Lagrangian Interpolation

After reading this chapter, you should be able to:

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

Lagrangian Method

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n *in* $f_n(x)$ *stands for the* n^{th} *order polynomial that approximates the function* $y = f(x)$ *given at* $n+1$ *data points as* $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, *and*

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ *is a weighting function that includes a product of* $n-1$ *terms with terms of* $j = i$ *omitted. The application of Lagrangian interpolation will be clarified using an example.*

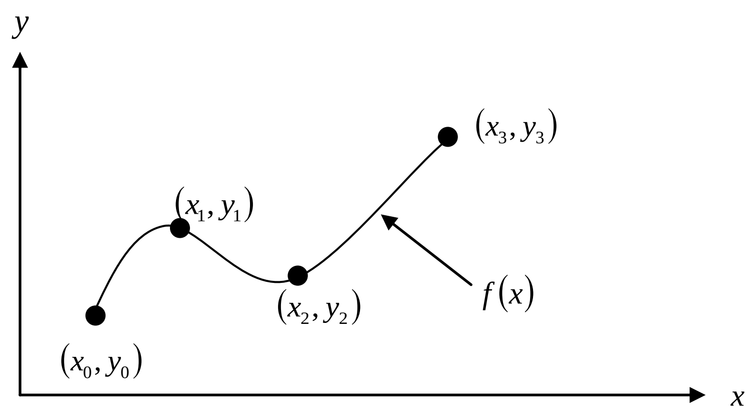

Figure 1 Interpolation of discrete data.

Example 1
The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 *Velocity as a function of time.*

t *(s)*	$v(t)$ *(m/s)*
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ *seconds using a first order Lagrange polynomial.*

Solution
For first order polynomial interpolation (also called linear interpolation), the velocity is given by

$$\begin{aligned} v(t) &= \sum_{i=0}^{1} L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) \end{aligned}$$

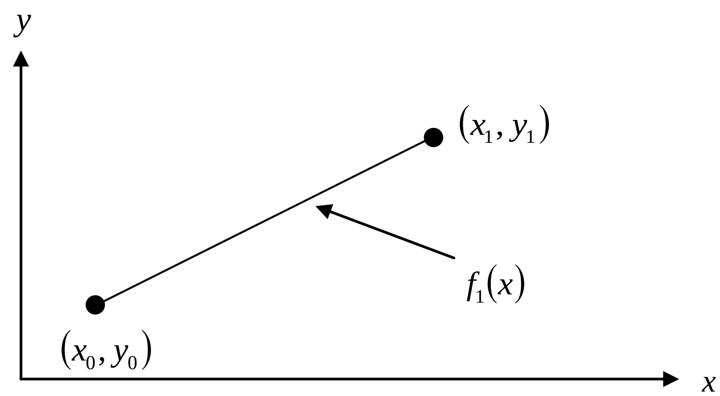

Figure 2 *Linear interpolation.*

Since we want to find the velocity at $t = 16$*, and we are using a first order polynomial, we need to choose the two data points that are closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The two points are* $t_0 = 15$ *and* $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$
$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

Hence

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$
$$= \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35), \quad 15 \le t \le 20$$
$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35) = 0.8(362.78) + 0.2(517.35) = 393.69 \text{ m/s}$$

You can see that $L_0(t) = 0.8$ *and* $L_1(t) = 0.2$ *are like weightages given to the velocities at* $t = 15$ *and* $t = 20$ *to calculate the velocity at* $t = 16$.

Quadratic Interpolation

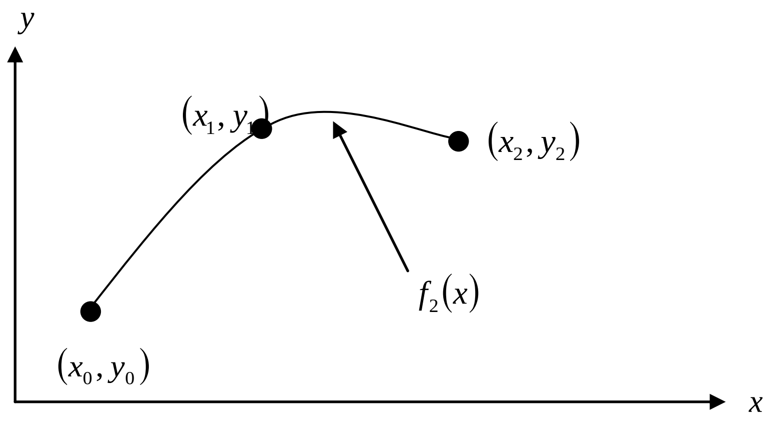

Figure 3 *Quadratic interpolation.*

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 *Velocity as a function of time.*

t *(s)*	$v(t)$ *(m/s)*
0	*0*
10	*227.04*
15	*362.78*
20	*517.35*
22.5	*602.97*
30	*901.67*

a) Determine the value of the velocity at $t = 16$ *seconds with second order polynomial interpolation using Lagrangian polynomial interpolation.*

b) Find the absolute relative approximate error for the second order polynomial approximation.

Solution

a) For second order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$

Since we want to find the velocity at $t = 16$*, and we are using a second order polynomial, we need to choose the three data points that are closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The three points are* $t_0 = 10,\ t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j} = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Hence

$$v(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)v(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)v(t_1) + \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)v(t_2),\ t_0 \le t \le t_2$$

$$v(16) = \frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04) + \frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78) + \frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35)$$

$$= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35) = 392.19 \text{ m/s}$$

b) The absolute relative approximate error $|\in_a|$ *for the second order polynomial is calculated by considering the result of the first order polynomial (Example 1) as the previous approximation.*

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| \times 100 = 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 *Velocity as a function of time*

t *(s)*	$v(t)$ *(m/s)*
0	*0*
10	*227.04*
15	*362.78*
20	*517.35*
22.5	*602.97*
30	*901.67*

a) Determine the value of the velocity at $t = 16$ *seconds using third order Lagrangian polynomial interpolation.*

b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{s}$ *to* $t = 16\text{s}$.
d) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{s}$.

Solution
a) For third order polynomial interpolation (also called cubic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$
$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$

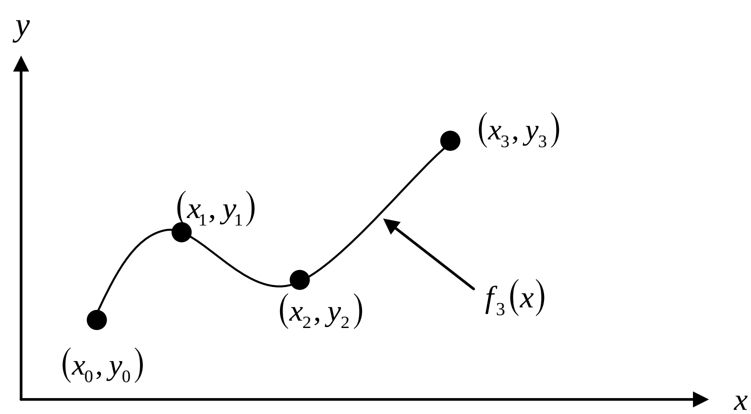

Figure 4 *Cubic interpolation.*

Since we want to find the velocity at $t = 16$*, and we are using a third order polynomial, we need to choose the four data points closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The four points are* $t_0 = 10,\ t_1 = 15,\ t_2 = 20$ *and* $t_3 = 22.5$.
Then

$$t_0 = 10,\ v(t_0) = 227.04$$
$$t_1 = 15,\ v(t_1) = 362.78$$
$$t_2 = 20,\ v(t_2) = 517.35$$
$$t_3 = 22.5,\ v(t_3) = 602.97$$

gives

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{3}\frac{t-t_j}{t_0-t_j}=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

$$L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{3}\frac{t-t_j}{t_1-t_j}=\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)$$

$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{3}\frac{t-t_j}{t_2-t_j}=\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)$$

$$L_3(t)=\prod_{\substack{j=0\\ j\neq 3}}^{3}\frac{t-t_j}{t_3-t_j}=\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)$$

Hence

$$\begin{aligned}v(t)=&\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_1)\\ &+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2)+\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3),\ t_0\le t\le t_3\end{aligned}$$

$$\begin{aligned}v(16)=&\frac{(16-15)(16-20)(16-22.5)}{(10-15)(10-20)(10-22.5)}(227.04)+\frac{(16-10)(16-20)(16-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)\\ &+\frac{(16-10)(16-15)(16-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)\\ &+\frac{(16-10)(16-15)(16-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97)\end{aligned}$$

$$\begin{aligned}&=(-0.0416)(227.04)+(0.832)(362.78)+(0.312)(517.35)+(-0.1024)(602.97)\\ &=392.06\ \text{m/s}\end{aligned}$$

b) The absolute percentage relative approximate error, $|\in_a|$ for the value obtained for $v(16)$ can be obtained by comparing the result with that obtained using the second order polynomial (Example 2)

$$|\in_a|=\left|\frac{392.06-392.19}{392.06}\right|\times 100=0.033269\%$$

c) The distance covered by the rocket between $t=11\text{s}$ *to* $t=16\text{s}$ *can be calculated from the interpolating polynomial as*

$$v(t)=\frac{(t-15)(t-20)(t-22.5)}{(10-15)(10-20)(10-22.5)}(227.04)+\frac{(t-10)(t-20)(t-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+\frac{(t-10)(t-15)(t-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+\frac{(t-10)(t-15)(t-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97), 10\le t\le 22.5$$

$$=\frac{(t^2-35t+300)(t-22.5)}{(-5)(-10)(-12.5)}(227.04)+\frac{(t^2-30t+200)(t-22.5)}{(5)(-5)(-7.5)}(362.78)$$
$$+\frac{(t^2-25t+150)(t-22.5)}{(10)(5)(-2.5)}(517.35)+\frac{(t^2-25t+150)(t-20)}{(12.5)(7.5)(2.5)}(602.97)$$

$$=(t^3-57.5t^2+1087.5t-6750)(-0.36326)+(t^3-52.5t^2+875t-4500)(1.9348)$$
$$+(t^3-47.5t^2+712.5t-3375)(-4.1388)+(t^3-45t^2+650t-3000)(2.5727)$$
$$=-4.245+21.265t+0.13195t^2+0.00544t^3,\ 10\le t\le 22.5$$

Note that the polynomial is valid between $t=10$ *and* $t=22.5$ *and hence includes the limits of* $t=11$ *and* $t=16$.

So

$$s(16)-s(11)=\int_{11}^{16}v(t)dt=\int_{11}^{16}(-4.245+21.265t+0.13195t^2+0.00544t^3)dt$$
$$=\left[-4.245t+21.265\frac{t^2}{2}+0.13195\frac{t^3}{3}+0.00544\frac{t^4}{4}\right]_{11}^{16}=1605\text{ m}$$

d) The acceleration at $t=16$ *is given by*

$$a(16)=\frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t)=-4.245+21.265t+0.13195t^2+0.00544t^3,\ 10\le t\le 22.5$$
$$a(t)=\frac{d}{dt}v(t)=\frac{d}{dt}\left(-4.245+21.265t+0.13195t^2+0.00544t^3\right)$$
$$=21.265+0.26390t+0.01632t^2,\ 10\le t\le 22.5$$

$$a(16)=21.265+0.26390(16)+0.01632(16)^2=29.665\text{ m/s}^2$$