Simultaneous Linear Equations

Direct Method (Cramer's Rule):-

After reading this chapter, you should be able to:

1. *solve a set of simultaneous linear equations using Cramer's Rule,*
2. *find the determinant of a matrix*

How is a set of equations solved by using Cramer's Rule?

Cramer's Rule is another solution technique that is best suited to small numbers of equations.

Cramer's Rule consists of two steps

1. *Determinant of the denominator: calculate the determinant of the coefficient matrix.*
2. *Determinant of the numerator: calculate the determinant by replacing the column of the unknown in question by the constants $b_1, b_2, \ldots, b_n$.*

Determinant

The determinant can be illustrated for a set of three equations

$$[A][X] = [B]$$

Where [A] is the coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant D of this system is formed from the coefficients of the equation as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Although the determinant and the coefficient matrix [A] are composed of the same elements, they are completely different mathematical concepts. In contrast to a matrix, the determinant is a single number.

The numerical value for the determinant can be computed as

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where the 2 by 2 determinants are called minors.

Cramer's Rule

The rule states that each unknown in a of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of the unknown in question by the constants b_1, b_2, …, b_n . The value of x_1, x_2 and x_3 will be computed as

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Example 1

Use Cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Solution

The determinant D can be written as

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

The minors are

$$d_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1 * 0.5 - 1.9 * 0.3 = -0.07$$

$$d_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5 * 0.5 - 1.9 * 0.1 = 0.06$$

$$d_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5 * 0.3 - 1 * 0.1 = 0.05$$

These minors will be used to calculate the determinant D

$$D = 0.3 * -0.07 - 0.52 * 0.06 + 1 * 0.05 = -0.0022$$

Now calculate the value of x_1

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.044 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

Now calculate the value of x_2

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

Now calculate the value of x_3

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ -0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$