Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.

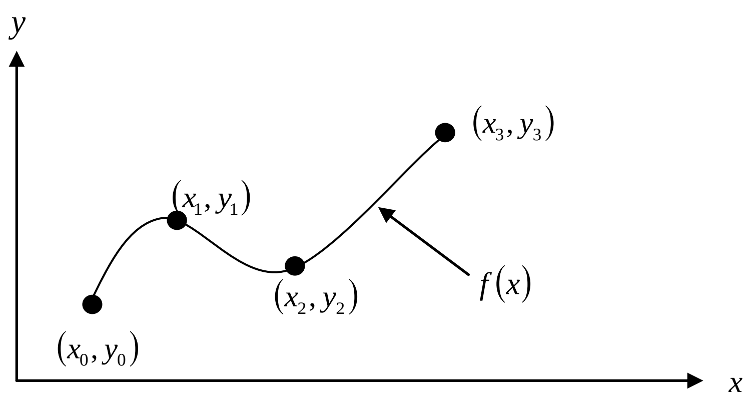

Figure 1 *Interpolation of discrete data.*

<u>Linear Interpolation</u>

Given (x_0, y_0) *and* (x_1, y_1), *fit a linear interpolant through the data. Noting* $y = f(x)$ *and* $y_1 = f(x_1)$, *assume the linear interpolant* $f_1(x)$ *is given by (Figure 2)*

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

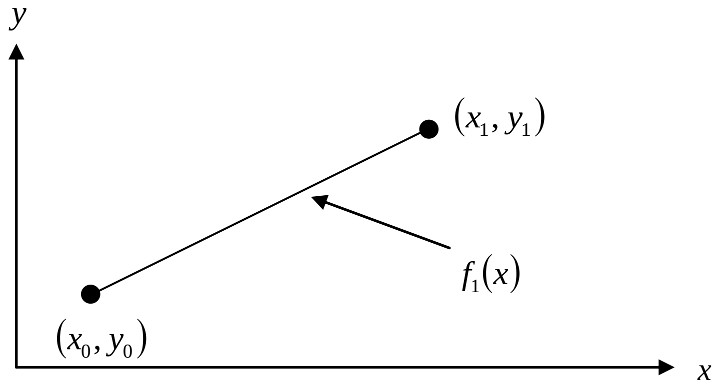

Figure 2 *Linear interpolation.*

Example 1

The upward velocity of a rocket is given as a function of time in Table 1 (Figure 3).

Table 1 *Velocity as a function of time.*

t (s)	$v(t)$ (m/s)
0	*0*
10	*227.04*
15	*362.78*
20	*517.35*
22.5	*602.97*
30	*901.67*

Determine the value of the velocity at $t = 16$ seconds using first order polynomial interpolation by Newton's divided difference polynomial method.

Solution

For linear interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0)$$

Since we want to find the velocity at $t = 16$*, and we are using a first order polynomial, we need to choose the two data points that are closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The two points are* $t = 15$ *and* $t = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{517.35 - 362.78}{20 - 15} = 30.914$$

Hence

$$\begin{aligned} v(t) &= b_0 + b_1(t - t_0) \\ &= 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20 \end{aligned}$$

At $t = 16,$

$$v(16) = 362.78 + 30.914(16 - 15) = 393.69 \text{ m/s}$$

If we expand

$$v(t) = 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20$$

we get

$$v(t) = -100.93 + 30.914t, \qquad 15 \le t \le 20$$

and this is the same expression as obtained in the direct method.

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), *and* (x_2, y_2), *fit a quadratic interpolant through the data. Noting* $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, *and* $y_2 = f(x_2)$, *assume the quadratic interpolant* $f_2(x)$ *is given by*

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$\begin{aligned} f_2(x_0) = f(x_0) &= b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

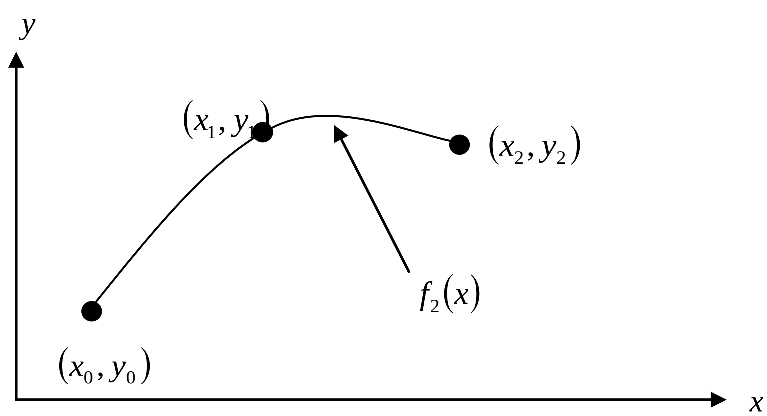

Figure 3 *Quadratic interpolation.*

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 *Velocity as a function of time.*

t (s)	$v(t)$ (m/s)
0	*0*
10	*227.04*
15	*362.78*
20	*517.35*
22.5	*602.97*
30	*901.67*

Determine the value of the velocity at $t = 16$ *seconds using second order polynomial interpolation using Newton's divided difference polynomial method.*

Solution

For quadratic interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

Since we want to find the velocity at $t = 16,$ *and we are using a second order polynomial, we need to choose the three data points that are closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The three points are* $t_0 = 10,$ $t_1 = 15,$ *and* $t_2 = 20$.

Then

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

gives

$$b_0 = v(t_0) = 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10} = 0.37660$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16-10) + 0.37660(16-10)(16-15) = 392.19 \text{ m/s}$$

If we expand

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15), \ \ 10 \le t \le 20$$

we get

$$v(t) = 12.05 + 17.733t + 0.37660t^2, \quad 10 \le t \le 20$$

This is the same expression obtained by the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that $b_0, b_1,$ *and* b_2 *are finite divided differences.* $b_0, b_1,$ *and* b_2 *are the first, second, and third finite divided differences, respectively. We denote the first divided difference by*

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, *and* $f[x_2, x_1, x_0]$ *are called bracketed functions of their variables enclosed in square brackets.*

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ *data points,* $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n)$, *as*

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$
$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 *Velocity as a function of time.*

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ *seconds with third order polynomial interpolation using Newton's divided difference polynomial method.*

b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\,\text{s}$ *to* $t = 16\,\text{s}$.

c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\,\text{s}$.

Solution

a) For a third order polynomial, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Since we want to find the velocity at $t = 16$, *and we are using a third order polynomial, we need to choose the four data points that are closest to* $t = 16$ *that also bracket* $t = 16$ *to evaluate it. The four data points are* $t_0 = 10,\ t_1 = 15,\ t_2 = 20,$ *and* $t_3 = 22.5$.

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$b_0 = v[t_0] = v(t_0) = 227.04$$

$$b_1 = v[t_1, t_0] = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148$$

$$b_2 = v[t_2, t_1, t_0] = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{517.35 - 362.78}{20 - 15} = 30.914, \quad v[t_1, t_0] = 27.148$$

$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0} = \frac{30.914 - 27.148}{20 - 10} = 0.37660$$

$$b_3 = v[t_3, t_2, t_1, t_0] = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$

$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2} = \frac{602.97 - 517.35}{22.5 - 20} = 34.248$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{517.35 - 362.78}{20 - 15} = 30.914$$

$$v[t_3,t_2,t_1]=\frac{v[t_3,t_2]-v[t_2,t_1]}{t_3-t_1}=\frac{34.248-30.914}{22.5-15}=0.44453$$

$$v[t_2,t_1,t_0]=0.37660$$

$$b_3=\frac{v[t_3,t_2,t_1]-v[t_2,t_1,t_0]}{t_3-t_0}=\frac{0.44453-0.37660}{22.5-10}=5.4347\times 10^{-3}$$

Hence

$$\begin{aligned} v(t) &= b_0+b_1(t-t_0)+b_2(t-t_0)(t-t_1)+b_3(t-t_0)(t-t_1)(t-t_2) \\ &= 227.04+27.148(t-10)+0.37660(t-10)(t-15) \\ &\quad +5.5347\times 10^{-3}(t-10)(t-15)(t-20) \end{aligned}$$

At $t=16,$

$$\begin{aligned} v(16) &= 227.04+27.148(16-10)+0.37660(16-10)(16-15) \\ &\quad +5.5347\times 10^{-3}(16-10)(16-15)(16-20) \end{aligned} \qquad =392.06\ \text{m/s}$$

b) The distance covered by the rocket between $t=11\,\text{s}$ *and* $t=16\,\text{s}$ *can be calculated from the interpolating polynomial*

$$\begin{aligned} v(t) &= 227.04+27.148(t-10)+0.37660(t-10)(t-15) \\ &\quad +5.5347\times 10^{-3}(t-10)(t-15)(t-20) \\ &= -4.2541+21.265t+0.13204t^2+0.0054347t^3, \qquad 10\le t\le 22.5 \end{aligned}$$

Note that the polynomial is valid between $t=10$ *and* $t=22.5$ *and hence includes the limits of* $t=11$ *and* $t=16$.

So

$$s(16)-s(11)=\int_{11}^{16} v(t)\,dt$$

$$=\int_{11}^{16}(-4.2541+21.265t+0.13204t^2+0.0054347t^3)\,dt \;=1605\ \text{m}$$

c) The acceleration at $t=16$ *is given by*

$$a(16)=\frac{d}{dt}v(t)\Big|_{t=16}=\frac{d}{dt}\left(-4.2541+21.265t+0.13204t^2+0.0054347t^3\right)$$

$$=21.265+0.26408t+0.016304t^2$$

$$a(16)=21.265+0.26408(16)+0.016304(16)^2=29.664\ \text{m/s}^2$$