Simultaneous Linear Equation

Iterative Method (Successive Over Relaxation Method (SOR))

After reading this chapter, you should be able to:

1. *solve a set of equations using the SOR method,*
2. *recognize the advantages and pitfalls of the SOR method, and*
3. *determine under what conditions the SOR method always converges.*

What is the algorithm for the SOR method?

Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown as follows

$$x_1{}^{r+1} = x_1^r + \frac{w}{a_{11}}(b_1 - a_{11}\, x_1{}^r - a_{12}x_2{}^r - a_{13}\, x_3{}^r)$$

$$x_2{}^{r+1} = x_2^r + \frac{w}{a_{22}}(b_2 - a_{21}x_1{}^{r+1} - a_{22}\, x_2{}^r - a_{23}\, x_3{}^r)$$

$$x_3{}^{r+1} = x_3^r + \frac{w}{a_{33}}(b_3 - a_{31}x_1{}^{r+1} + a_{32}x_2{}^{r+1} - a_{33}\, x_3{}^r)$$

Where the value of **w** *is in the rang 1< **W** <2* ***or*** *0< **W** <1*

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i,

x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified error, the iterations are stopped.

Example 1

The upward velocity of a rocket is given at three different times in the following table

Table 1 *Velocity vs. time data.*

Time, t (s)	*Velocity, v (m/s)*
5	*106.8*
8	*177.2*
12	*279.2*

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12$$

Find the values of $a_1,\ a_2,$ and a_3 using the SOR method with W = 0.95. Assume an initial guess of the solution as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and conduct two iterations.

Solution

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2),$ and (t_3, v_3) where from the above table

$$t_1 = 5, \quad v_1 = 106.8$$
$$t_2 = 8, \quad v_2 = 177.2$$
$$t_3 = 12, \ v_3 = 279.2$$

Requiring that $v(t) = a_1 t^2 + a_2 t + a_3$ passes through the three data points gives

$$v(t_1) = v_1 = a_1 t_1^2 + a_2 t_1 + a_3$$
$$v(t_2) = v_2 = a_1 t_2^2 + a_2 t_2 + a_3$$
$$v(t_3) = v_3 = a_1 t_3^2 + a_2 t_3 + a_3$$

Substituting the data $(t_1,v_1),(t_2,v_2),\text{and}\,(t_3,v_3)$ *gives*

$$a_1(5^2)+a_2(5)+a_3=106.8$$
$$a_1(8^2)+a_2(8)+a_3=177.2$$
$$a_1(12^2)+a_2(12)+a_3=279.2$$

or

$$25a_1+5a_2+a_3=106.8$$
$$64a_1+8a_2+a_3=177.2$$
$$144a_1+12a_2+a_3=279.2$$

The coefficients $a_1, a_2,\text{and}\; a_3$ *for the above expression are given by*

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

Rewriting the equations gives

$$a_1^{r+1}=a_1^r+\left(\frac{w}{25}\right)\left(106.8-25a_1^r-5a_2^r-a_3^r\right)$$
$$a_2^{r+1}=a_2^r+\left(\frac{w}{8}\right)\left(177.2-64a_1^{r+1}-8a_2^r-a_3^r\right)$$
$$a_3^{r+1}=a_3^r+\left(\frac{w}{1}\right)\left(279.2-144a_1^{r+1}-12a_2^{r+1}-a_3^r\right)$$

<u>Iteration 1</u>

Given the initial guess of the solution vector as

$$\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 5\end{bmatrix}$$

we get

$$a_1^{r+1}=1+\left(\frac{0.95}{25}\right)\left(106.8-25*1-5*2-5\right)=3.5384$$
$$a_2^{r+1}=2+\left(\frac{0.95}{8}\right)\left(177.2-64*3.5384-8*2-5\right)=-6.3431$$
$$a_3^{r+1}=5+\left(\frac{0.95}{1}\right)\left(279.2-144*3.5384-12*-6.3431-5\right)=-146.252$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{3.5384 - 1}{3.5384}\right| \times 100 = 71.739\%$$

$$|\in_a|_2 = \left|\frac{-6.3431 - 2}{-6.3431}\right| \times 100 = 131.530\%$$

$$|\in_a|_3 = \left|\frac{-146.252 - 5}{-146.252}\right| \times 100 = 103.419\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.5384 \\ -6.3431 \\ -146.252 \end{bmatrix}$$

and the maximum absolute relative approximate error is 131.530%.

<u>Iteration 2</u>

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.5384 \\ -6.3431 \\ -146.252 \end{bmatrix}$$

Now we get

$$a_1^{r+1} = 3.5384 + \left(\frac{0.95}{25}\right)(106.8 - 25 * 3.5384 - 5 * -6.3431 - (-146.252)) = 10.9981$$

$$a_2^{r+1} = -6.3431 + \left(\frac{0.95}{8}\right)(177.2 - 64 * 10.9981 - 8 * -6.3431 - (-146.252)) = -45.4926$$

$$a_3^{r+1} = -146.252 + \left(\frac{0.95}{1}\right)(279.2 - 144 * 10.9981 - 12 * -45.4926 - (-146.252)) = -727.994$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{10.9981 - 3.5384}{10.9981}\right| \times 100 = 67.827\%$$

$$|\in_a|_2 = \left|\frac{-45.4926 - (-6.3431)}{-45.4926}\right| \times 100 = 86.057\%$$

$$\left|\in_a\right|_3=\left|\frac{-727.994-(-146.252)}{-727.994}\right|\times 100=79.910\%$$

At the end of the second iteration the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} 10.9981 \\ -45.4926 \\ -727.994 \end{bmatrix}$$

and the maximum absolute relative approximate error is 86.057%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	a_1	$\left\|\in_a\right\|_1\%$	a_2	$\left\|\in_a\right\|_2\%$	a_3	$\left\|\in_a\right\|_3\%$
1	*3.5384*	*71.739*	*–6.3431*	*131.530*	*–146.252*	*103.419*
2	*10.9981*	*67.827*	*–45.4926*	*86.057*	*–727.994*	*79.910*
3	*40.9157*	*73.120*	*–205.742*	*77.889*	*–3022.96*	*75.918*
4	*160.0678*	*74.439*	*–846.783*	*75.703*	*–12129.9*	*75.078*
5	*633.8852*	*74.748*	*–3398.40*	*75.083*	*–48315*	*74.894*
6	*2517.417*	*74.820*	*–13543.8*	*74.908*	*–192133*	*74.853*

As seen in the above table, the solution estimates are not converging to the true solution of

$$a_1=0.29048$$
$$a_2=19.690$$
$$a_3=1.0857$$

The above system of equations does not seem to converge. Why?
Well, a pitfall of most iterative methods is that they may or may not converge. However, the solution does always converge using the SOR method, if the class of system of equations is where the coefficient matrix is diagonally dominant, that is

$$\left|a_{ii}\right|\ge\sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}\right| \text{ for all } i$$

$$\left|a_{ii}\right|>\sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}\right| \text{ for at least one } i$$

If a system of equations has a coefficient matrix that is not diagonally dominant, it may or may not converge.

Example 2

Find the solution to the following set of equations using SOR method with w = 1.2?

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$
$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$
$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the SOR method.

Rewriting the equations, we get

$$x_1^{r+1} = x_1^r + (1.2/a_{11}) * (b_1 - 12 * x_1^r - 3 * x_2^r + 5 * x_3^r)$$

$$x_2^{r+1} = x_2^r + (1.2/a_{22}) * (b_2 - x_1^{r+1} - 5 * x_2^r - 3 * x_3^r)$$

$$x_3^{r+1} = x_3^r + (1.2/a_{33}) * (b_3 - 3 * x_1^{r+1} - 7 * x_2^{r+1} - 13 * x_3^r)$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

<u>Iteration 1</u>

$$x_1^{r+1} = 1 + (1.2/12) * (1 - 12*1 - 3*0 + 5*1) = 0.4000$$

$$x_2^{r+1} = 0 + (1.2/5) * (28 - 0.4 - 5*0 - 3*1) = 5.9040$$

$$x_3^{r+1} = 1 + (1.2/13) * (76 - 3*0.4 - 7*5.904 - 13*1) = 2.8897$$

The absolute relative approximate error at the end of the first iteration is

$$\left|\in_a\right|_1 = \left|\frac{0.4000 - 1}{0.4000}\right| \times 100 = 150\%$$

$$\left|\in_a\right|_2 = \left|\frac{5.9040 - 0}{5.9040}\right| \times 100 = 100\%$$

$$\left|\in_a\right|_3 = \left|\frac{2.8897 - 1}{2.8897}\right| \times 100 = 65.395\%$$

The maximum absolute relative approximate error is 150%

<u>Iteration 2</u>

$$x_1^{r+1} = 0.4000 + (1.2/12) * (1 - 12*0.4000 - 3*5.9040 + 5*2.8897) = -0.3063$$

$$x_2^{r+1} = 5.904 + (1.2/5) * (28 - (-0.3063) - 5*5.904 - 3*2.8897) = 3.5321$$

$$x_3^{r+1} = 2.8897 + (1.2/13) * (76 - 3*(-0.3063) - 7*3.5321 - 13*2.8897) = 4.2399$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\in_a\right|_1 = \left|\frac{(-0.3063) - 0.4000}{(-0.3063)}\right| \times 100 = 230.575\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.5321 - 5.9040}{3.5321}\right| \times 100 = 67.152\%$$

$$\left|\in_a\right|_3 = \left|\frac{4.2399 - 2.8897}{4.2399}\right| \times 100 = 31.846\%$$

The maximum absolute relative approximate error is 230.575%. This is greater than the value of 150% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	*0.4000*	*150*	*5.904*	*100*	*2.8897*	*65.395*
2	*-0.3063*	*230.575*	*3.5321*	*67.152*	*4.2399*	*31.846*
3	*1.2216*	*125.076*	*2.6676*	*32.408*	*4.1054*	*3.278*
4	*1.1081*	*10.245*	*2.9646*	*10.019*	*3.9718*	*3.363*
5	*0.9749*	*13.663*	*3.0334*	*2.266*	*3.9910*	*0.481*
6	*0.9905*	*1.576*	*3.0021*	*1.043*	*4.0031*	*0.302*

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$12x_1 + 3x_2 - 5x_3 = 1$$

find the solution using the SOR method with w = 1.2,

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess.

Solution

Rewriting the equations, we get

$$x_1^{r+1} = x_1^r + (1.2/3) * \left(76 - 3 * x_1^r - 7 * x_2^r - 13 * x_3^r\right)$$

$$x_2^{r+1} = x_2^r + (1.2/5) * \left(28 - x_1^{r+1} - 5 * x_2^r - 3 * x_3^r\right)$$

$$x_3^{r+1} = x_3^r + (1.2/-5) * \left(1 - 12 * x_1^{r+1} - 3 * x_2^{r+1} + 5 * x_3^r\right)$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Iteration	x_1	$\lvert\in_a\rvert_1 \%$	x_2	$\lvert\in_a\rvert_2 \%$	x_3	$\lvert\in_a\rvert_3 \%$
1	*25*	*96*	*8.53*10^{-16}*	*100*	*71.560*	*98.603*
2	*-346.712*	*107.211*	*38.4077*	*100*	*-985.429*	*107.262*
3	*5116.432*	*106.776*	*-519.396*	*107.395*	*14558.2*	*106.769*
4	*-75241.2*	*106.8*	*7686.590*	*106.757*	*-214072*	*106.801*
5	*1106732*	*106.799*	*-113014*	*106.801*	*3148833*	*106.799*
6	*1.6*10^{7}*	*106.799*	*1662363*	*106.798*	*-4.6*10^{7}*	*106.799*

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix is not diagonally dominant as

$$|a_{11}| = |3| = 3 \le |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the SOR method may or may not converge.

However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.

Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases.