I. NUCLEAR SHELL MODEL

A. Single particle orbit

In the single particle shell model, the nucleons in to nucleus are assumed to move in a common potential $V(r)$ similar with atomic electrons, where

$$V(r) = \frac{ze^2}{r}$$

One might then expect to find a shell structure in nuclei, most of the nucleons are paired so that a pair of nucleons contribute **zero** spin and magnetic moment. The pair of nucleons thus from an inertcore, and therefor the spin and magnetic moment of even-even nuclei is **zero**, in case of odd-nuclei the properties of the nucleus is characterized by unpaired proton or neutron. To determine how the nucleons fill the various quantum states, we should specify the mean potential to which each nucleon is subjected.

As a starting point, we may confine ourselves to spherical nuclei and remark that the average force on a nucleon of the center to zero. In a central potential, the orbital angular momentum of each nucleon is a constant of motion, each quantum number **l** there is a series of energy of levels which we shall distinguish by the quantum number **n** associated with number of nodes of the radial wave function. The space of the energy levels depends on the form of the potential. The extreme case for which calculations can be mode are

The Harmonic oscillator $V_{ho} = -V_0 + \frac{1}{2}mw^2r^2$

The uniform Square well $V_{sq} = -V_0, \quad r < R \qquad V_{sq} = \infty, \quad r > R$

The Woods-Saxon potential $V_{ws}(r) = -V_0/(1 + e^{[(r-R)/a]})$ (drawn here for R = 10a).

The nuclear potential is expected to be in between the first two extremes and we desire to arrive at the correct single particle quantum states by interpolating between the two.

B. Harmonic Oscillator Potential Well

$V(r) = \frac{1}{2}mw^2r^2 \qquad \psi_{nlm}(\vec{r}) = U_{nl}(r)Y_{lm}(\Theta, \Phi) = \frac{R_{nl}(r)}{r}Y_{lm}(\Theta, \Phi)$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

The energy eigenvalue corresponding to the eigenfunction

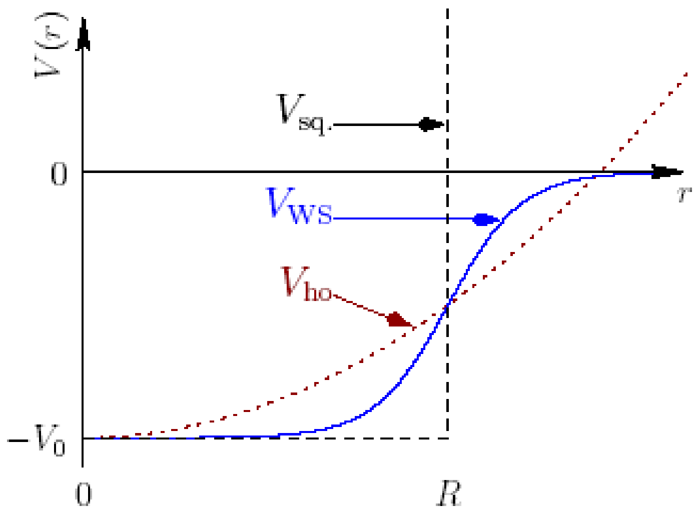

FIG. 1: Radial dependence of different mean potenials.

$$E_{nl} = \hbar\omega_0(2n + l - \frac{1}{2}) = \hbar\omega_0(N + \frac{3}{2}) = E_n - - - - - - - - -(1)$$

where $N = 2(n - 1) + l$ represents the total number of oscillator quantum excited.

with

$$N = 0, 1, 2, ..., n = 1, 2, 3, ...l = 0, 1, 2, ...$$

$$m = -l, ..., l - - - - - - - - -(2)$$

C. Spin-orbit Energy

As shown above the H.O. potential is a reasonable starting point for understanding the structure of single-particle state in nuclei. However, deviation are found beyond $N_{max} = 3$.

To correct for this , additional term must introduced.

If the potential that binds a nucleon to central well has a term that depends on the coupling between $\mathbf{s}$ and $\mathbf{l}$ (intrinsic spin and orbital angular momentum), the single particle energy will be a function of the $\mathbf{j}$ of a state as well. Since $j = s + l$ with two possible state

$\mathbf{s}$ is parallel to $\mathbf{l} \rightarrow \mathbf{j = l + \frac{1}{2}}$

$\mathbf{s}$ is anti-parallel to $\mathbf{l} \rightarrow \mathbf{j = l - \frac{1}{2}}$

Let $\mathbf{a}$ be the strength of spin-orbit term then

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + as.l$$

The parameter $\mathbf{a}$ may depend on the nucleon number A when the spin -orbit term is included the single particle energy

$$E_{Nlj} = (N + \frac{3}{2})\hbar\omega_0 + \frac{a}{2}l \qquad \leftrightarrow j = l + \frac{1}{2}$$
$$E_{Nlj} = (N + \frac{3}{2})\hbar\omega_0 - \frac{a}{2}(l + \frac{1}{2}) \qquad \leftrightarrow j = l - \frac{1}{2}$$

The splitting in the energy between $\mathbf{j = l + \frac{1}{2}}$ and $\mathbf{j = l - \frac{1}{2}}$ is related with value of $\mathbf{a}$

For $\mathbf{a < 0}$ the $\mathbf{j = l + \frac{1}{2}}$ is lowered in energy and the amount of depression increases with increasing l. $\mathbf{j = l + \frac{1}{2}}$ state for larger $\mathbf{l}$ may be pushed down in energy by a amount comparable to $\hbar\omega_0$. The $\mathbf{j = l + \frac{1}{2}}$ states of the largest $\mathbf{l}$ in a shell with $\mathbf{N}$ oscillator quanta may be moved closer to the group of states belonging to the $(\mathbf{N - 1})$ shell below. For example, because of spin-orbit splitting, we find that $j = \frac{9}{2}$ state for $\mathbf{l = 4}$ shell is depressed sufficiently to closed in energy to the $\mathbf{N = 3}$ group. Specified single paretical states by their label $\mathbf{N}$, $\mathbf{l}$, $\mathbf{j}$. and corresponding to these quantum number, we use the a single letter $\mathbf{s, p, f, g, h, i, j}$ for $\mathbf{l = 0, 1, 2, 3, 4, 5}$, ... as shown in the figure and the examples.

TABLE I: Classification of nuclear states

N	$E/\hbar\omega_0$	n	l	$\mid nl>$	π	No.= 2(2l+1)	J	$\mid nl_J>$	No.=(2J+1)	Total No.
0	3/2	1	0	$\mid 1s>$	+	2	1/2	$\mid 1s_{1/2}>$	2	2
1	5/2	1	1	$\mid 1p>$	-	6	3/2	$\mid 1p_{3/2}>$	4	6
							1/2	$\mid 1p_{1/2}>$	2	8
2	7/2	1	2	$\mid 1d>$	+	10	5/2	$\mid 1d_{5/2}>$	6	14
							3/2	$\mid 1d_{3/2}>$	4	18
		2	0	$\mid 2s>$	+	2	1/2	$\mid 2s_{1/2}>$	2	20
3	9/2	1	3	$\mid 1f>$	-	14	7/2	$\mid 1f_{7/2}>$	8	28
							5/2	$\mid 1f_{5/2}>$	6	34
		2	1	$\mid 2p>$	-	6	3/2	$\mid 2p_{3/2}>$	4	38
							1/2	$\mid 2p_{1/2}>$	2	40
4	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

The configuration of a real nuclide (which of course has both neutrons and protons) describes the filling of its energy levels (sub-shells), for protons and for neutrons, in order, with the notation $(nlj)^k$ for each sub-shell, where k is the occupancy of the given sub-shell. Sometimes, for brevity, the completely filled sub-shells are not listed, and if the highest sub-shell is nearly filled, k can be given as a negative number, indicating how far from being filled that sub-shell is. Using the ordering diagram above, and remembering that the maximum occupancy of each sub-shell is $2j+1$, we predict, for example, the configuration for ${}^{17}_{8}$O to be: $(1s_{1/2})^2$ $(1p_{3/2})^4$($1p1/2)^2$ for the protons
and $(1s_{1/2})^2$ $(1p_{3/2})^4$ $(1p1/2)^2$ $(1d_{5/2})^1$ for the neutrons

Notice that all the proton sub-shells are filled, and that all the neutrons are in filled sub-shells except for the last one, which is in a sub-shell on its own. Most of the ground state properties of ${}^{17}_{8}$O can therefore be found from just stating the neutron configuration as $(1d_{5/2})^1$.

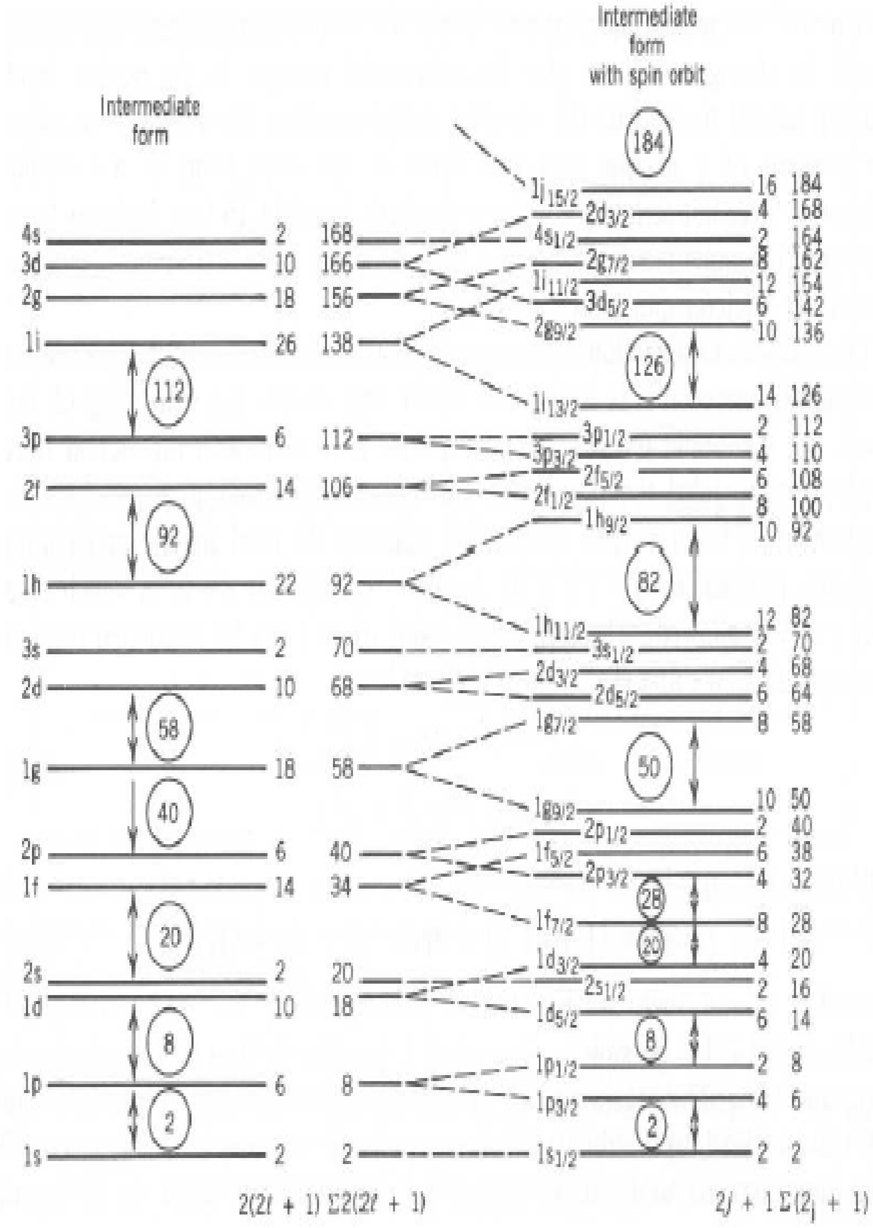

FIG. 2: Single particle energy Levels.

The number of states D_N (i.e, the maximum number of protons and neutrons a harmonic oscillator shell can accommodate is given by

$$D_N = 2 \sum_{allwod(l)} (2l+1) = 2 \sum_{k=1}^{N+1} k = (N+1)(N+2) - - - - - - - - -(3)$$

the factor 2 is account the two possible orientations of nucleon intrinsic spin. The total number of states D_{max}, up to maximum number of Harmonic oscillator quantum N_{max} is given by a sum overall N-value to N_{max}

$$D_{max} = \sum_{N=0}^{N_{max}} D_N = \frac{1}{3}(N_{max}+1)(N_{max}+2)(N_{max}+3) - - - - - - - - -(4)$$

for

$$N_{max} >>> 1 \qquad D_{max} = \frac{1}{3}(N_{max}+2)^3$$

Then the values of $D_{max} = 2, 8, 20, 40, 70, 112,$................................. corresponding to $N_{max} = 0, 1, 2, 3, 4, 5, 6,$ ......

The H. O. frequency ω_0 may be related to the size of the nucleon and to the nucleons number A of the nucleus. The expectation value of r^2 in the state $\mathbf{N\hbar\omega_0}$ can be obtained from the expectation value of the H.O. potential energy

$$< \frac{1}{2}m\omega_0^2 r^2 >_N = \frac{1}{2}(N + \frac{3}{2})\hbar\omega_0$$

The factor $\frac{1}{2}$ in right hand side comes from the fact that, for particle in a three-dimension H. O. potential , the average potential energy is **half** of total energy, the $< r^2 >$ in the state N

$$< r^2 >_N = \frac{\hbar}{m\omega_0}(N + \frac{3}{2}) - - - - - - - - -(5)$$

The mean- square radius of nucleus made of A given by

$$< R^2 > = \frac{2}{A} \sum_{N=0}^{N_{max}} D_N < r^2 >_N$$

using eqs. (4) and (5)

$$< R^2 > = \frac{2}{A} \sum_{N=0}^{N_{max}} (n+1)(N+2)(N+\frac{3}{2})\frac{\hbar}{m\omega_0} - - - - - - - - -(6)$$

the factor 2 from the fact we must consider proton and assuming that neutron and proton number are equal for simplicity. Using mathematical identity

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{k=1}^{n} k^3 = (\frac{n(n+1)}{2})^2$$

The result is

$$\sum_{N}^{N_{max}} (N+1)(N+2)(N+\frac{3}{2}) = \frac{1}{4}(N_{max}+1)(N_{max}+2)^2(N_{max}+3)$$

for $N_{max} >>> 1$

$$\sum_{N}^{N_{max}} (N+1)(N+2)(N+\frac{3}{2}) = \frac{1}{4}(N_{max}+2)^4$$

In the limit of large N_{max} we obtain the result

$$< R^2 >= \frac{2}{A}\frac{\hbar}{m\omega_0}\frac{1}{4}(N_{max}+2)^4$$

$$\hbar\omega_0 = \frac{1}{A}\frac{\hbar^2}{m < R^2 >}\frac{1}{2}(N_{max}+2)^4 - - - - - - - - -(7)$$

The number **A** can be also expressed in terms of N_{max} using (4)

$$A = 2\sum_{N=0}^{N_{max}} D_N = \frac{2}{3}(N_{max}+2)^3$$

factor 2 account for H. O. state can take a neutron as well as proton.

$$(N_{max}+2) = \frac{3}{2}A^{1/3} - - - - - - - - -(8)$$

combining the result of (6) and (8) we obtain

$$\hbar\omega_0 = \frac{1}{A}\frac{\hbar^2}{m < r^2 >}\frac{1}{2}(\frac{3}{2}A)^{4/3}$$

$$\hbar\omega_0 = \frac{\hbar^2}{m\frac{3}{5}(r_0A^{1/3})^2}\frac{3}{4}(\frac{3}{2}A)^{1/3}$$

$$\hbar\omega_0 = \frac{5}{4}(\frac{3}{2})^{1/3}\frac{\hbar^2}{mr_0^2}A^{-1/3} \approx 41A^{-1/3}MeV - - - - - - - - -(9)$$

we use $< r^2 >= \frac{3}{5}(r_0A^{1/3})^2$ adopted a constant density sphere model and $r_0 = 1.2\ fm$.

Problems

Q1 Write down the shell-model configuration of the nucleus $^7_3$Li and hence find its spin, parity and magnetic moment (in nuclear magnetons). Give the two most likely configurations for the first excited state, assuming that only protons are excited.

Q2 A certain odd-parity shell-model state can hold up to a maximum of 16 nucleons; what are its values of j and l?

Q3 The ground state of the radioisotope ${}^{17}_{9}\mathrm{F}$ has spin-parity $J^\pi = \frac{5}{2}^+$ and the first excited state has $J^\pi = \frac{1}{2}^-$. Suggest two possible configurations for the latter state.

Q4 What are the configurations of the ground states of the nuclei ${}^{93}_{41}\mathrm{Nb}$ and ${}^{33}_{16}\mathrm{S}$ and what values are predicted in the single-particle shell model for their spins, parities and magnetic dipole moments?