Transcendental Functions

Objectives

- Know what Trigonometric Functions and its derivatives are.
- Know what Inverse Trigonometric Functions and its derivatives are.
- Know what Natural Logarithm Functions and its derivatives are.
- Know what Exponential Functions and its derivatives are.
- Know what Functions a^u and log_a^u and its derivatives are.

A) Trigonometric Functions

Circles of Radius r

- <u>Theorem</u>.

  For an angle θ in standard position, let $P = (x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then

$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}, y \neq 0 \quad \tan\theta = \frac{y}{x}, x \neq 0$$

$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}, x \neq 0 \quad \cot\theta = \frac{x}{y}, y \neq 0$$

Exact Values for Quadrantal Angles

Quadrantal Angles							
θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$0°$	0	1	0	Not defined	1	Not defined
$\frac{\pi}{2}$	$90°$	1	0	Not defined	1	Not defined	0
π	$180°$	0	-1	0	Not defined	-1	Not defined
$\frac{3\pi}{2}$	$270°$	-1	0	Not defined	-1	Not defined	0

Exact Values for Standard Angles

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Properties of the Trigonometric Functions

Domains of Trigonometric Functions

- Domain of sine and cosine functions is the set of all real numbers
- Domain of tangent and secant functions is the set of all real numbers, except odd integer multiples of $\frac{\pi}{2} = 90°$
- Domain of cotangent and cosecant functions is the set of all real numbers, except integer multiples of $\pi = 180°$

Ranges of Trigonometric Functions

- Sine and cosine have range $[-1, 1]$
  - $-1 \leq \sin\theta \leq 1$; $|\sin\theta| \leq 1$
  - $-1 \leq \cos\theta \leq 1$; $|\cos\theta| \leq 1$
- Range of cosecant and secant is $(-\infty, -1] \cup [1, \infty)$
  - $|\csc\theta| \geq 1$
  - $|\sec\theta| \geq 1$
- Range of tangent and cotangent functions is the set of all real numbers

Periods of Trigonometric Functions

- Periodic Properties:

$$\sin(\theta + 2\pi) = \sin\,\theta$$
$$\cos(\theta + 2\pi) = \cos\,\theta$$
$$\tan(\theta + \pi) = \tan\,\theta$$
$$\csc(\theta + 2\pi) = \csc\,\theta$$
$$\sec(\theta + 2\pi) = \sec\,\theta$$
$$\cot(\theta + \pi) = \cot\,\theta$$

- Sine, cosine, cosecant and secant have period 2π
- Tangent and cotangent have period π

Signs of the Trigonometric Functions

- $P = (x, y)$ corresponding to angle θ
  - Definitions of functions, where defined

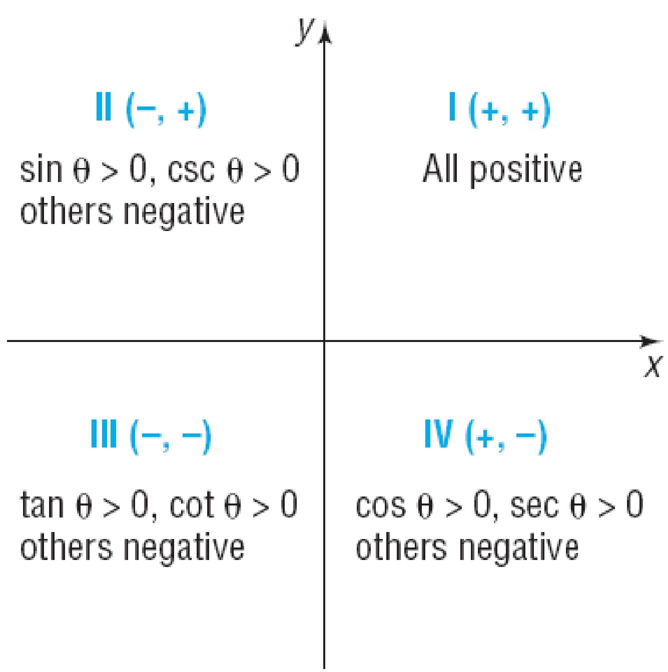
$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x}$$

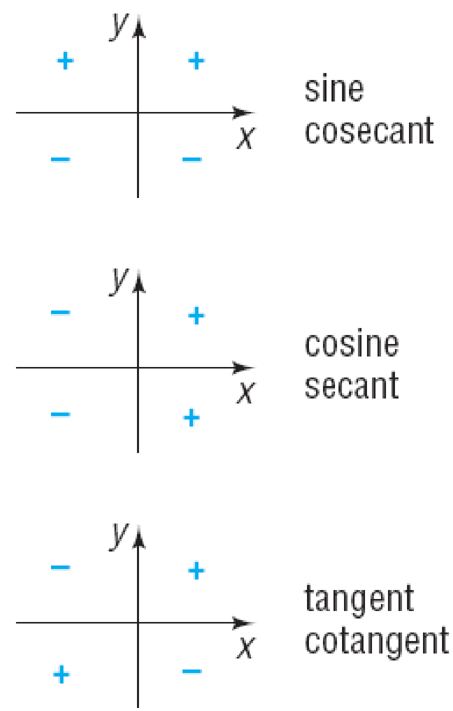
$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x} \qquad \cot\theta = \frac{x}{y}$$

- Find the signs of the functions
  - Quadrant I: $x > 0$, $y > 0$
  - Quadrant II: $x < 0$, $y > 0$
  - Quadrant III: $x < 0$, $y < 0$
  - Quadrant IV: $x > 0$, $y < 0$

Signs of the Trigonometric Functions

Quadrant of θ	$\sin \theta$, $\csc \theta$	$\cos \theta$, $\sec \theta$	$\tan \theta$, $\cot \theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative





Pythagorean Identities

- Unit circle: $x^2 + y^2 = 1$
- $(\sin\,\theta)^2 + (\cos\,\theta)^2 = 1$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Even-Odd Properties

- A function f is *even* if $f(-\theta) = f(\theta)$ for all θ in the domain of f
- A function f is *odd* if $f(-\theta) = -f(\theta)$ for all θ in the domain of f

Even-Odd Properties

- Theorem. [Even-Odd Properties]

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$
$$\csc(-\theta) = -\csc(\theta)$$
$$\sec(-\theta) = \sec(\theta)$$
$$\cot(-\theta) = -\cot(\theta)$$

- Cosine and secant are even functions
- The other functions are odd functions

Fundamental Trigonometric Identities

- Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \tan^2\theta + 1 = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$$

- Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Graphs of the Sine and Cosine Functions

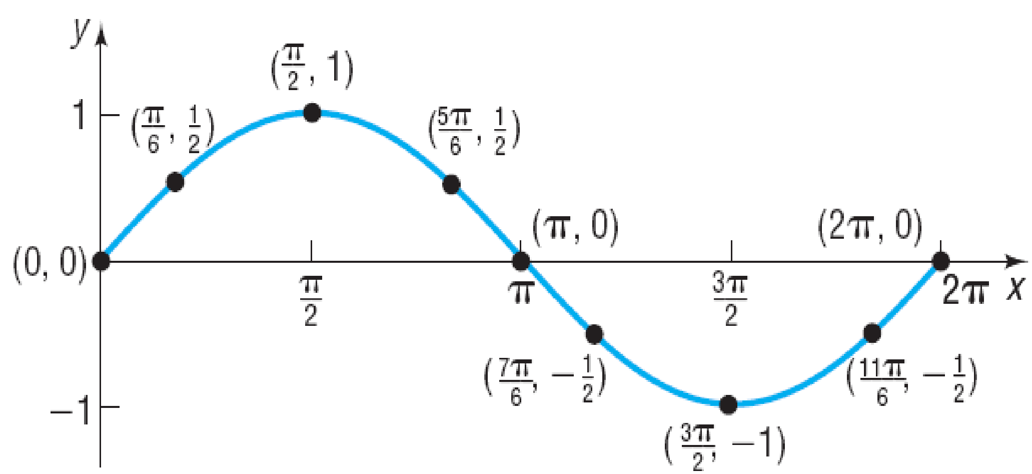
Graphing Trigonometric Functions

- Graph in xy-plane
- Write functions as
  - $y = f(x) = \sin x$
  - $y = f(x) = \cos x$
  - $y = f(x) = \tan x$
  - $y = f(x) = \csc x$
  - $y = f(x) = \sec x$
  - $y = f(x) = \cot x$
- Variable x is an angle, measured in radians
  - Can be any real number

Graphing the Sine Function

- *Periodicity*: Only need to graph on interval $[0, 2\pi]$ (One *cycle*)
- Plot points and graph

x	$y = \sin x$	(x, y)
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
π	0	$(\pi, 0)$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
2π	0	$(2\pi, 0)$



Properties of the Sine Function

- Domain: All real numbers
- Range: [–1, 1]
- Odd function
- Periodic, period 2π
- x-intercepts: …, -2π, $-\pi$, 0, π, 2π, 3π, …
- y-intercept: 0
- Maximum value: $y = 1$, occurring at

$$x = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$

- Minimum value: $y = -1$, occurring at

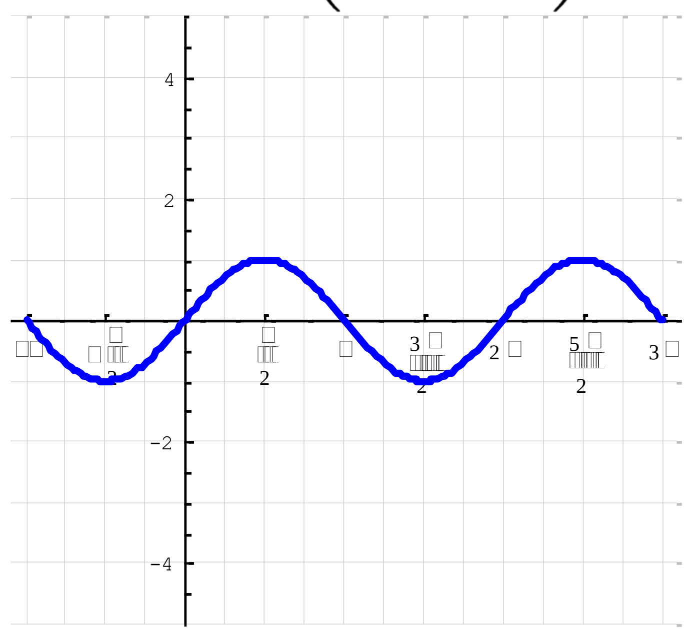
$$x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$$

Transformations of the Graph of the Sine Functions

- Example.

  Problem: Use the graph of $y = \sin x$ to graph $y = -4\sin\left(x + \frac{\pi}{4}\right)$

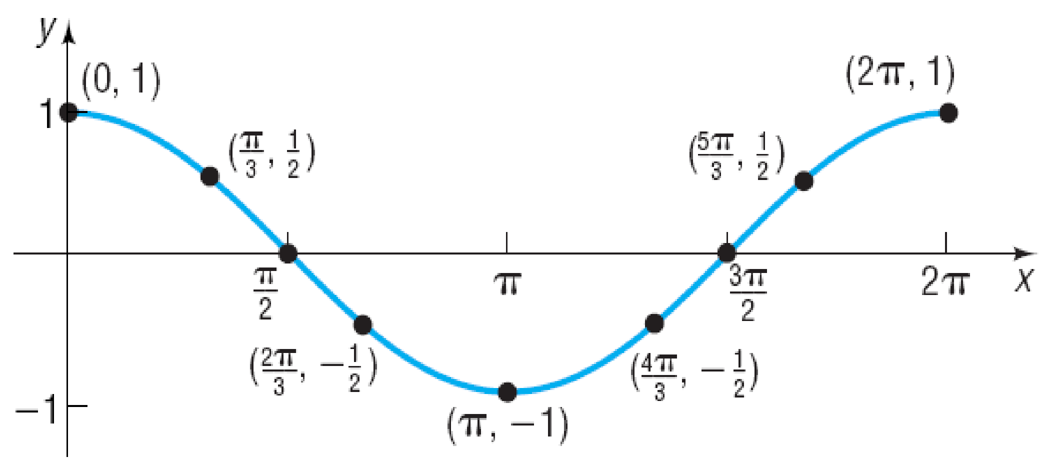
  Answer:



Graphing the Cosine Function

- Periodicity: Again, only need to graph on interval $[0, 2\pi]$ (One *cycle*)
- Plot points and graph

x	$y = \cos x$	(x, y)
0	1	$(0, 1)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3}, \frac{1}{2}\right)$
2π	1	$(2\pi, 1)$



Properties of the Cosine Function

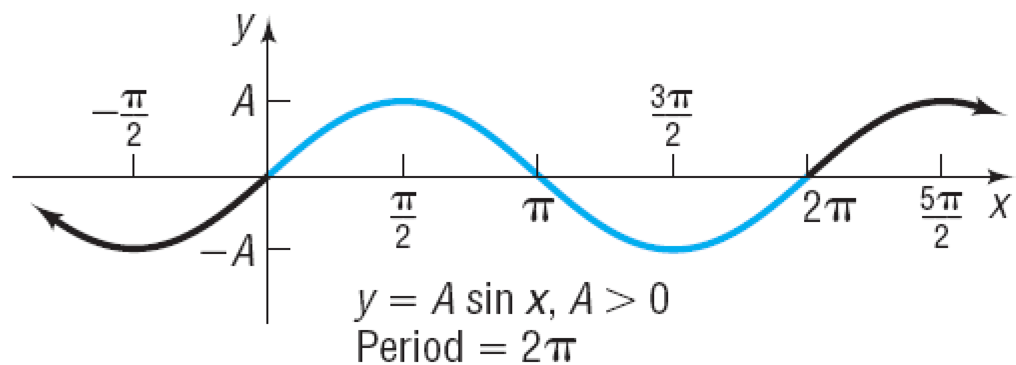
- Domain: All real numbers
- Range: [–1, 1]
- Even function
- Periodic, period 2π
- x-intercepts: $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$
- y-intercept: 1
- Maximum value: $y = 1$, occurring at

  $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$
- Minimum value: $y = -1$, occurring at

  $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$

Amplitude and Period of Sinusoidal Functions

- *Cycle*: One period of $y = \sin(\omega x)$ or $y = \cos(\omega x)$



Amplitude and Period of Sinusoidal Functions

- <u>Theorem</u>. If $\omega > 0$, the amplitude and period of $y = A\sin(\omega x)$ and $y = A\cos(\omega\, x)$ are given by

$$\text{Amplitude} = |A|$$

$$\text{Period} = T = \frac{2\pi}{\omega}.$$

Amplitude and Period of Sinusoidal Functions

- Example.

  Problem: Determine the amplitude and period of $y = -2\cos(\pi x)$

  Answer:

Graphing Sinusoidal Functions

- One cycle contains four important subintervals
- For $y = \sin x$ and $y = \cos x$ these are

$$\left[0, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \pi\right], \left[\pi, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, 2\pi\right]$$

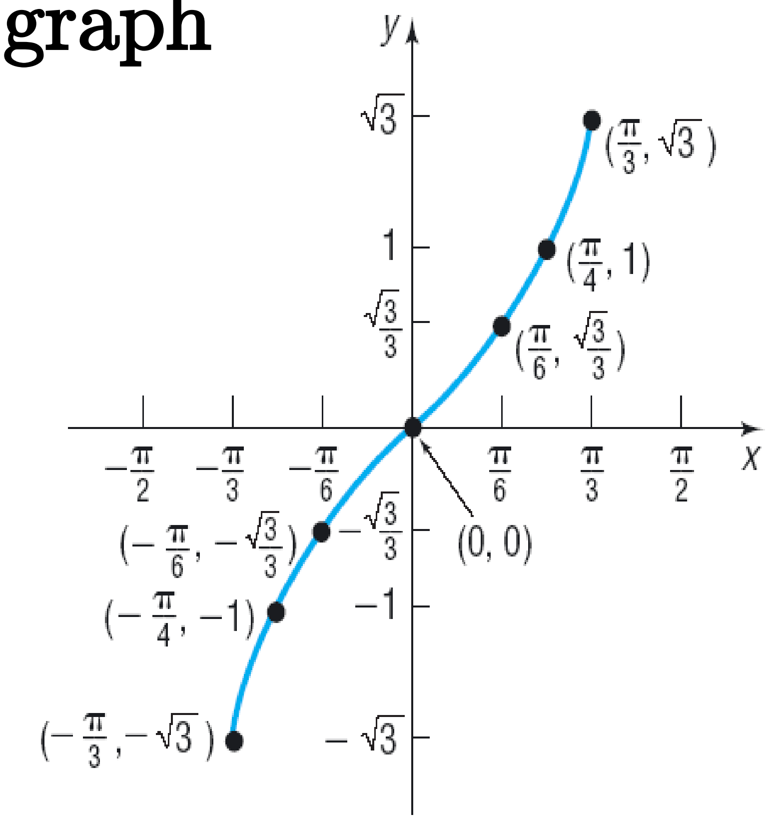
- Gives five key points on graph

Graphs of the Tangent, Cotangent, Cosecant and Secant Functions

Graphing the Tangent Function

- Periodicity: Only need to graph on interval $[0, \pi]$
- Plot points and graph

x	$y = \tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.73$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -0.58$	$\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx 0.58$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$



Properties of the Tangent Function

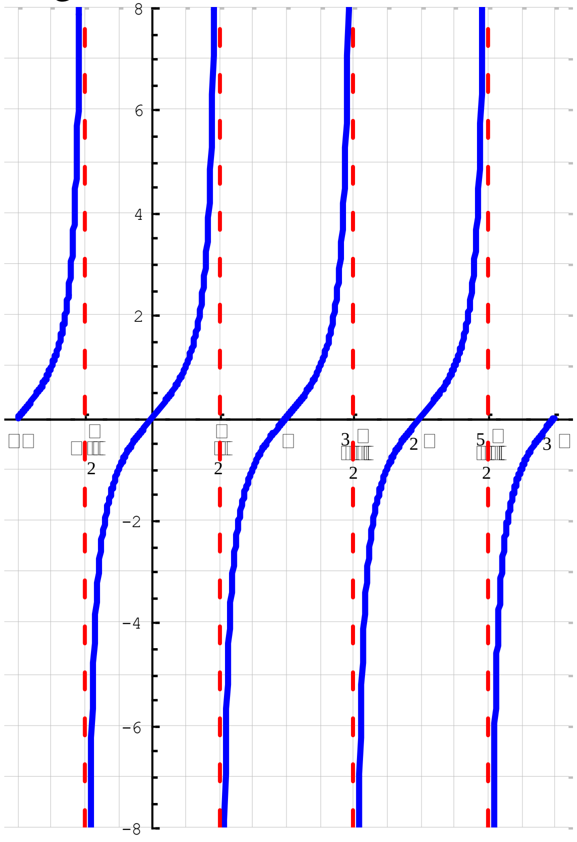
- Domain: All real numbers, except odd multiples of $\frac{\pi}{2}$
- Range: All real numbers
- Odd function
- Periodic, period π
- x-intercepts: …, –2π, –π, 0, π, 2π, 3π, …
- y-intercept: 0
- Asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Transformations of the Graph of the Tangent Functions

- Example.

  Problem: Use the graph of $y = \tan x$ to graph $y = -2\tan\left(\frac{x}{3}\right)$

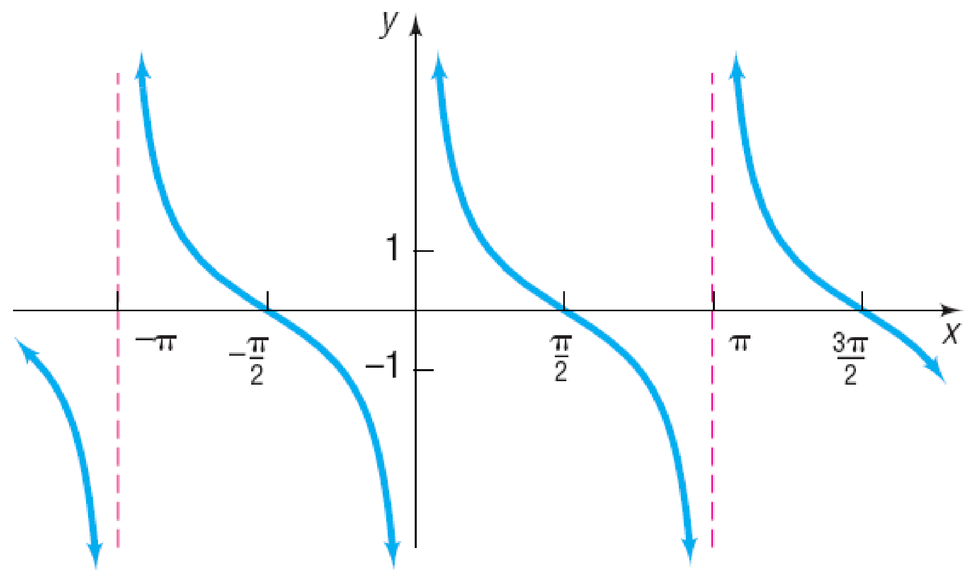
  Answer:



Graphing the Cotangent Function

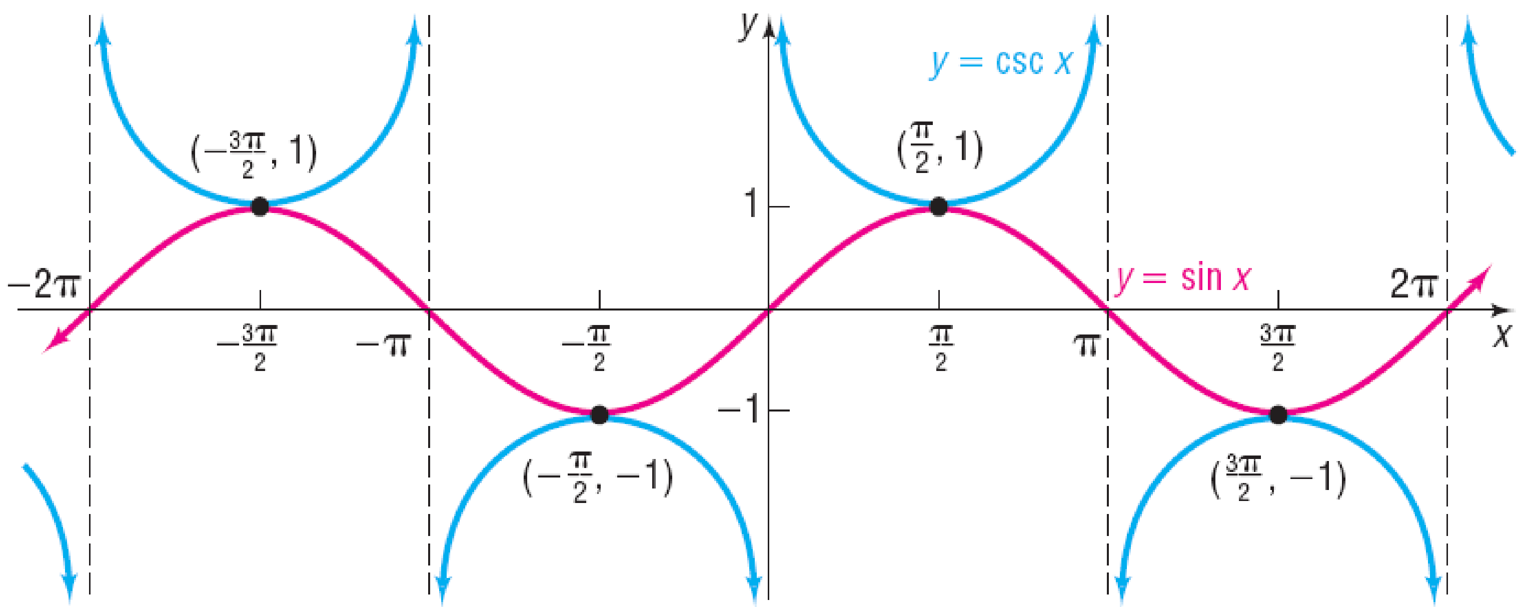
- Periodicity: Only need to graph on interval $[0, \pi]$

x	$y = \cot x$	(x, y)
$\frac{\pi}{6}$	$\sqrt{3}$	$\left(\frac{\pi}{6}, \sqrt{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3}$	$\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3}$	$\left(\frac{2\pi}{3}, -\frac{\sqrt{3}}{3}\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\left(\frac{5\pi}{6}, -\sqrt{3}\right)$



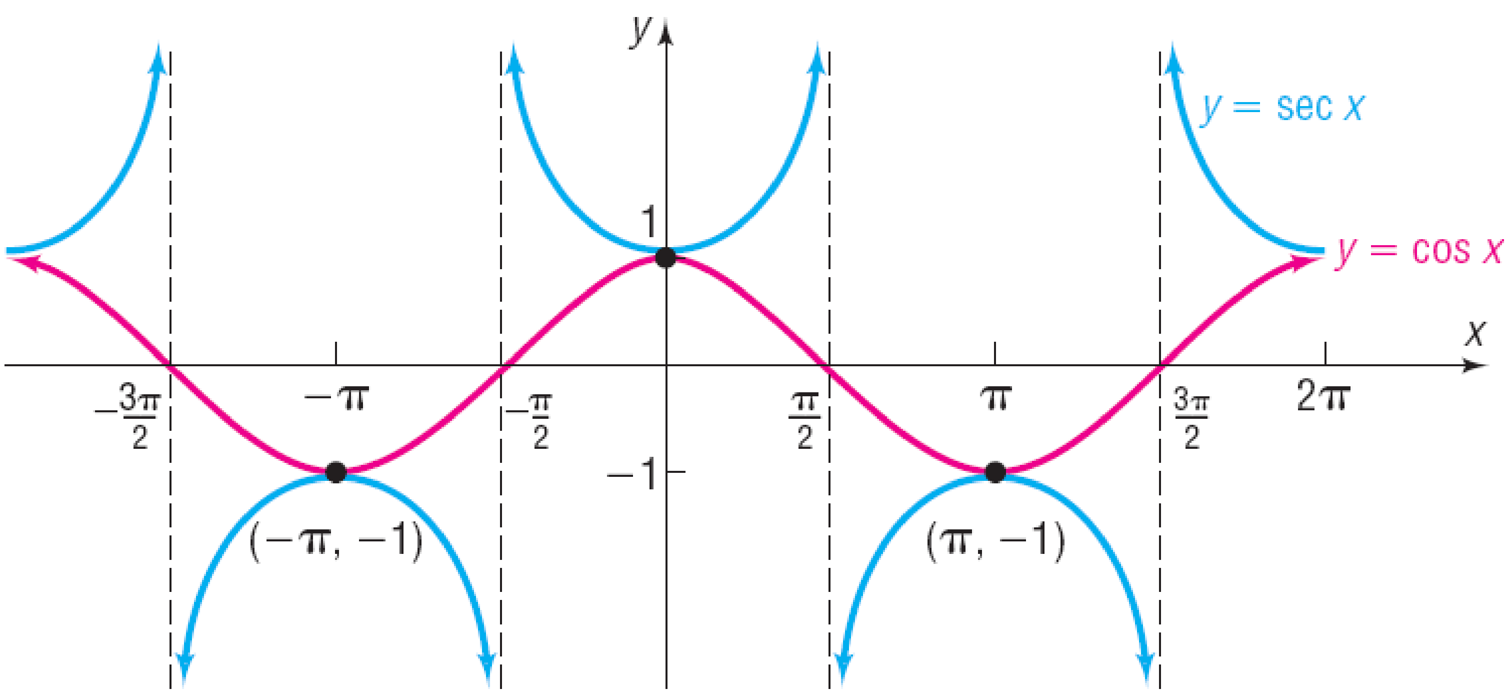
Graphing the Cosecant and Secant Functions

- Use reciprocal identities
- Graph of $y = \csc x$



Graphing the Cosecant and Secant Functions

- Use reciprocal identities
- Graph of $y = \sec x$



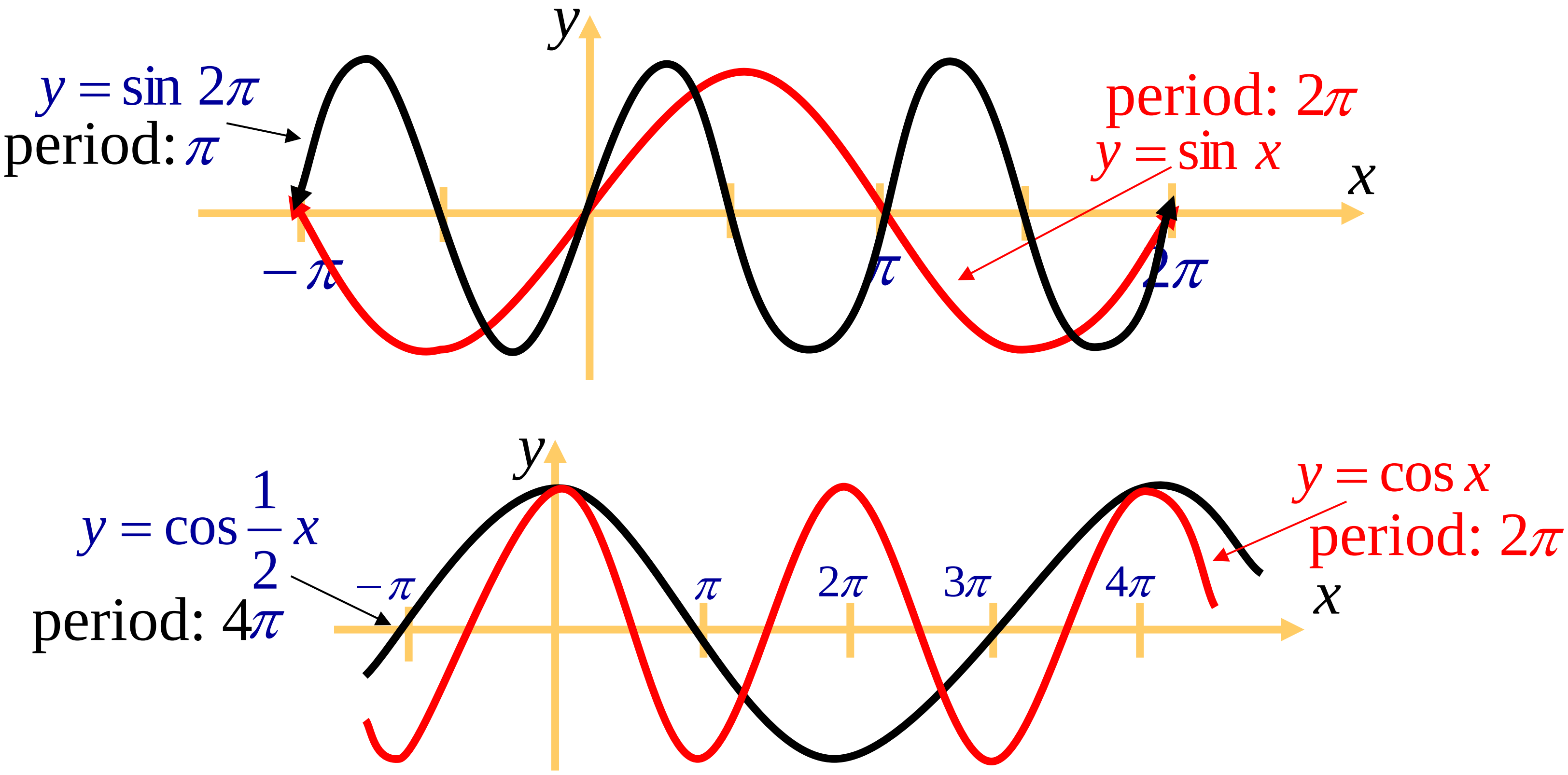
Transformations of the Graph of the Trigonometric Functions

If A, C, B & ω *Are real numbers* , $\omega > 0$,
$y = A\sin(\omega x + C)$+B, ..Than,

- The range of function is $\mp A$.
- If A< 1 than, the function will reflect.
- The period $T = \frac{2\pi}{\omega}$ for (sin, cos, sec & csc).
- The period $T = \frac{\pi}{\omega}$ for (tan & cot).
- If ω> 1 than, the period will shrink horizontally.
- If 0<ω< 1 than, the period will stretch horizontally.

- The start of period $x_{start} = \frac{-C}{\omega}$.
- If $C > 0$, the period will move rightward.
- If $C > 0$, the period will move leftward.
- The end of period $x_{end} = x_{start} + T$.
- If B > 0, the function will move upward.
- If B < 0, the function will move downward.

Examples:



y
$y = \sin 2\pi$
period: π
period: 2π
$y = \sin\ x$
x
$-\pi$
π
2π
y
$y = \cos\frac{1}{2}x$
period: 4π
$-\pi$
π
2π
3π
4π
$y = \cos x$
period: 2π
x

Derivatives of Trigonometric Functions

When we talk about the function *f* defined for all real numbers *x* by $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x.
A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot

$$f(x) = \sin x$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin \mathrm{h} - \sin x}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$

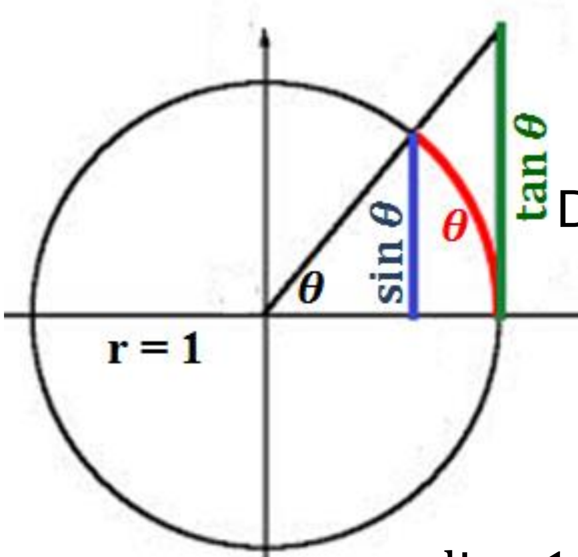
$$= \lim_{h\to 0}\sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

$$\lim_{h \to 0} \frac{\sin h}{h} = ?$$

For first quadrant all,
$\sin\theta$, θ , *and* $\tan\theta$
are positive so we can write

$$\sin\theta < \theta < \tan\theta$$



Divide it by $\sin\theta$ $\quad 1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

Take the inverse $\quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

$$\lim_{\theta \to 0^+} 1 = 1 \qquad \& \qquad \lim_{\theta \to 0^+} \cos\theta = 1$$

By the Squeeze Theorem, we have: $\displaystyle \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

However, the function (sin θ)/θ is an even function.
So, its right and left limits must be equal.
Hence, we have:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0}\frac{\cos h-1}{h}=?$$

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right)$$

$$=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

$$=-\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right)$$

$$=-\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1}=-1\cdot\left(\frac{0}{1+1}\right)=0$$

$$\lim_{h\to 0}\frac{\cos h-1}{h}=0$$

$$f'(x) = \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1$$

$$= \cos x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

Using the same methods as in the case of finding derivative of $\sin x$, we can prove:

$$\frac{d}{dx}(\cos x) = -\sin x$$

The tangent function can also be differentiated by using the definition of a derivative.
However, it is easier to use the Quotient Rule together with formulas for derivatives of $\sin x$ & $\cos x$ as follows.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

The derivatives of the remaining trigonometric functions — csc, sec, and cot — can also be found easily using the Quotient Rule.

All together:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example:

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$

For what values of *x* does the graph of *f* have a horizontal tangent?

$$f'(x) = \frac{(1+\tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

$$\tan^2 x - \sec^2 x = \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}$$

$$= -\frac{\cos^2 x}{\cos^2 x} = -1$$

B) Inverse Trigonometric functions

Arcsine

Arccosine

Lets review inverse functions

Find the inverse of $f(x) = 3x + 6$

$y = 3x + 6$

Inverse functions switch domain and range.

$x = 3y + 6$ (solve for y) So,

$$x – 6 = 3y$$

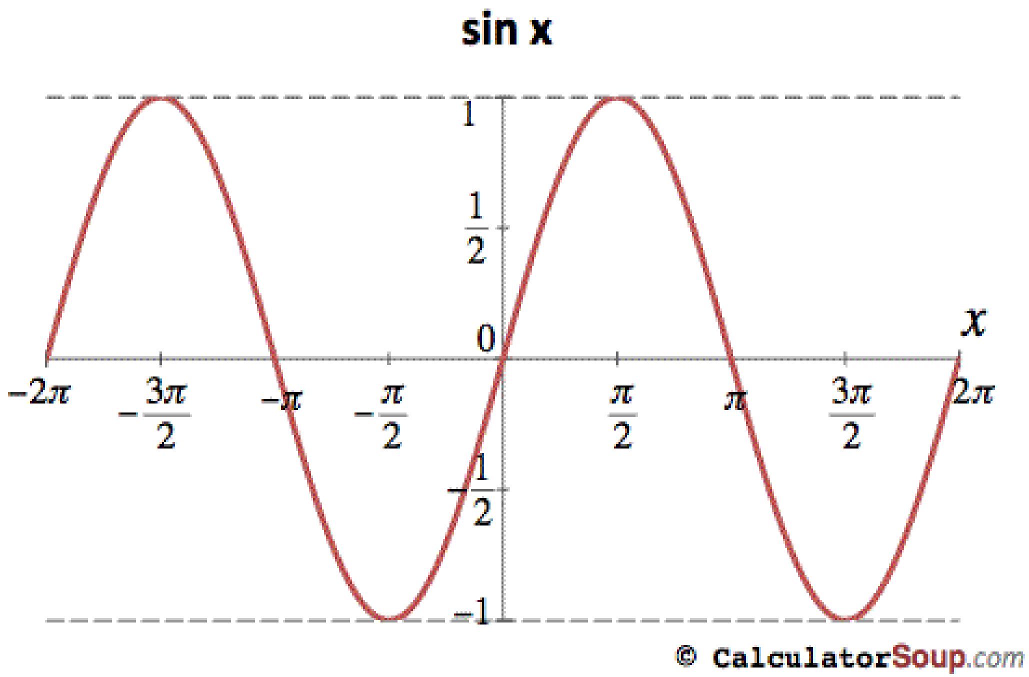
$$⅓ x – 2 = y$$

$$f^{-1}(x) = ⅓ x - 2$$

What is the domain and range of the Sine function

Domain: All real numbers

Range: - 1 to 1



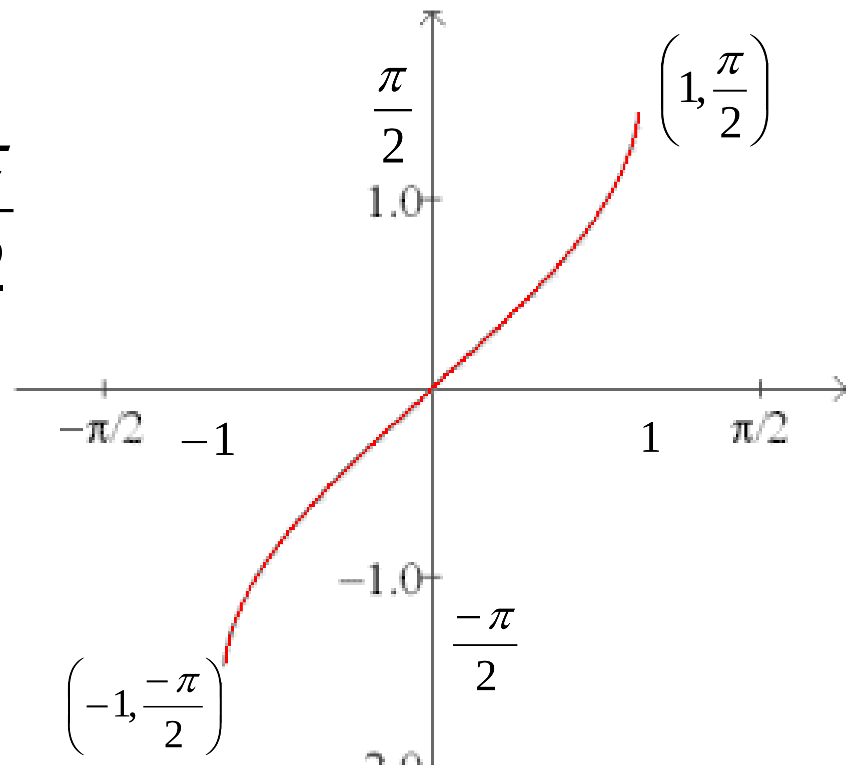
What is the domain and range of the Inverse of the Sine function

The inverse's Domain would be -1 to 1;

Yet the Range is not all real numbers.

Range

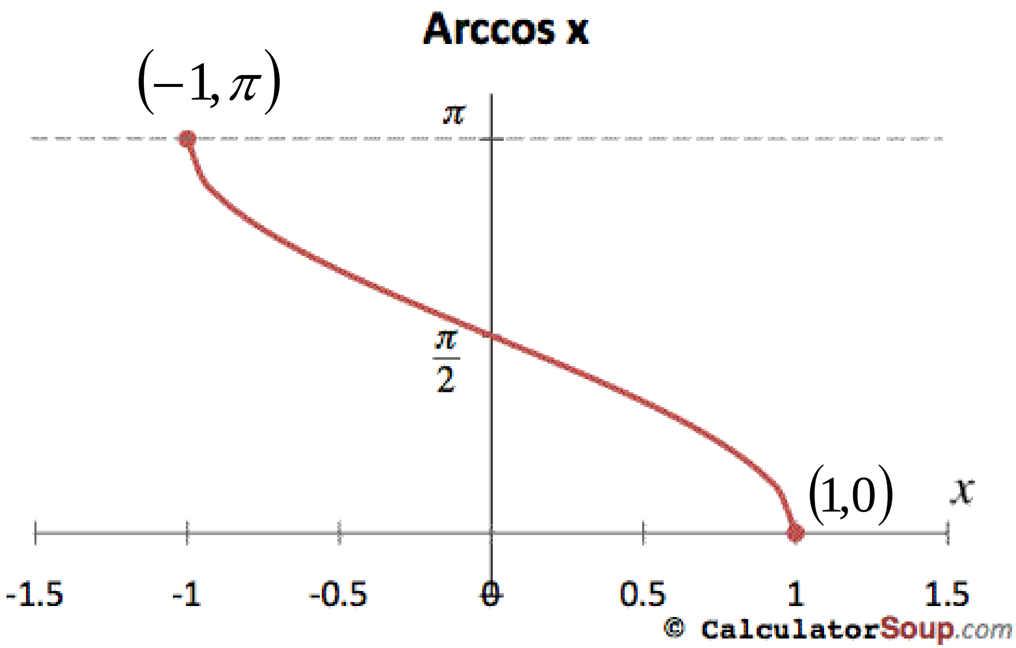
$$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$$



Inverse of the Cosine

Domain: [-1,1]

Range: $[\pi, 0]$

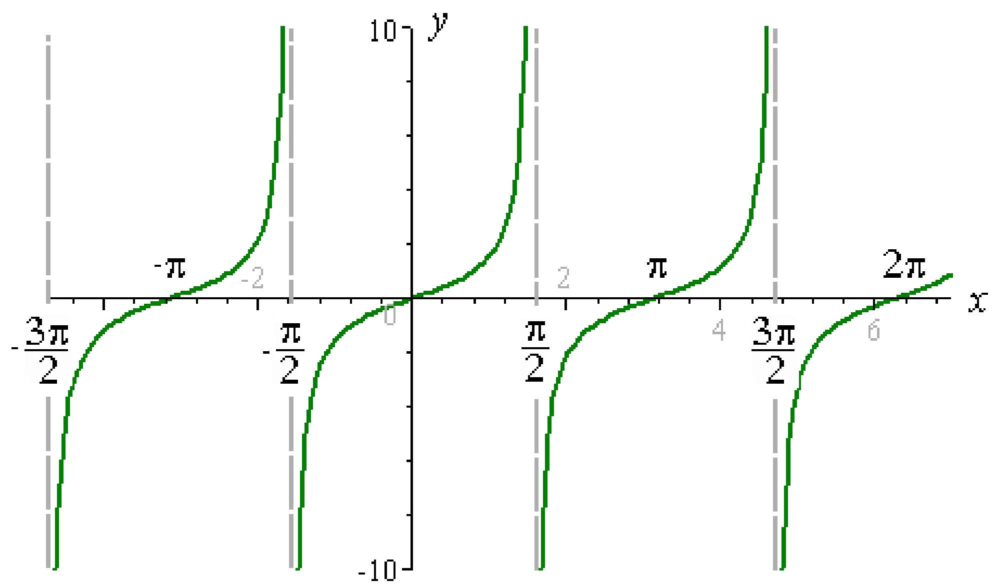


The Tangent function

Domain: All real numbers except $\dfrac{\pi}{2} + n\pi$

Where n is a integer

Range: All real numbers

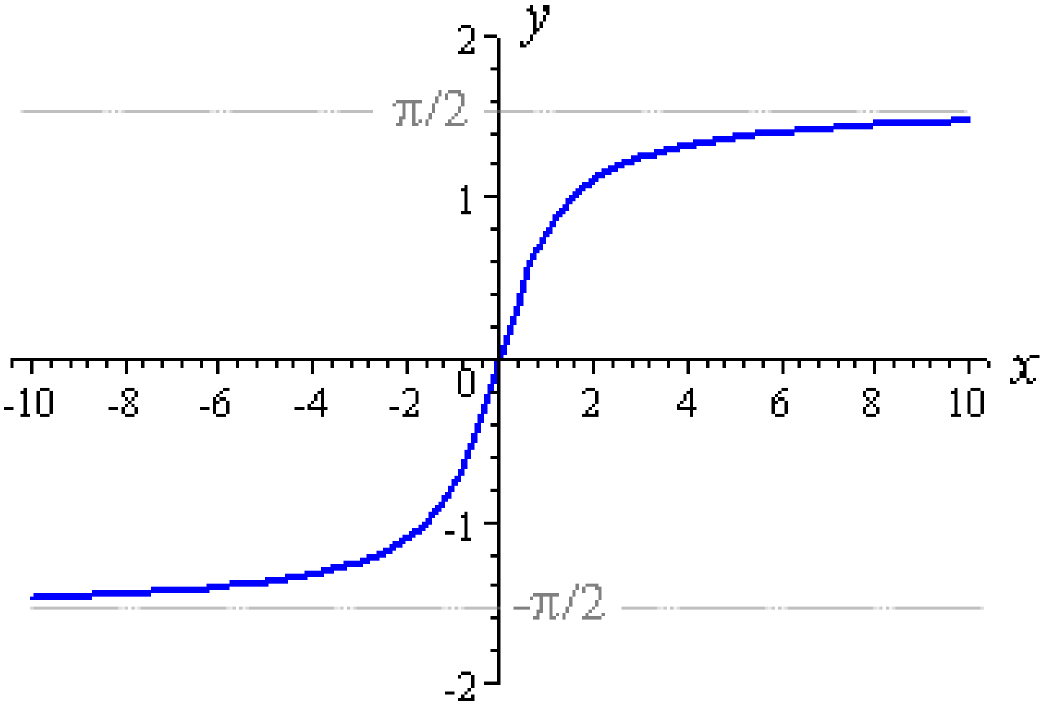


Inverse of Tangent function

Domain: All real numbers

Range:

$$\frac{-\pi}{2} < y < \frac{\pi}{2}$$



Definition of Arcsine

The arc sine is the inverse function of the sine. What is the angle that has a sine equal to a given number

$$\frac{\sqrt{2}}{2}$$

$$\arcsin\frac{\sqrt{2}}{2}=\frac{\pi}{4}$$

Since,

$$\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

Examples

Find the exact value.

For these problems

All answers are in the

First Quadrant.

$$\arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$\arcsin \frac{1}{2} = \frac{\pi}{6}$$

$$\arctan 1 = \frac{\pi}{4}$$

Examples

Be careful to make sure it is in the Range

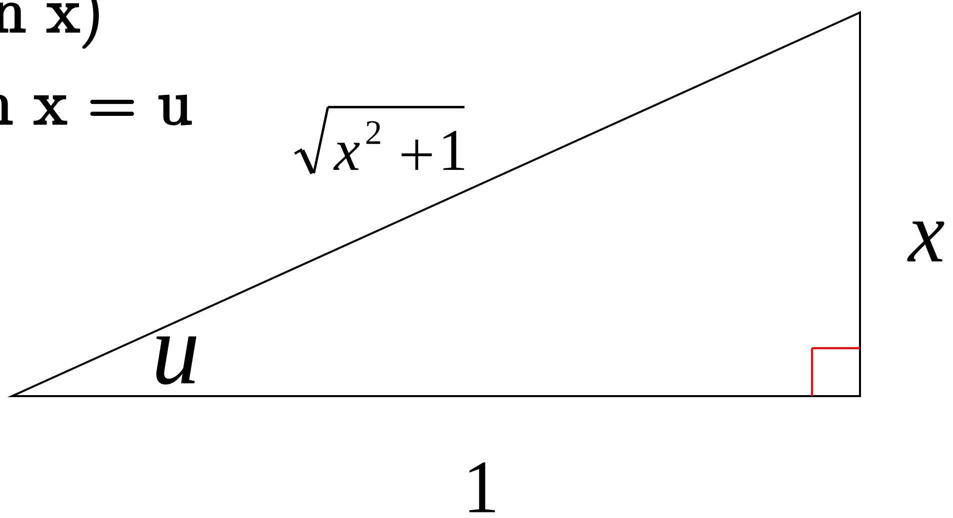
$$\arcsin\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$$

$$\arccos\left(\cos\frac{7\pi}{6}\right)=\frac{5\pi}{6}$$

Solve using a triangle

Cot(arctan x)

Let arctan x = u



Cot u =

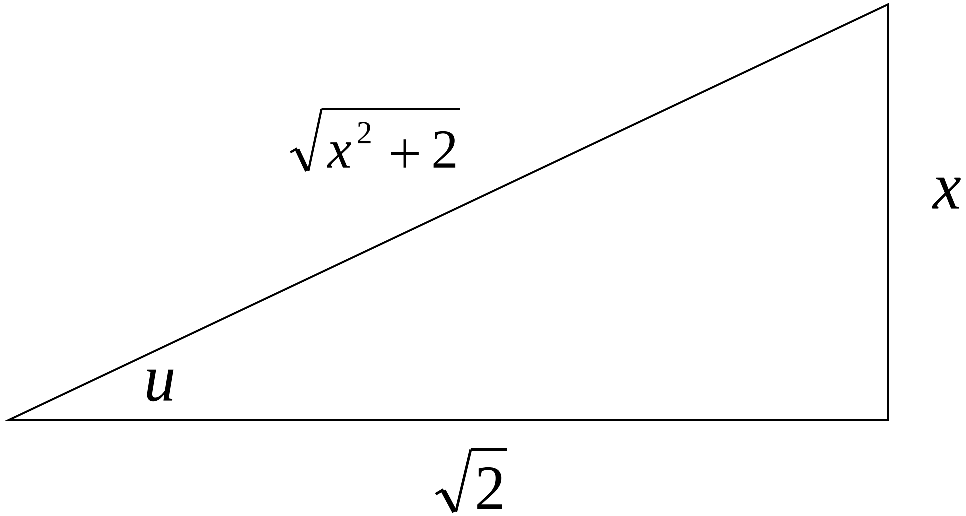
$$\frac{1}{x}$$

Solve

$$\csc\left(\arctan\frac{x}{\sqrt{2}}\right)$$

Let

$$u = \arctan\frac{x}{\sqrt{2}}, then\text{K}\ \tan u = \frac{x}{\sqrt{2}}$$



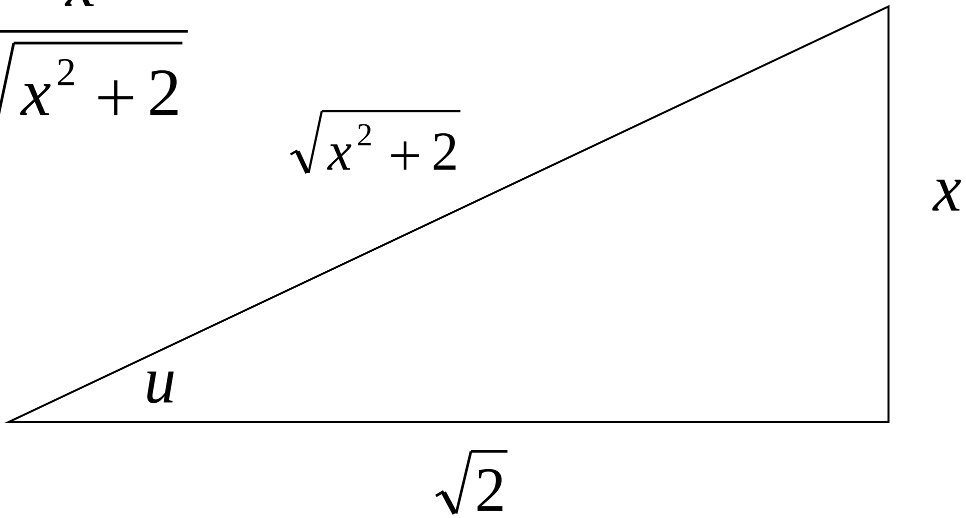
Solve

$$\csc\left(\arctan\frac{x}{\sqrt{2}}\right)$$

So $$\csc u = \frac{1}{\sin u}$$

$$\sin u = \frac{x}{\sqrt{x^2+2}}$$

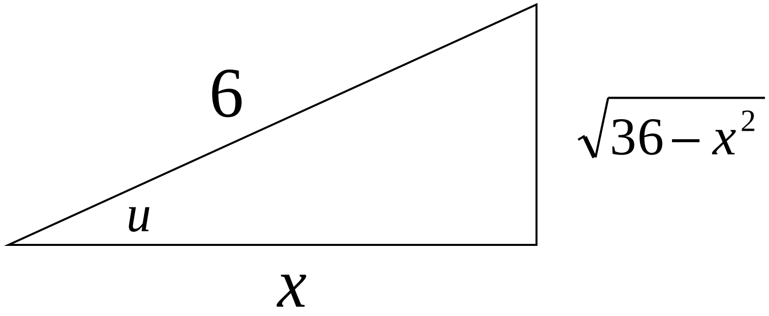
$$\csc u = \frac{1}{\dfrac{x}{\sqrt{x^2+2}}}$$



$$\csc u = \frac{\sqrt{x^2+2}}{x}$$

Show that $$\arcsin\frac{\sqrt{36-x^2}}{6}=\arccos\frac{x}{6}$$

Using arccos



$$x^2+?^2=6^2$$

$$?^2=36-x^2$$

$$?=\sqrt{36-x^2}$$

$$\sin u=\frac{\sqrt{36-x^2}}{6}$$

$$\cos u=\frac{x}{6}$$

Derivatives of Inverse Trigonometric Functions

The next theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos *u*, arccot *u*, and arccsc *u* are the *negatives* of the derivatives of arcsin *u*, arctan *u*, and arcsec *u*, respectively.

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example:

Differentiating Inverse Trigonometric Functions

a. $$\frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$$

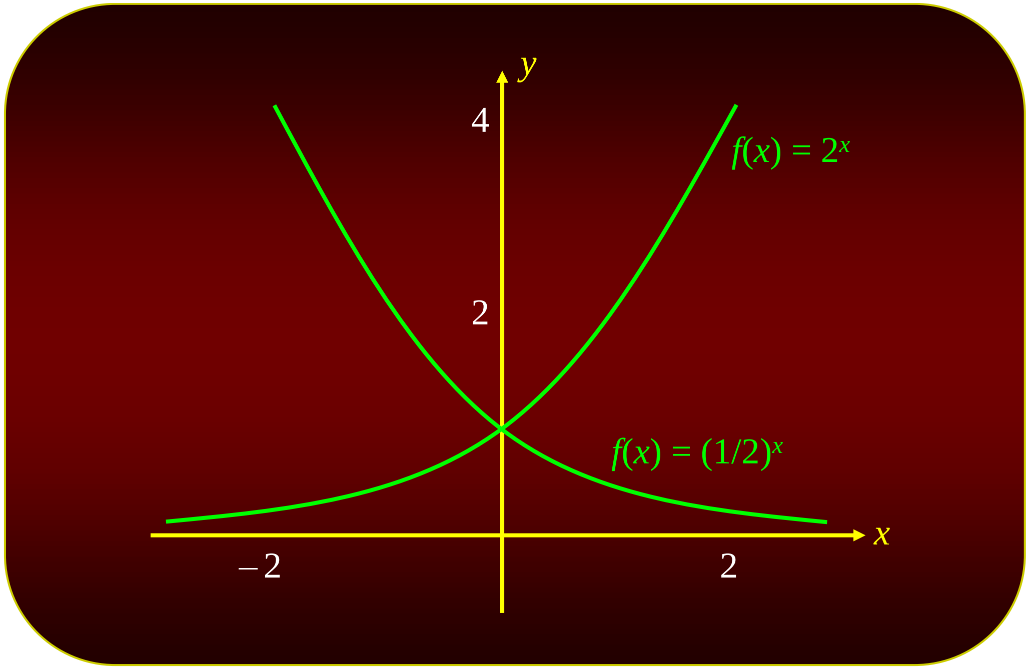
b. $$\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$$

c. $$\frac{d}{dx}\left[\arcsin\sqrt{x}\right] = \frac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

d. $$\frac{d}{dx}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \frac{2}{\sqrt{e^{4x}-1}}$$

The absolute value sign is not necessary because $e^{2x} > 0$.

C) Exponential Functions



Exponential Function

- The function defined by

$$f(x) = b^x \qquad (b > 0,\ b \neq 1)$$

  is called an exponential function with base b and exponent x.
- The domain of f is the set of all real numbers.

Example

- The exponential function with base 2 is the function

$$f(x) = 2^x$$

with domain $(-\infty, \infty)$.

- The values of $f(x)$ for selected values of x follow:

$$f(3) = 2^3 = 8$$

$$f\left(\frac{3}{2}\right) = 2^{3/2} = 2 \cdot 2^{1/2} = 2\sqrt{2}$$

$$f(0) = 2^0 = 1$$

Example

- The exponential function with base 2 is the function

$$f(x) = 2^x$$

with domain $(-\infty, \infty)$.

- The values of $f(x)$ for selected values of x follow:

$$f(-1) = 2^{-1} = \frac{1}{2}$$

$$f\left(-\frac{2}{3}\right) = 2^{-2/3} = \frac{1}{2^{2/3}} = \frac{1}{\sqrt[3]{4}}$$

Laws of Exponents

- Let a and b be positive numbers and let x and y be real numbers. Then,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $\left(b^x\right)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Examples

- Let $f(x) = 2^{2x-1}$. Find the value of x for which $f(x) = 16$.

<u>Solution</u>

- We want to solve the equation

$$2^{2x-1} = 16 = 2^4$$

- But this equation holds if and only if

$$2x - 1 = 4$$

giving $x = \dfrac{5}{2}$.

Examples

- Sketch the graph of the exponential function $f(x) = 2^x$.

<u>Solution</u>

- First, recall that the domain of this function is the set of real numbers.
- Next, putting $x = 0$ gives $y = 2^0 = 1$, which is the y-intercept.

  (There is no x-intercept, since there is no value of x for which $y = 0$)

Examples

- Sketch the graph of the exponential function $f(x) = 2^x$.

<u>Solution</u>

- Now, consider a few values for x:

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32

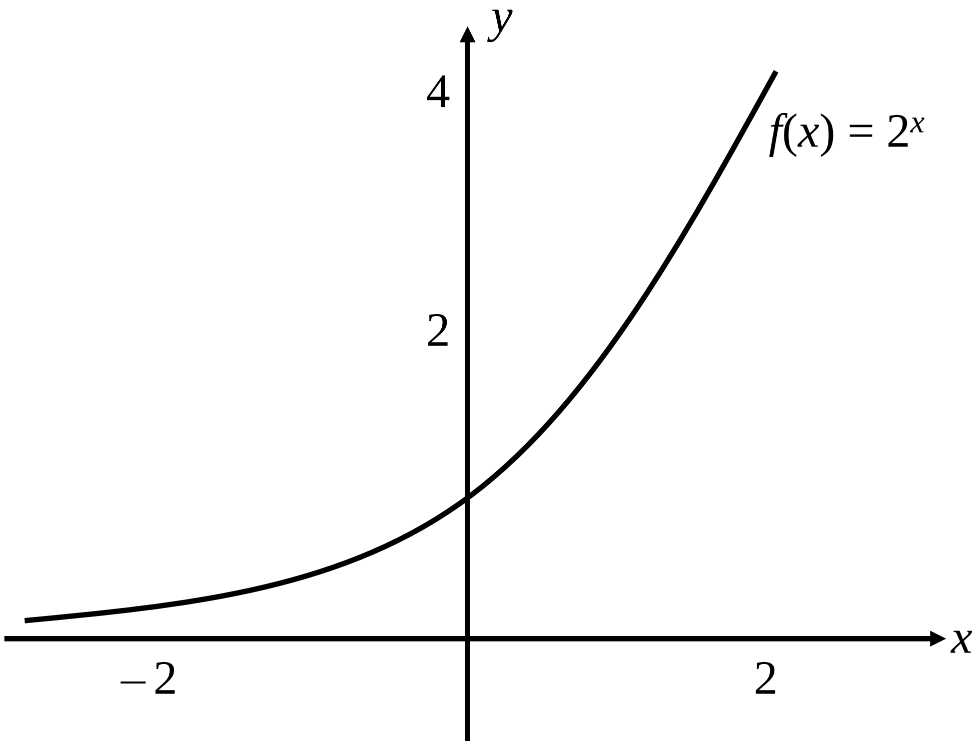
- Note that 2^x approaches zero as x decreases without bound:
  - There is a horizontal asymptote at $y = 0$.
- Furthermore, 2^x increases without bound when x increases without bound.
- Thus, the range of f is the interval $(0, \infty)$.

Examples

- Sketch the graph of the exponential function $f(x) = 2^x$.

Solution

- Finally, sketch the graph:



Examples

- Sketch the graph of the exponential function $f(x) = (1/2)^x$.

Solution

- First, recall again that the domain of this function is the set of real numbers.
- Next, putting $x = 0$ gives $y = (1/2)^0 = 1$, which is the y-intercept.

  (There is no x-intercept, since there is no value of x for which $y = 0$)

Examples

- Sketch the graph of the exponential function $f(x) = (1/2)^x$.

<u>Solution</u>

- Now, consider a few values for x:

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	32	16	8	4	2	1	1/2	1/4	1/8	1/16	1/32

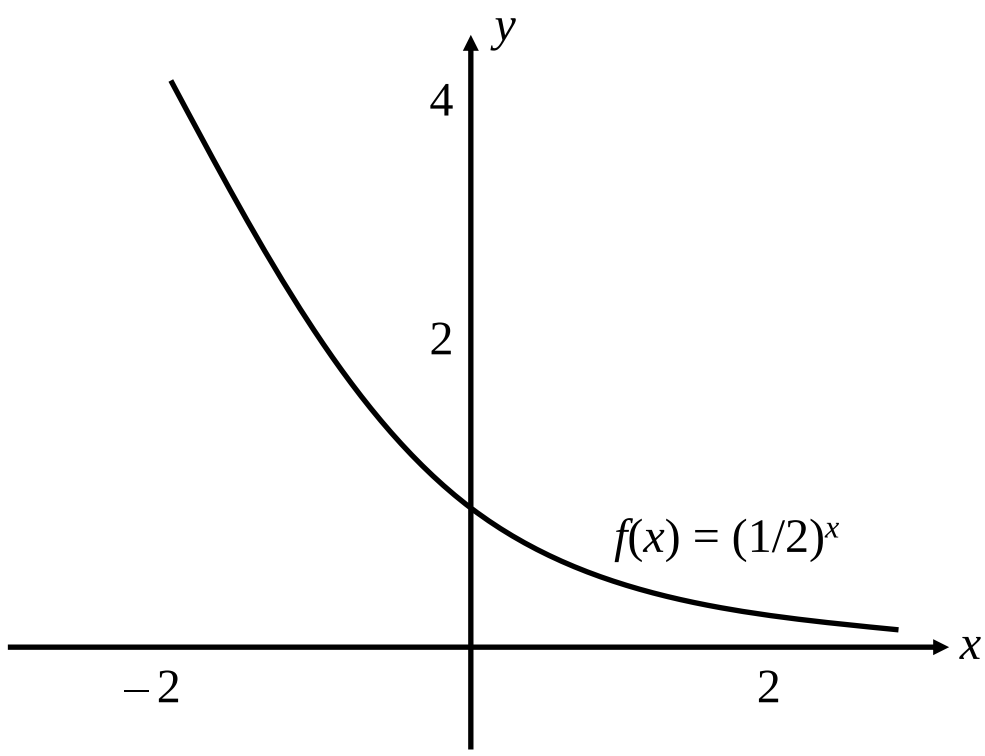
- Note that $(1/2)^x$ increases without bound when x decreases without bound.
- Furthermore, $(1/2)^x$ approaches zero as x increases without bound: there is a horizontal asymptote at $y = 0$.
- As before, the range of f is the interval $(0, \infty)$.

Examples

- Sketch the graph of the exponential function $f(x) = (1/2)^x$.

Solution

- Finally, sketch the graph:

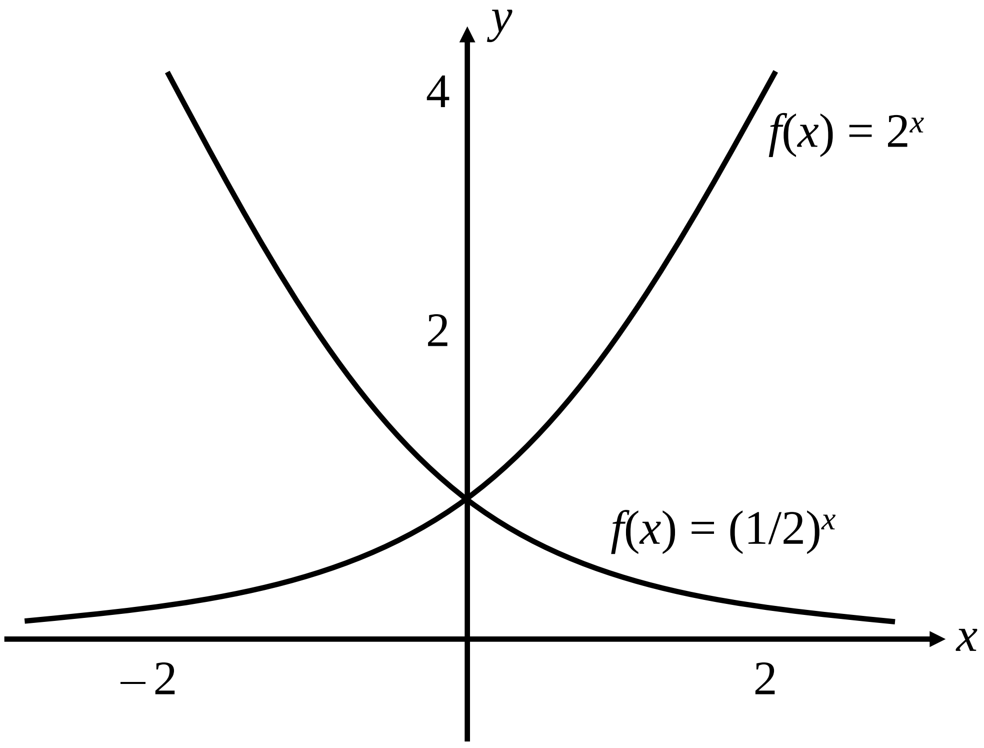


Examples

- Sketch the graph of the exponential function $f(x) = (1/2)^x$.

Solution

- Note the symmetry between the two functions:



Properties of Exponential Functions

- The exponential function $y = b^x \quad (b > 0,\ b \neq 1)$ has the following properties:

1. Its domain is $(-\infty, \infty)$.
2. Its range is $(0, \infty)$.
3. Its graph passes through the point $(0, 1)$
4. It is continuous on $(-\infty, \infty)$.
5. It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

The Base e

- Exponential functions to the base e, where e is an irrational number whose value is $2.7182818\ldots$, play an important role in both theoretical and applied problems.
- It can be shown that

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

Examples

- Sketch the graph of the exponential function $f(x) = e^x$.

<u>Solution</u>

- Since $e^x > 0$ it follows that the graph of $y = e^x$ is similar to the graph of $y = 2^x$.
- Consider a few values for x:

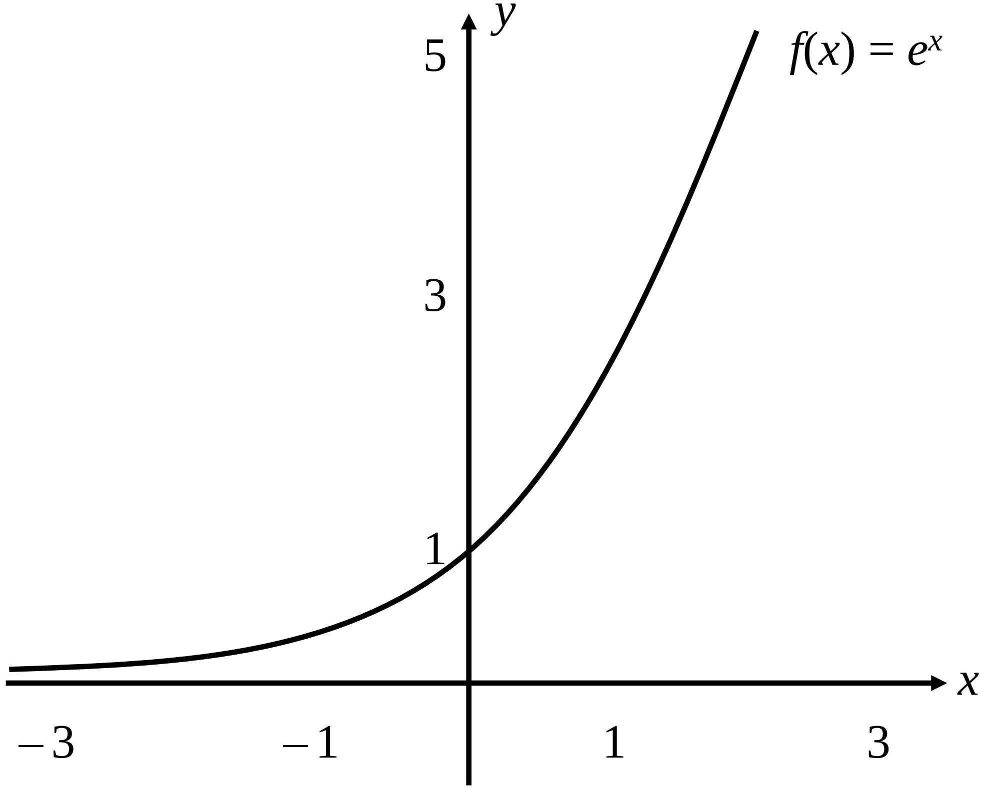
x	– 3	– 2	– 1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.09

Examples

- Sketch the graph of the exponential function $f(x) = e^x$.

<u>Solution</u>

- Sketching the graph:



Examples

- Sketch the graph of the exponential function $f(x) = e^{-x}$.

<u>Solution</u>

- Since $e^{-x} > 0$ it follows that $0 < 1/e < 1$ and so $f(x) = e^{-x} = 1/e^{x} = (1/e)^{x}$ is an exponential function with base less than 1.
- Therefore, it has a graph similar to that of $y = (1/2)^{x}$.
- Consider a few values for x:

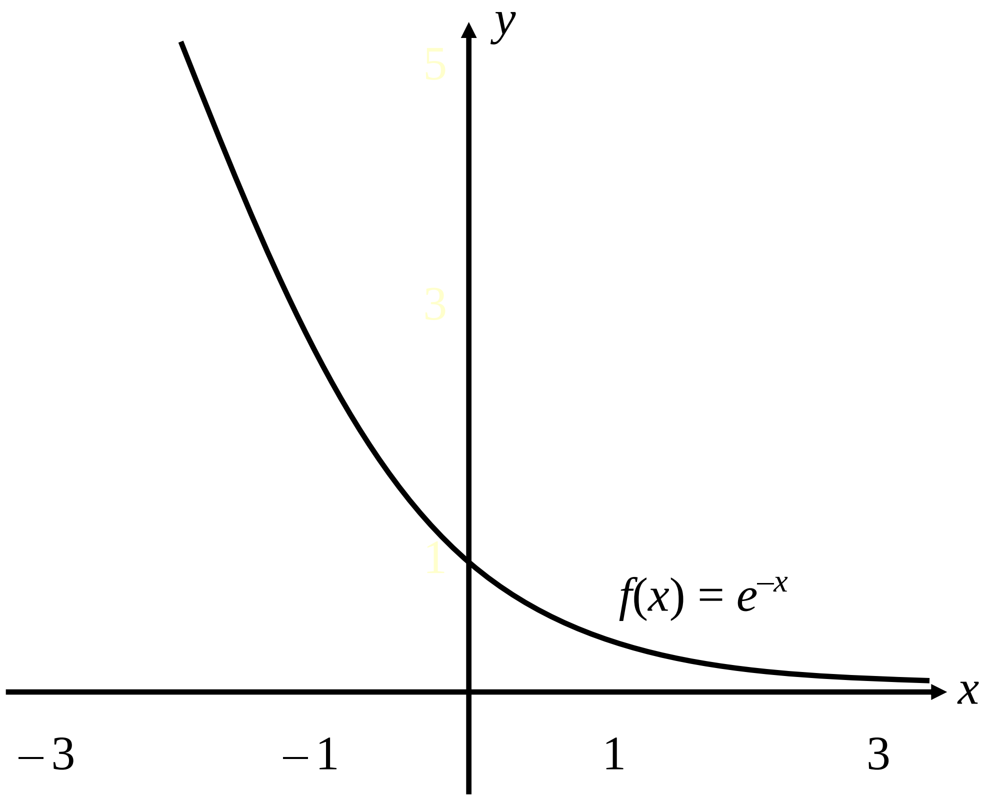
x	– 3	– 2	– 1	0	1	2	3
y	20.09	7.39	2.72	1	0.37	0.14	0.05

Examples

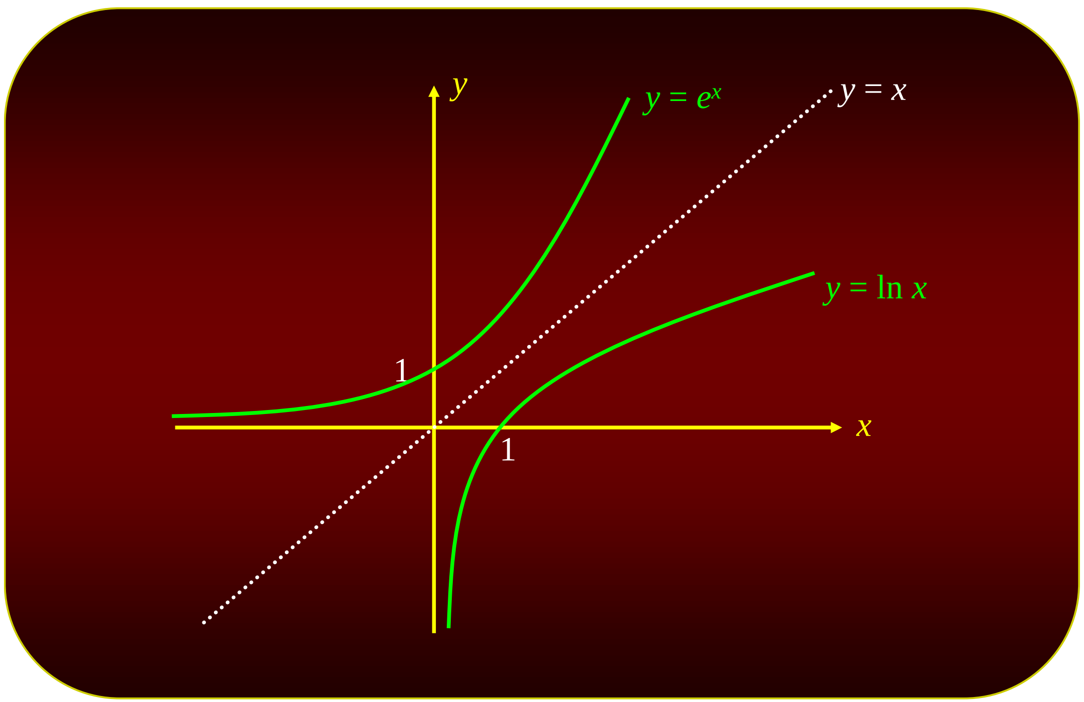
- Sketch the graph of the exponential function $f(x) = e^{-x}$.

<u>Solution</u>

- Sketching the graph:



D) Logarithmic Functions



Logarithms

- We've discussed exponential equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

- But what about solving the same equation for y?
- You may recall that y is called the logarithm of x to the base b, and is denoted $\log_b x$.
    - Logarithm of x to the base b

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

Examples

- **Solve** $\log_3 x = 4$ **for** x:

<u>Solution</u>

- **By definition,** $\log_3 x = 4$ **implies** $x = 3^4 = 81.$

- **Solve** $\log_{16} 4 = x$ **for** x:

<u>Solution</u>

- $\log_{16} 4 = x$ **is equivalent to** $4 = 16^x = (4^2)^x = 4^{2x}$, **or**

  $4^1 = 4^{2x}$,

  from which we deduce that

-

Examples

- Solve $\log_{16} 4 = x$ for x:

<u>Solution</u>

- $\log_{16} 4 = x$ is equivalent to $4 = 16^x = (4^2)^x = 4^{2x}$,

  or $4^1 = 4^{2x}$,

  from which we deduce that

$$2x = 1$$

$$x = \frac{1}{2}$$

Examples

- Solve $\log_x 8 = 3$ for x:

<u>Solution</u>

- By definition, we see that $\log_x 8 = 3$ is equivalent to

$$8 = 2^3 = x^3$$

$$x = 2$$

Logarithmic Notation

$\log\, x = \log_{10}\, x$ **Common logarithm**

$\ln\, x = \log_e\, x$ **Natural logarithm**

Laws of Logarithms

- If m and n are positive numbers, then

1. $$\log_b mn = \log_b m + \log_b n$$
2. $$\log_b \frac{m}{n} = \log_b m - \log_b n$$
3. $$\log_b m^n = n \log_b m$$
4. $$\log_b 1 = 0$$
5. $$\log_b b = 1$$

Examples

- Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find

$$\begin{aligned}\log 15 &= \log 3 \cdot 5 \\ &= \log 3 + \log 5 \\ &\approx 0.4771 + 0.6990 \\ &= 1.1761\end{aligned}$$

Examples

- **Given that** $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, **and** $\log 5 \approx 0.6990$, **use the laws of logarithms to find**

$$\begin{aligned}\log 7.5 &= \log(15/2)\\ &= \log(3\cdot 5/2)\\ &= \log 3+\log 5-\log 2\\ &\approx 0.4771+0.6990-0.3010\\ &= 0.8751\end{aligned}$$

Examples

- Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find

$$\begin{aligned} \log 81 &= \log 3^4 \\ &= 4 \log 3 \\ &\approx 4(0.4771) \\ &= 1.9084 \end{aligned}$$

Examples

- **Given that** $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, **and** $\log 5 \approx 0.6990$, **use the laws of logarithms to find**

$$\begin{aligned} \log 50 &= \log 5 \cdot 10 \\ &= \log 5 + \log 10 \\ &\approx 0.6990 + 1 \\ &= 1.6990 \end{aligned}$$

Examples

- Expand and simplify the expression:

$$\log_3 x^2 y^3 = \log_3 x^2 + \log_3 y^3$$
$$= 2\log_3 x + 3\log_3 y$$

Examples

- Expand and simplify the expression:

$$\log_2 \frac{x^2+1}{2^x} = \log_2\left(x^2+1\right) - \log_2 2^x$$
$$= \log_2\left(x^2+1\right) - x\log_2 2$$
$$= \log_2\left(x^2+1\right) - x$$

Examples

- Expand and simplify the expression:

$$\ln\frac{x^2\sqrt{x^2-1}}{e^x} = \ln\frac{x^2(x^2-1)^{1/2}}{e^x}$$

$$= \ln x^2 + \ln(x^2-1)^{1/2} - \ln e^x$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x\ln e$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x$$

Examples

- Use the properties of logarithms to solve the equation for x:

$$\log_3(x+1) - \log_3(x-1) = 1$$

$$\log_3 \frac{x+1}{x-1} = 1$$

Law 2

$$\frac{x+1}{x-1} = 3^1 = 3$$

Definition of logarithms

$$x+1 = 3(x-1)$$

$$x+1 = 3x-3$$

$$4 = 2x$$

$$x = 2$$

Examples

- Use the properties of logarithms to solve the equation for x:

$$\log x + \log(2x-1) = \log 6$$

$$\log x + \log(2x-1) - \log 6 = 0$$

$$\log \frac{x(2x-1)}{6} = 0$$

Laws 1 and 2

$$\frac{x(2x-1)}{6} = 10^0 = 1$$

Definition of logarithms

$$x(2x-1) = 6$$

$$2x^2 - x - 6 = 0$$

$$(2x+3)(x-2) = 0$$

$$x = 2$$

$x = -\frac{3}{2}$ is out of the domain of $\log x$, so it is discarded.

Logarithmic Function

- The function defined by

$$f(x) = \log_b x \qquad (b > 0,\ b \neq 1)$$

  is called the logarithmic function with base b.
- The domain of f is the set of all positive numbers.

Properties of Logarithmic Functions

- The logarithmic function

$$y = \log_b x \qquad (b > 0,\ b \neq 1)$$

has the following properties:

1. Its domain is $(0,\ \infty)$.
2. Its range is $(-\infty,\ \infty)$.
3. Its graph passes through the point $(1,\ 0)$.
4. It is continuous on $(0,\ \infty)$.
5. It is increasing on $(0,\ \infty)$ if b > 1
   and decreasing on $(0,\ \infty)$ if b ¡ 1.

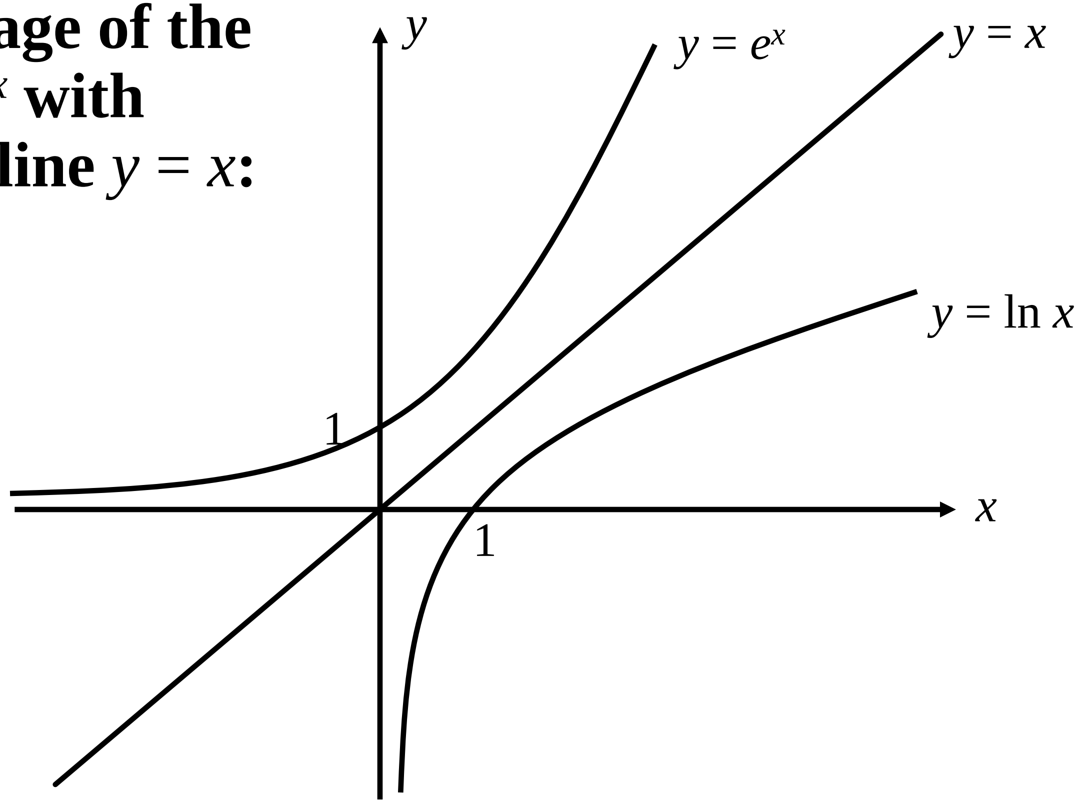
Example

- Sketch the graph of the function $y = \ln x$.

<u>Solution</u>

- We first sketch the graph of $y = e^x$.

- **The required graph is the mirror image of the graph of $y = e^x$ with respect to the line $y = x$:**



Properties Relating Exponential and Logarithmic Functions

- **Properties relating e^x and ln x:**

$$e^{\ln x} = x \qquad (x > 0)$$

$$\ln e^x = x \qquad (\text{for any real number } x)$$

Examples

- Solve the equation $2e^{x+2} = 5$.

<u>Solution</u>

- Divide both sides of the equation by 2 to obtain:

$$e^{x+2} = \frac{5}{2} = 2.5$$

- Take the natural logarithm of each side of the equation and solve:

$$\ln e^{x+2} = \ln 2.5$$

$$(x+2)\ln e = \ln 2.5$$

$$x + 2 = \ln 2.5$$

$$x = -2 + \ln 2.5$$

$$x \approx -1.08$$

Examples

- Solve the equation $5 \ln x + 3 = 0$.

<u>Solution</u>

- Add -3 to both sides of the equation and then divide both sides of the equation by 5 to obtain:

$$5 \ln x = -3$$

$$\ln x = -\frac{3}{5} = -0.6$$

and so:

$$e^{\ln x} = e^{-0.6}$$

$$x = e^{-0.6}$$

$$x \approx 0.55$$

Derivative of Exponential and Logarithmic Functions

1- Derivative of exponential function $f(x) = e^x$

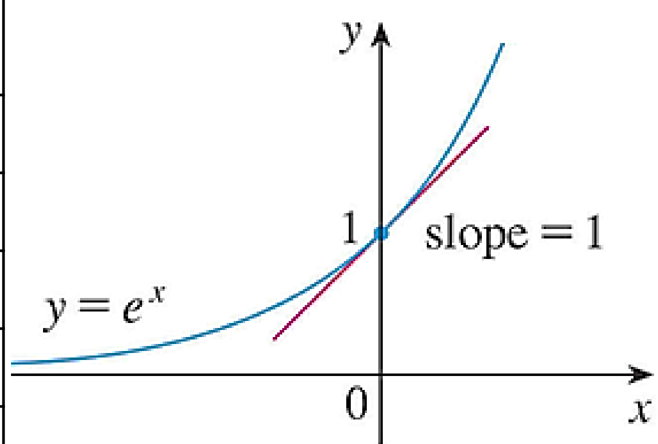
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}} = e^x f'(0)$$

By definition, this is derivative $f'(0)$, what is the slope of e^x at $x = 1$.

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

h	$\frac{e^h-1}{h}$
1	1.71828
0.1	1.05171
0.01	1.00502
0.001	1.00050
0.0001	1.00005
0.00001	1.00001



$$\frac{d}{dx}e^x = e^x$$

example:

Differentiate the function $y = e^{\tan x}$

To use the Chain Rule, we let $u = \tan x$.

Then, we have $y = e^u$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \frac{du}{dx} = e^{\tan x} \sec^2 x$$

example:

Find y' if $y = e^{-4x} \sin 5x$.

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4)$$
$$= e^{-4x}(5\cos 5x - 4\sin 5x)$$

chain rule:

$$\frac{d}{dx}e^{u} = e^{u}\frac{du}{dx}$$

We can now use this formula to find the derivative of a^{x}

$$y = a^{x} \quad \Longrightarrow \quad \ln y = \ln a^{x} = x \ln a \quad \Longrightarrow a^{x} = e^{x \ln a}$$

$$\frac{d}{dx}a^{x} = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a}\frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a$$

$$\frac{d}{dx}a^{x} = a^{x} \cdot \ln a$$

2- Derivative of Natural Logarithm Function

$$y = \ln x$$

$$e^y = x \implies \frac{d}{dx} e^y = \frac{d}{dx} x \implies$$

$$e^y \frac{dy}{dx} = 1 \implies x \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

example:

Differentiate $y = \ln(x^3 + 1)$.

To use the Chain Rule, we let $u = x^3 + 1$.

Then, $y = \ln u$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$$

example:

Find: $\frac{d}{dx}\ln(\sin x)$

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

example: Differentiate $f(x)=\sqrt{\ln x}$

$$f'(x)=\tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x)=\frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}=\frac{1}{2x\sqrt{\ln x}}$$

example: $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}=?$

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}=\frac{1}{\frac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}=\frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1-(x+1)\left(\tfrac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$=\frac{x-2-\tfrac{1}{2}(x+1)}{(x+1)(x-2)}=\frac{x-5}{2(x+1)(x-2)}$$

If we first simplify the given function using the laws of logarithms, the differentiation becomes easier

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}=\frac{d}{dx}\left[\ln(x+1)-\tfrac{1}{2}\ln(x-2)\right]=\frac{1}{x+1}-\frac{1}{2}\left(\frac{1}{x-2}\right)$$

example:

Find $f'(x)$ if $f(x) = \ln|x|$.

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus, $f'(x) = 1/x$ for all $x \neq 0$.

The result is worth remembering:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

3- Derivative of Logarithm Function

a logarithmic function with base *a* in terms of the natural logarithmic function:

$$\log_a x = \frac{\ln x}{\ln a}$$

Since ln *a* is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

example:

$$\frac{d}{dx}\log_{10}(2+\sin x)$$

$$= \frac{1}{(2+\sin x)\ln 10}\frac{d}{dx}(2+\sin x) = \frac{\cos x}{(2+\sin x)\ln 10}$$

IMPORTANT and UNUSUAL: If you have a daunting task to find derivative in the case of a function raised to the function $\left(x^x, x^{\sin x} \ ..., \right)$, or a crazy product, quotient, chain problem you do a simple trick:
FIRST find logarithm , ln, so you'll have sum instead of product, and product instead of exponent. Life will be much, much easier.

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation *y* = *f*(*x*) and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to *x*.
3. Solve the resulting equation for *y*'.

example:

Differentiate: $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln\left(x^2+1\right) - 5\ln(3x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Since we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn’t used logarithmic differentiation the resulting calculation would have been horrendous.

example:

$y = x^{\sin x} \quad y' = ?$

$$\ln y = (\sin x)\ln x \;\Rightarrow\; \frac{1}{y}\,y' = (\cos x)\ln x + \frac{\sin x}{x}$$

$$y' = (\ln x)x^{\sin x}\cos x + (\sin x)\; x^{\sin x - 1}$$

Try: $y = (\sin x)^x$