Simultaneous Linear Equation

Why do we need another method to solve a set of simultaneous linear equations?

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

Iterative Method (Jacobi Method)

After reading this chapter, you should be able to:

1- *solve a set of equations using the Jacobi method,*
2- *recognize the advantages and pitfalls of the Jacobi method, and*
3- *determine under what conditions the Jacobi method always converges.*

What is the algorithm for the Jacobi method?

Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1{}^{r+1} = (b_1 - a_{12}x_2{}^r - a_{13}\, x_3{}^r)/a_{11}$$
$$x_2{}^{r+1} = (b_2 - a_{21}x_1{}^r - a_{23}\, x_3{}^r)/a_{22}$$
$$x_3{}^{r+1} = (b_3 - a_{31}x_1{}^r + a_{32}x_2{}^r)/a_{33}$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates.

At the end of each iteration, one calculates the absolute relative approximate error for each x_i *as*

$$|\epsilon_{ai}| = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| * 100$$

where x_i^{new} *is the recently obtained value of* x_i *,*

x_i^{old} *is the previous value of* x_i *.*

When the absolute relative approximate error for each x_i *is less than the pre-specified error, the iterations are stopped.*

Example 1

The upward velocity of a rocket is given at three different times in the following table

Table 1 *Velocity vs. time data.*

Time, t *(s)*	*Velocity,* v *(m/s)*
5	*106.8*
8	*177.2*
12	*279.2*

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12$$

Find the values of $a_1,\ a_2,$ and a_3 *using the Jacobi method. Assume an initial guess of the solution as*

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and conduct two iterations.

Solution

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2), \text{and } (t_3, v_3)$ *where from the above table*

$$t_1 = 5, \quad v_1 = 106.8$$
$$t_2 = 8, \quad v_2 = 177.2$$
$$t_3 = 12, \quad v_3 = 279.2$$

Requiring that $v(t) = a_1 t^2 + a_2 t + a_3$ *passes through the three data points gives*

$$v(t_1) = v_1 = a_1 t_1^2 + a_2 t_1 + a_3$$
$$v(t_2) = v_2 = a_1 t_2^2 + a_2 t_2 + a_3$$
$$v(t_3) = v_3 = a_1 t_3^2 + a_2 t_3 + a_3$$

Substituting the data $(t_1, v_1), (t_2, v_2), \text{and } (t_3, v_3)$ *gives*

$$a_1(5^2) + a_2(5) + a_3 = 106.8$$
$$a_1(8^2) + a_2(8) + a_3 = 177.2$$
$$a_1(12^2) + a_2(12) + a_3 = 279.2$$

or

$$25a_1 + 5a_2 + a_3 = 106.8$$
$$64a_1 + 8a_2 + a_3 = 177.2$$
$$144a_1 + 12a_2 + a_3 = 279.2$$

The coefficients $a_1, a_2, \text{and } a_3$ *for the above expression are given by*

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Rewriting the equations gives

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$
$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$
$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Iteration 1

Given the initial guess of the solution vector as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25} = \frac{106.8 - 5(2) - (5)}{25} = 3.672$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8} = \frac{177.2 - 64(1) - (5)}{8} = 13.525$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1} = \frac{279.2 - 144(1) - 12(2)}{1} = 111.2$$

The absolute relative approximate error for each x_i *then is*

$$\left|\in_a\right|_1 = \left|\frac{3.672 - 1}{3.672}\right| \times 100 = 72.76\%$$

$$\left|\in_a\right|_2 = \left|\frac{13.525 - 2}{13.525}\right| \times 100 = 85.21\%$$

$$\left|\in_a\right|_3 = \left|\frac{111.2 - 5}{111.2}\right| \times 100 = 95.50\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.672 \\ 13.525 \\ 111.2 \end{bmatrix}$$

and the maximum absolute relative approximate error is 95.50 %.

Iteration 2

The estimate of the solution vector at the end of Iteration 1 is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.672 \\ 13.525 \\ 111.2 \end{bmatrix}$$

Now we get

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25} = \frac{106.8 - 5(13.525) - (111.2)}{25} = -2.881$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8} = \frac{177.2 - 64(3.672) - (111.2)}{8} = -21.126$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1} = \frac{279.2 - 144(3.672) - 12(13.525)}{1} = -411.868$$

The absolute relative approximate error for each x_i *then is*

$$|\epsilon_a|_1 = \left|\frac{-2.881 - 3.672}{-2.881}\right| \times 100 = 277.456\%$$

$$|\epsilon_a|_2 = \left|\frac{-21.126 - 13.525}{-21.126}\right| \times 100 = 164.021\%$$

$$|\epsilon_a|_3 = \left|\frac{-411.868 - 111.2}{-411.868}\right| \times 100 = 126.99\%$$

At the end of the second iteration the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -2.881 \\ -21.126 \\ -411.868 \end{bmatrix}$$

and the maximum absolute relative approximate error is 277.456%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	a_1	$\left\|\in_a\right\|_1\%$	a_2	$\left\|\in_a\right\|_2\%$	a_3	$\left\|\in_a\right\|_3\%$
1	3.6720	72.767	13.525	85.213	111.2	95.504
2	–2.881	227.456	–21.126	164.021	–411.868	126.999
3	24.972	111.537	96.682	121.851	947.576	143.465
4	–52.967	147.146	–296.073	132.655	–4476.952	121.166
5	242.565	121.836	1005.505	129.445	11459.324	139.068
6	–655.201	137.021	–3350.778	130.008	–46716.064	124.530

As seen in the above table, the solution estimates are not converging to the true solution of

$$a_1 = 0.29048$$
$$a_2 = 19.690$$
$$a_3 = 1.0857$$

The above system of equations does not seem to converge. Why?

Well, a pitfall of most iterative methods is that they may or may not converge. However, the solution to a certain classes of systems of simultaneous equations does always converge using the Jacobi method. This class of system of equations is where the coefficient matrix $[A]$ *in* $[A][X]=[C]$ *is diagonally dominant, that is*

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \text{for all } i$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \text{for at least one } i$$

If a system of equations has a coefficient matrix that is not diagonally dominant, it may or may not converge.

Example 2

Find the solution to the following system of equations using the Jacobi method.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Jacobi method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration 1

$$x_1 = \frac{1-3(0)+5(1)}{12} = 0.5000$$

$$x_2 = \frac{28-(1)-3(1)}{5} = 4.8000$$

$$x_3 = \frac{76-3(1)-7(0)}{13} = 5.6154$$

The absolute relative approximate error at the end of the first iteration is

$$|\in_a|_1 = \left|\frac{0.5000-1}{0.5000}\right| \times 100 = 100.00\%$$

$$|\in_a|_2 = \left|\frac{4.8000-0}{4.8000}\right| \times 100 = 100.00\%$$

$$|\in_a|_3 = \left|\frac{5.6154-1}{5.6154}\right| \times 100 = 82.192\%$$

The maximum absolute relative approximate error is 100.00%

Iteration 2

$$x_1 = \frac{1-3(4.8000)+5(5.6154)}{12} = 1.2231$$

$$x_2 = \frac{28-(0.5000)-3(5.6154)}{5} = 2.1308$$

$$x_3 = \frac{76-3(0.5000)-7(4.8000)}{13} = 3.1462$$

At the end of second iteration, the absolute relative approximate error is

$$|\in_a|_1 = \left|\frac{1.2231-0.5000}{1.2231}\right| \times 100 = 59.120\%$$

$$\left|\in_a\right|_2 = \left|\frac{2.1308 - 4.8000}{2.1308}\right| \times 100 = 125.268\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.1462 - 5.6154}{3.1462}\right| \times 100 = 78.482\%$$

The maximum absolute relative approximate error is 125.268%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	*0.5000*	*100.00*	*4.8000*	*100.00*	*5.6154*	*82.192*
2	*1.2231*	*59.120*	*2.1308*	*125.268*	*3.1462*	*78.482*
3	*0.8616*	*41.957*	*3.4677*	*38.553*	*4.4165*	*28.763*
4	*1.0566*	*18.455*	*2.7778*	*24.836*	*3.7801*	*16.836*
5	*0.9639*	*9.617*	*3.1206*	*10.985*	*4.1066*	*7.951*
6	*1.0143*	*4.969*	*2.9433*	*6.024*	*3.9434*	*4.139*

The results are converging to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

Find the solution using the Jacobi method, the following values are the initial guess

$$x_1 = 1, \qquad x_2 = 0, \qquad x_3 = 1$$

Solution

Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	*21.000*	*95.238*	*4.800*	*100*	*2.200*	*54.545*
2	*4.600*	*356.522*	*0.080*	*5900*	*53.080*	*95.855*
3	*–204.867*	*102.245*	*–27.168*	*100.295*	*10.888*	*387.509*
4	*41.544*	*593.132*	*40.041*	*167.851*	*–508.181*	*102.143*
5	*2134.022*	*98.053*	*302.199*	*86.750*	*123.529*	*511.383*
6	*–1215.1*	*275.626*	*–495.322*	*161.011*	*5302.773*	*97.670*

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant.

However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.
Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases.