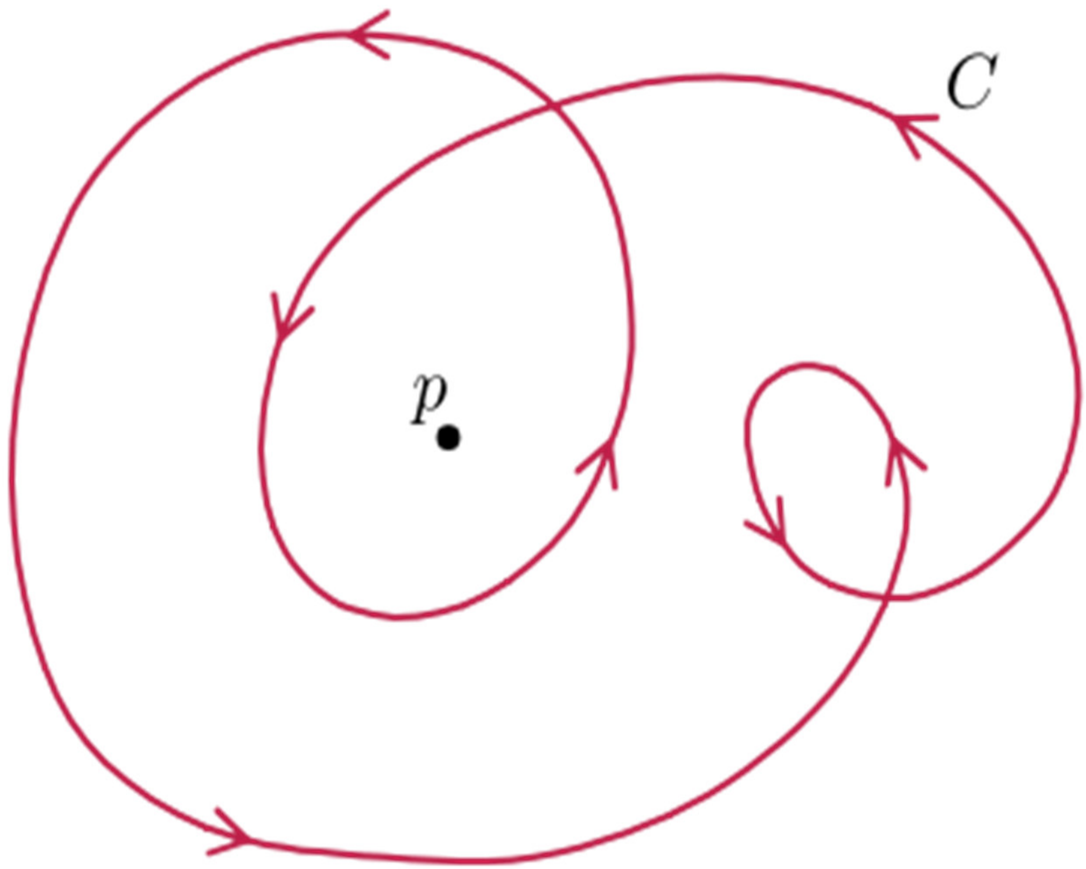
Topological Degree of Maps:

Topological degree theory is a generalization of the winding number of a curve in the complex plane. It can be used to estimate the number of solutions of an equation. When one solution of an equation is easily found, degree theory can often be used to prove existence of a second, nontrivial, solution. the winding number of a closed curve in the plane around a given point is an integer representing the total number of times that curve travels counterclockwise around the point. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise.



This curve has winding number two around the point *p*.

There are different types of degree for different types of maps: e.g. for maps between Banach spaces there is the Brouwer degree in $\mathbf{R}^n$, the Leray-Schauder degree for compact mappings in normed spaces.

Definition of the Degree of a Mapping

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $f : \bar{\Omega} \to \mathbb{R}^n$ be a mapping which satisfies

- $$f \in C^1(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n), \tag{1}$$
- $y \in \mathbb{R}^n$ is such that
$$y \notin f(\partial\Omega), \tag{2}$$
- if $x \in \Omega$ is such that $f(x) = y$ then
$$f'(x) = Df(x) \tag{3}$$
is nonsingular.

Let

$$f(x) = y \tag{4}$$

Definition *Let f satisfy (1), (2), (3). Define*

$$\mathrm{d}(f, \Omega, y) = \sum_{i=1}^{k} \mathrm{sgn}\ \det\ f'(x_i)$$

where $x_1, \cdots, x_k$ are the solutions of (4) in Ω and

$$\mathrm{sgn}\ \det\ f'(x_i) = \begin{cases} +1, \ \textit{if} \det\ f'(x_i) > 0 \\ -1, \ \textit{if} \det\ f'(x_i) < 0, \quad i = 1, \cdots, k. \end{cases}$$

If (4) has no solutions in Ω we let $\mathrm{d}(f, \Omega, y) = 0$.

The Brouwer degree $\mathrm{d}(f,\Omega,y)$ to be defined for mappings $f \in C(\bar{\Omega},\mathbb{R}^n)$ which satisfy (2) will coincide with the number just defined in case f satisfies (1), (2), (3).

Lemma *Let f_1 and f_2 satisfy (1), (2), (3) and let $\epsilon > 0$ be such that*

$$|f_i(x) - y| > 7\epsilon, \quad x \in \partial\Omega, \quad i = 1, 2,$$

$$|f_1(x) - f_2(x)| < \epsilon, \quad x \in \bar{\Omega},$$

then

$$\mathrm{d}(f_1,\Omega,y) = \mathrm{d}(f_2,\Omega,y).$$

PROOF. assume that $y = 0$, since by Definition

$$\mathrm{d}(f,\Omega,y) = \mathrm{d}(f - y,\Omega,0).$$

let $g \in C^1[0,\infty)$ be such that

$$g(s) = 1, \quad 0 \le s \le 2\epsilon$$

$$0 \leq g(r) \leq 1, \quad 2\epsilon \leq r < 3\epsilon$$

$$g(r) = 0, \quad 3\epsilon \leq r < \infty.$$

Consider

$$f_3(x) = [1 - g(|f_1(x)|)]f_1(x) + g(|f_1(x)|)f_2(x),$$

then

$$f_3 \in C^1(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n)$$

and

$$|f_i(x) - f_k(x)| < \epsilon, \quad i, k = 1, 2, 3, \; x \in \bar{\Omega}$$

$$|f_i(x)| > 6\epsilon, \quad x \in \partial\Omega, \quad i = 1, 2, 3.$$

Let $\phi_i \in C[0, \infty)$, $i = 1, 2$ be continuous and be such that

$$\phi_1(t) = 0, \quad 0 \leq t \leq 4\epsilon, \quad 5\epsilon \leq t \leq \infty$$

$$\phi_2(t) = 0, \quad \epsilon \leq t < \infty, \quad \phi_2(0) = 0$$

$$\int_{\mathbb{R}^n} \phi_i(|x|)dx = 1, \quad i = 1, 2.$$

We note that

$$f_3 \equiv f_1, \quad \text{if} \quad |f_1| > 3\epsilon$$

$$f_3 \equiv f_2, \quad \text{if} \quad |f_1| < 2\epsilon.$$

Therefore

$$\phi_1(|f_3(x)|)\text{det } f_3'(x) = \phi_1(|f_1(x)|)\text{det } f_1'(x)$$

$$\phi_2(|f_3(x)|)\text{det } f_3'(x) = \phi_2(|f_2(x)|)\text{det } f_2'(x).$$

Properties of the Brouwer Degree

Proposition (Solution property) *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $y \notin f(\partial\Omega)$ and assume that $\mathrm{d}(f, \Omega, y) \neq 0$. Then the equation*

$$f(x) = y$$

has a solution in Ω.

Proposition (Continuity property) *Let* $f \in C(\bar{\Omega}, \mathbb{R}^n)$ *and* $y \in \mathbb{R}^n$ *be such that* $\mathrm{d}(f, \Omega, y)$ *is defined. Then there exists* $\epsilon > 0$ *such that for all* $g \in C(\bar{\Omega}, \mathbb{R}^n)$ *and* $\hat{y} \in \mathbb{R}$ *with* $\|f - g\| + |y - \hat{y}| < \epsilon$

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, \hat{y}).$$

Proposition (Homotopy invariance property) *Let* $f, g \in C(\bar{\Omega}, \mathbb{R}^n)$ *with* $f(x)$ *and* $g(x) \neq y$ *for* $x \in \partial\Omega$ *and let* $h : [a, b] \times \bar{\Omega} \to \mathbb{R}^n$ *be continuous such that* $h(t, x) \neq y$, $(t, x) \in [a, b] \times \partial\Omega$. *Further let* $h(a, x) = f(x)$, $h(b, x) = g(x)$, $x \in \bar{\Omega}$. *Then*

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, y);$$

more generally, $\mathrm{d}(h(t, \cdot), \Omega, y) = \text{constant}$ *for* $a \leq t \leq b$.

Corollary *Let* $f \in C(\bar{\Omega}, \mathbb{R}^n)$ *be such that* $\mathrm{d}(f, \Omega, y)$ *is defined. Let* $g \in C(\bar{\Omega}, \mathbb{R}^n)$ *be such that* $|f(x) - g(x)| < |f(x) - y|$, $x \in \partial\Omega$. *Then* $\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, y)$.

Proof. For $0 \leq t \leq 1$ and $x \in \partial\Omega$ we have that

$$|y - tg(x) - (1 - t)f(x)| = |(y - f(x)) - t(g(x) - f(x))|$$

$$\geq |y - f(x)| - t|g(x) - f(x)| > 0 \quad \text{since} \quad 0 \leq t \leq 1,$$

hence $h : [0,1] \times \bar{\Omega} \to \mathbb{R}^n$ given by $h(t,x) = tg(x) + (1-t)f(x)$ satisfies the conditions of **Homotopy invariance property** and the conclusion follows from that proposition.

Proposition (Additivity property) *Let Ω be a bounded open set which is the union of m disjoint open sets $\Omega_1, \cdots, \Omega_m$, and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ be such that $y \notin f(\partial\Omega_i)$, $i = 1, \cdots, m$. Then*

$$\mathrm{d}(f, \Omega, y) = \sum_{i=1}^{m} \mathrm{d}(f, \Omega_i, y).$$

Proposition (Excision property) *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and let K be a closed subset of $\bar{\Omega}$ such that $y \notin f(\partial\Omega \cup K)$. Then*

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(f, \Omega \setminus K, y).$$

Proposition (Cartesian product formula) *Assume that $\Omega = \Omega_1 \times \Omega_2$ is a bounded open set in $\mathbb{R}^n$ with Ω_1 open in $\mathbb{R}^p$ and Ω_2 open in $\mathbb{R}^q$, $p + q = n$. For $x \in \mathbb{R}^n$ write $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^q$. Suppose that $f(x) = (f_1(x_1), f_2(x_2))$ where $f_1 : \bar{\Omega}_1 \to \mathbb{R}^p$, $f_2 : \bar{\Omega}_2 \to \mathbb{R}^q$ are continuous. Suppose $y = (y_1, y_2) \in \mathbb{R}^n$ is such that $y_i \notin f_i(\partial\Omega_i), i = 1, 2$. Then*

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(f_1, \Omega_1, y_1)\mathrm{d}(f_2, \Omega_2, y_2).$$

PROOF. Using an approximation argument, we may assume that f, f_1 and f_2 satisfy also (1) and (3) (interpreted appropriately). For such functions we have

$$\begin{aligned}
\mathrm{d}(f,\Omega,y) &= \sum_{x\in f^{-1}(y)} \text{sgn det } f'(x) \\
&= \sum_{x\in f^{-1}(y)} \text{sgn det } \begin{pmatrix} f_1'(x_1) & 0 \\ \\ 0 & f_2'(x_2) \end{pmatrix} \\
&= \sum_{\substack{x_i \in f^{-1}(y_i) \\ i=1,2}} \text{sgn det } f_1'(x_1) \text{ sgn det } f_2'(x_2) \\
&= \prod_{i=1}^{2} \sum_{x_i\in f_i^{-1}(y_i)} \text{sgn det } f_i'(x_i) = \mathrm{d}(f_1,\Omega_1,y_1)\mathrm{d}(f_2,\Omega_2,y_2).
\end{aligned}$$

The theorems of Borsuk and Brouwer

Theorem (Borsuk) *Let Ω be a symmetric bounded open neighborhood of $0 \in \mathbb{R}^n$ (i.e. if $x \in \Omega$, then $-x \in \Omega$) and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be an odd mapping (i.e. $f(x) = -f(-x)$). Let $0 \notin f(\partial\Omega)$, then $\mathrm{d}(f, \Omega, 0)$ is an odd integer.*

Theorem (Brouwer fixed point theorem) *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$, $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, be such that $f : \bar{\Omega} \to \bar{\Omega}$. Then f has a fixed point in Ω, i.e. there exists $x \in \bar{\Omega}$ such that $f(x) = x$.*

Proof. Assume f has no fixed points in $\partial\Omega$. Let $h(t, x) = x - tf(x)$, $0 \le t \le 1$. Then $h(t, x) \neq 0$, $0 \le t \le 1$, $x \in \partial\Omega$ and thus $\mathrm{d}(h(t, 0), \Omega, 0) = \mathrm{d}(h(0, 0), \Omega, 0)$ by the homotopy property. Since $\mathrm{d}(\mathrm{id}, \Omega, 0) = 1$ it follows from the solution property that the equation $x - f(x) = 0$ has a solution in Ω. $\square$