Simultaneous Linear Equations

Direct Method (Gauss Gordan Elimination):-

After reading this chapter, you should be able to:

solve a set of simultaneous linear equations using Gauss Gordan elimination,

How is a set of equations solved by Gauss Gordan Elimination Method?

The gauss gordan method is a variation of gauss elimination. The major difference is that when an unknown is eliminating, it is eliminated from all other equations. In addition, all rows are normalized by dividing them by their pivot elements. Thus, the elimination steps results in an identity matrix rather than an upper triangle matrix. The approach is designed to solve a general set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nn}x_n = b_n$$

Gauss Gordan elimination consists of two steps

1. *Forward Elimination of Unknowns: In this step, the unknown is eliminated in each equation starting with the first equation. This way, the equations are reduced to equations with one unknown in each equation. In this method, the square matrix will be converting to identity matrix.*
2. *Normalizing: In this step, all rows are normalized by dividing them by their pivot elements*

Example 1

Find the values of x_1, x_2 and x_3 using the Gauss Gordan elimination method for the following set of simultaneous linear equations

$$2x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_3 = 7$$
$$x_1 - 3x_2 + 2x_3 = 0$$

Solution

Forward Elimination of Unknowns

Since there are three equations, there will be three steps of forward elimination of unknowns.

First step

Divide Row 1 by 2, to get the resulting equations as

$$\begin{bmatrix} 2 & -2 & 1 & \vdots & 3 \\ 3 & 1 & -1 & \vdots & 7 \\ 1 & -3 & 2 & & 0 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & -1 & 1/2 & & 3/2 \\ 3 & 1 & -1 & \vdots & 7 \\ 1 & -3 & 2 & & 0 \end{bmatrix}$$

Multiply Row 1 by -3 to get zero in Row 2 ***(a_{21})****,*

Multiply Row 1 by -1 to get zero in Row 3 ***(a_{31})****, the resulting equations will be*

$$\begin{bmatrix} 1 & -1 & 1/2 & & 3/2 \\ 3 & 1 & -1 & \vdots & 7 \\ 1 & -3 & 2 & & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1/2} & & \mathbf{3/2} \\ \mathbf{0} & \mathbf{4} & \mathbf{-5/2} & \vdots & \mathbf{5/2} \\ \mathbf{0} & \mathbf{-2} & \mathbf{3/2} & & \mathbf{-3/2} \end{bmatrix}$$

Second step

Now multiply Row 1, Row 2 and Row 3 by 2, to get the resulting equations as

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1/2} & & \mathbf{3/2} \\ \mathbf{0} & \mathbf{4} & \mathbf{-5/2} & \vdots & \mathbf{5/2} \\ \mathbf{0} & \mathbf{-2} & \mathbf{3/2} & & \mathbf{-3/2} \end{bmatrix} \xrightarrow{2R_1, 2R_2, 2R_3} \begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{1} & & \mathbf{3} \\ \mathbf{0} & \mathbf{8} & \mathbf{-5} & \vdots & \mathbf{5} \\ \mathbf{0} & \mathbf{-4} & \mathbf{3} & & \mathbf{-3} \end{bmatrix}$$

Divide Row 2 by 8 and then multiply it by 2 to get zero in Row 1 ***(a_{12})****,*

Divide Row 2 by 8 and then multiply it by 4 to get zero in Row 3 ***(a_{32})****, the resulting equations will be*

$$\begin{bmatrix} 2 & -2 & 1 & & 3 \\ 0 & 8 & -5 & \vdots & 5 \\ 0 & -4 & 3 & & -3 \end{bmatrix} \xrightarrow[R_3 + \left(\frac{4}{8}\right)R_2]{R_1 + \left(\frac{2}{8}\right)R_2} \begin{bmatrix} 2 & 0 & -1/4 & & 17/4 \\ 0 & 8 & -5 & \vdots & 5 \\ 0 & 0 & 1/2 & & -1/2 \end{bmatrix}$$

Now multiply Row 1 by 4 and Row 3 by 2, to get the resulting equations as

$$\begin{bmatrix} 2 & 0 & -1/4 & & 17/4 \\ 0 & 8 & -5 & \vdots & 5 \\ 0 & 0 & 1/2 & & -1/2 \end{bmatrix} \xrightarrow{4R_1, 2R_3} \begin{bmatrix} 8 & 0 & -1 & & 17 \\ 0 & 8 & -5 & \vdots & 5 \\ 0 & 0 & 1 & & -1 \end{bmatrix}$$

<u>*Third step*</u>

Now multiply Row 1 by 1 to get zero in Row 1 ***(a_{13})***

Multiply Row 3 by 5 to get zero in Row 2 ***(a_{23})****, the resulting equations will be*

$$\begin{bmatrix} \mathbf{8} & \mathbf{0} & \mathbf{-1} & \vdots & \mathbf{17} \\ \mathbf{0} & \mathbf{8} & \mathbf{-5} & \vdots & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \vdots & \mathbf{-1} \end{bmatrix} \xrightarrow[R_2 + 5R_3]{R_1 + R_3} \begin{bmatrix} 8 & 0 & 0 & \vdots & 16 \\ 0 & 8 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$$

Normalizing

Divide Row 1 and Row 2 by 8, the resulting equations will be (identity matrix)

$$\begin{bmatrix} \mathbf{8} & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{16} \\ \mathbf{0} & \mathbf{8} & \mathbf{0} & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \vdots & \mathbf{-1} \end{bmatrix} \xrightarrow{(\frac{1}{8})R_1, (\frac{1}{8})R_2} \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$$

From the first equation

$$x_1 = 2$$

From the second equation

$$x_2 = 0$$

From the third equation

$$x_3 = -1$$

Hence the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$