\Rightarrow \qquad D_f=[-2,+2]$

The domain of $g(x)$ ($D_g=R$)

So $\qquad D_f \cap D_g = [-2,2]$



Function	Formula	Domain
$f+g\ (x) =$	$\sqrt{4-x^2}+(3x+1)$	[-2,+2]
$f-g\ (x) =$	$\sqrt{4-x^2}-(3x+1)$	[-2,+2]
$g-f\ (x) =$	$(3x+1)-\sqrt{4-x^2}$	[-2,+2]
$f.g\ (x) =$	$\sqrt{4-x^2}.(3x+1)$	[-2,+2]
$\frac{f}{g}(x) =$	$\frac{\sqrt{4-x^2}}{3x+1}$	[-2,+2]\{-1/3}
$\frac{g}{f}(x) =$	$\frac{3x+1}{\sqrt{4-x^2}}$	(-2,+2)

Homework: Give the domains of f and g and the corresponding domains of $f+g$, $f-g$, $f.g$, f/g, and g/f for the following:

1. $f(x)=3x^2$; $\qquad g(x)=\frac{1}{2x-3}$.

2. $f(x) = x + \frac{1}{x}$; $g(x) = x - \frac{1}{x}$.

3. $f(x) = \sqrt{x+3}$; $g(x) = \sqrt{x+3}$.

4. $f(x) = x^3 + 3x$; $g(x) = 3x^2 + 1$.

5. $f(x) = \sqrt{x^2 + 4}$; $g(x) = 7x^2 + 1$