Integral

Objectives

- Know what definition of Indefinite Integral is.
- Know what Definite Integral is.
- Know what Length of a curve is.
- Know what Area under a curve is.
- Know what Volume of solids is.
- Know what Surface area is.

A) Indefinite Integral

- **The family of antiderivatives of a function *f* indicated by**

$$\int f(x)dx$$

- **The symbol is a stylized *S* to indicate summation**

Indefinite Integral

- The indefinite integral is a <u>family of functions</u>

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

- The **+ C** represents an arbitrary constant
  - The <u>constant of integration</u>

$$\int \left(3x^{-2} + 4\right) dx = -3x^{-1} + 4x + C$$

Properties of Indefinite Integrals

- **The power rule**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

- **The integral of a sum (difference) is the sum (difference) of the integrals**

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Properties of Indefinite Integrals

- **The derivative of the indefinite integral is the original function**

$$\frac{d}{dx}\int f(x)dx = f(x)$$

- **A constant can be factored out of the integral**

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

Examples

- **Determine the indefinite integrals as specified below**

$$\int 6\,dx \qquad\qquad \int (3x+5)\,dx$$

$$\int t^{1/4}\,dt \qquad\qquad \int \left(12y^3+6y^2-8y+5\right)dy$$

Indefinite Integrals of Exponential Functions

- $$\int e^{x}\,dx = e^{x} + C$$
- $$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$
- $$\int a^{x}\,dx = \frac{a^{x}}{\ln a} + C$$
- $$\int a^{kx}\,dx = \frac{a^{k\cdot x}}{k(\ln a)} + C$$

Examples

- **Use the exponential rules to determine these integrals**

$$\int -4e^{2v}\, dv$$

$$\int \left(v^2 - e^{3v}\right) dv$$

$$\int 3^{2x}\, dx$$

Indefinite Integral of x^{-1}

- The rule is $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$
- Try it out …

$$\int \left(\frac{9}{z} - 3e^{-0.4z} \right) dz$$

Indefinite Integrals of Trigonometric Functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

Example 1

- **Find the general indefinite integral**

$$\int (10x^4 - 2\sec^2 x)\, dx$$

$$\int (10x^4 - 2\sec^2 x)\, dx = 10\int x^4\, dx - 2\int \sec^2 x\, dx$$

$$= 10(x^5/5) - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

- **You should check this answer by differentiating it.**

Example 2

- **Evaluate**

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta \cot\theta \, d\theta = -\csc\theta + C$$

B) Definite Integral

Example 1

- **Evaluate**

$$\int_0^3 (x^3 - 6x)\,dx$$

$$\int_0^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 6\frac{x^2}{2}\Bigg]_0^3$$

$$= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right)$$

$$= \tfrac{81}{4} - 27 - 0 + 0 = -6.75$$

Example 2

- **Evaluate** $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx$

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx = 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1} x \Big]_0^2$$

$$= \tfrac{1}{2} x^4 - 3x^2 + 3\tan^{-1} x \Big]_0^2$$

$$= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1} 2 - 0$$

$$= -4 + 3\tan^{-1} 2$$

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx \approx -0.67855$$

Example 3

- **Evaluate** $$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$$

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$\int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

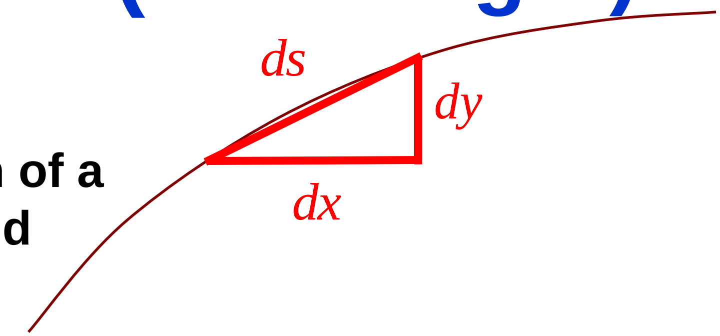
$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

$$= (2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}) - (2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1})$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$

C) Length of a curve (Arc Length)



If we want to approximate the length of a curve, over a short distance we could measure a straight line.

By the pythagorean theorem:

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$\int ds = \int \sqrt{dx^2 + dy^2}$$

We need to get dx out from under the radical.

$$S = \int \sqrt{\left(\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}\right) dx^2}$$

Length of Curve (Cartesian)

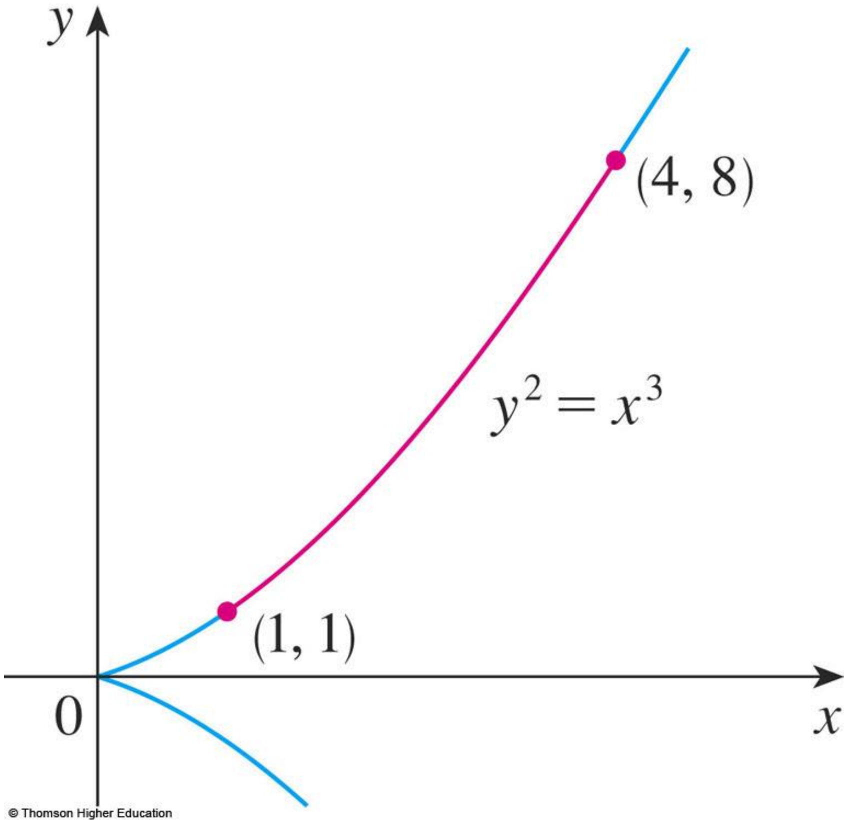
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- **If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y , we obtain its length as:**

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 1

Find the length of the arc of the semi-cubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8).



$$y = x^{3/2}$$

$$\frac{dy}{dx} = \tfrac{3}{2} x^{1/2}$$

-Thus, the arc length formula gives:

$$L \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \tfrac{9}{4} x}\, dx$$

- **If we substitute *u* = 1 + (9/4)*x*, then *du* = (9/4) *dx*.**

- **When *x* = 1, *u* = 13/4. When *x* = 4, *u* = 10.**

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big]_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Example 1

Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$$

- **Thus, the arc length function is given by:**

$$
\begin{aligned}
s(x) &= \int_1^x \sqrt{1+[f'(t)]^2}\,dt \\
&= \int_1^x \left(2t - \frac{1}{8t}\right) dt \\
&= t^2 + \tfrac{1}{8}\ln t \Big]_1^x \\
&= x^2 + \tfrac{1}{8}\ln x - 1
\end{aligned}
$$

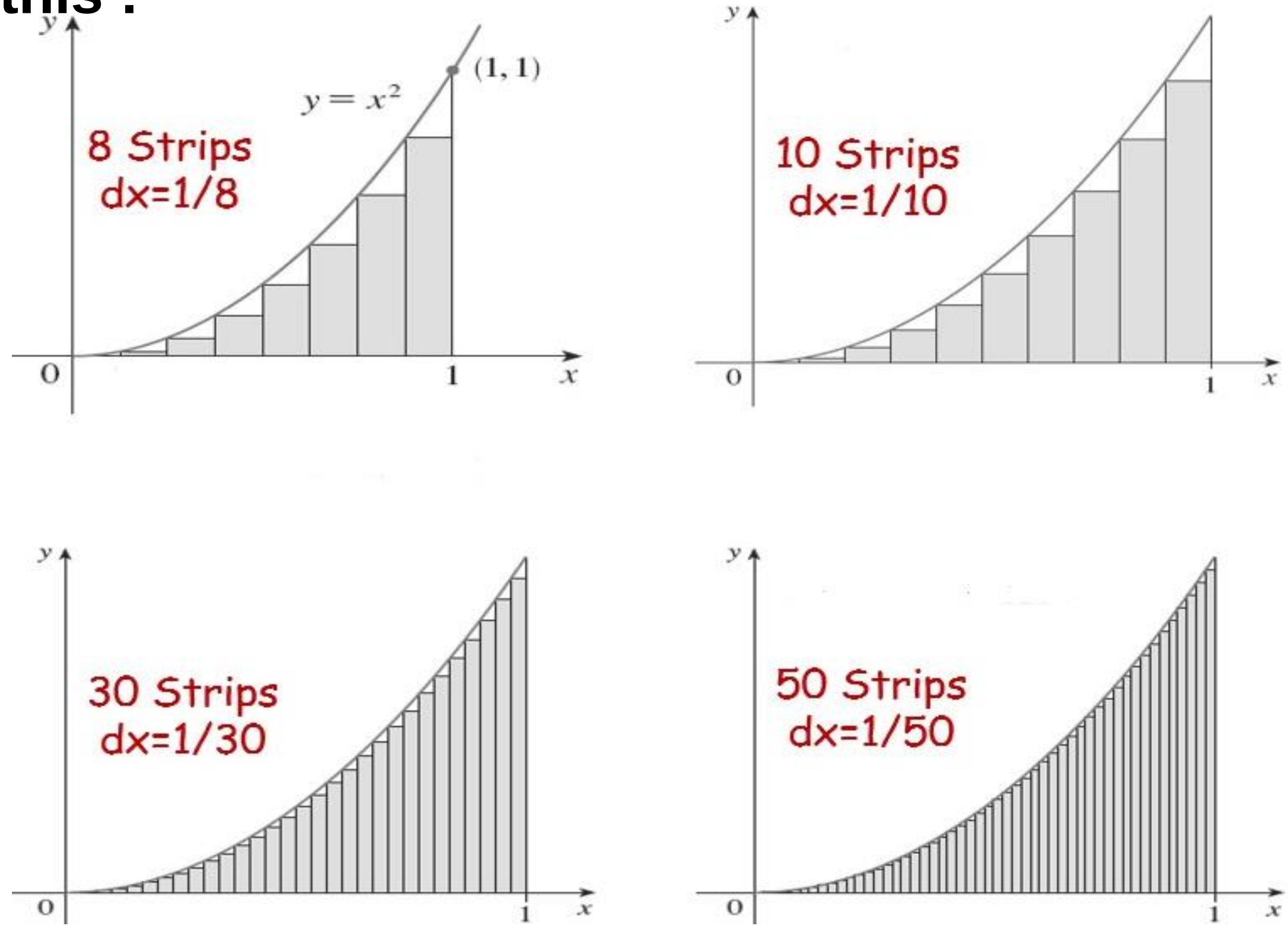
- **For instance, the arc length along the curve from (1, 1) to (3, *f*(3)) is:**

$$
\begin{aligned}
s(3) &= 3^2 + \tfrac{1}{8}\ln 3 - 1 \\
&= 8 + \frac{\ln 3}{8} \\
&\approx 8.1373
\end{aligned}
$$

D) Area under a Curve

1- Area between the curve and the x-axis

<u>Consider $y=x^2$</u>, If we wanted to find the area under this Curve between x=0 and x=1 we could use strips like this :

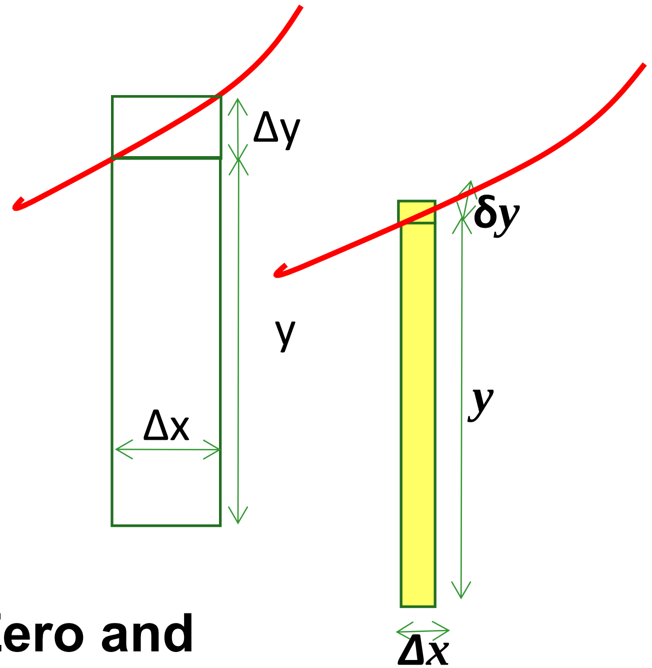


Finding the Area

If write the area of each of the individual rectangle as ΔA

Then the area of each individual rectangle is between

$$y\Delta x < \Delta A < (y + \Delta y)\Delta x$$



And So

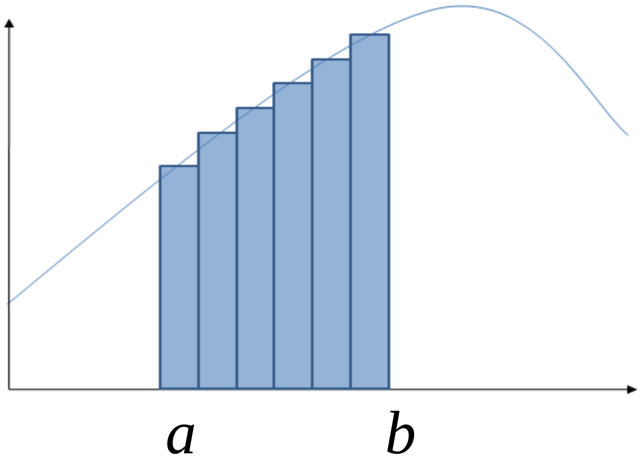
$$y < \frac{\Delta A}{\Delta x} < (y + \Delta y)$$

With increasing strips, Δx Δy tend to Zero and

we can write $\lim\limits_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$ **and** $\frac{dA}{dx} = y$

$$A = \int y\,dx$$

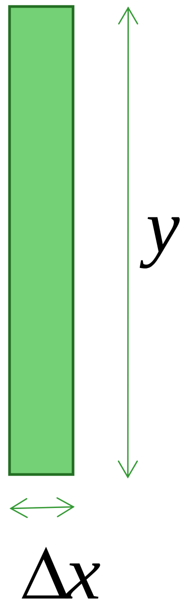
The area between a and b under the curve is



Area ~ $\sum y\Delta x$

As the number of rectangles increases the approximation to the area improves

$n \to \infty$ $\Delta x \to 0$

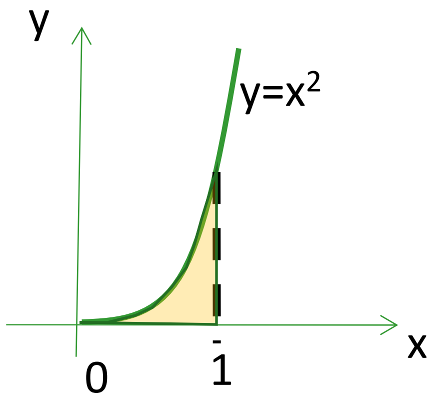


Area = $\lim_{\Delta x \to 0} \sum y\Delta x$

This Limit is written as $\int_a^b y\,dx$

Example 1

$$\int_{x=0}^{x=1} x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

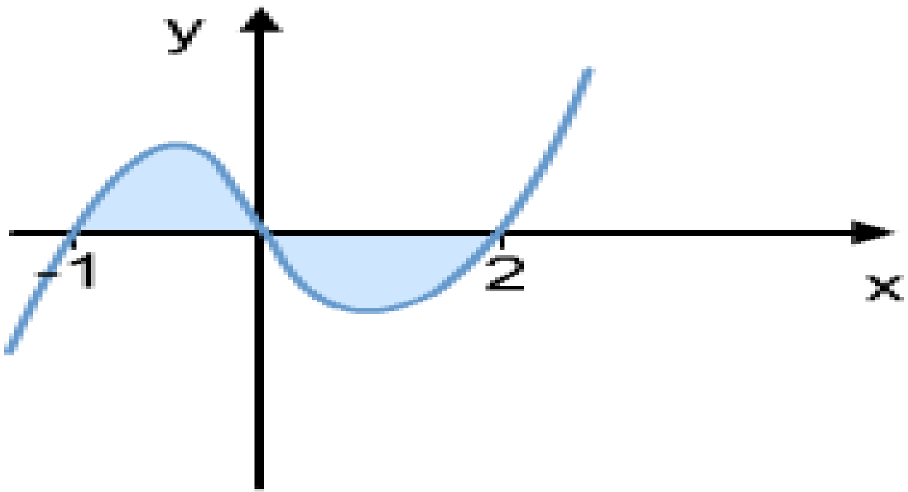


$$\left[\frac{1^3}{3} - \frac{0^3}{3}\right]_0^1 = \frac{1}{3}$$

2- Splitting Areas for Integration

Where a curve is below the x-axis the integral is negative

Therefore if the curve crosses the x axis we need to split the integration into separate parts.

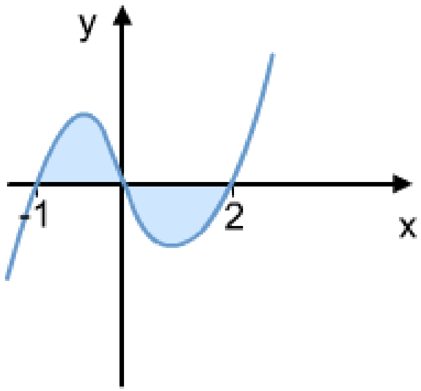


$$Area = \left| \int_0^2 f(x)\,dx \right| + \int_{-1}^0 f(x)\,dx$$

Example 2

Find the area enclosed by the x axis and the curve

$$y = x(x-2)(x+1)$$



y = 0 when x = 0
x = 2 and
x = -1

The curve is below the axis for 0 < x < 2

and above the axis for -1 < x < 0

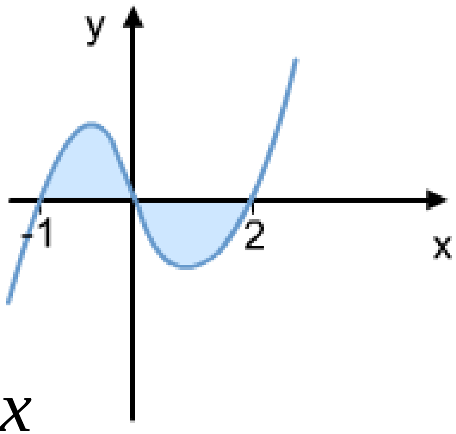
$$Area = \left| \int_0^2 x(x-2)(x+1)dx \right| + \int_{-1}^0 x(x-2)(x+1)\,dx$$

Find the area enclosed by the x axis and $y = x(x-2)(x+1)$

$$\int_0^2 x(x-2)(x+1)\,dx = \int_0^2 x^3 - x^2 - 2x\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2$$

$$= 4 - \frac{8}{3} - 4 - 0 = -\frac{8}{3} = \boxed{-2\frac{2}{3}}$$



$$\int_{-1}^0 x(x-2)(x+1)\,dx = \int_{-1}^0 x^3 - x^2 - 2x\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^0$$

$$= 0 - \left(\frac{1}{4} - \frac{-1}{3} - 1\right) = -\left(\frac{7}{12} - 1\right) = \boxed{\frac{5}{12}}$$

$$\boxed{Total\,area = 2\frac{2}{3} + \frac{5}{12} = 3\frac{1}{12}\,sq.units}$$

3- Area between the curve and the y-axis

Example 3

Find the area bounded by $x=y^2+2$, y-axis, y=1 and y=3

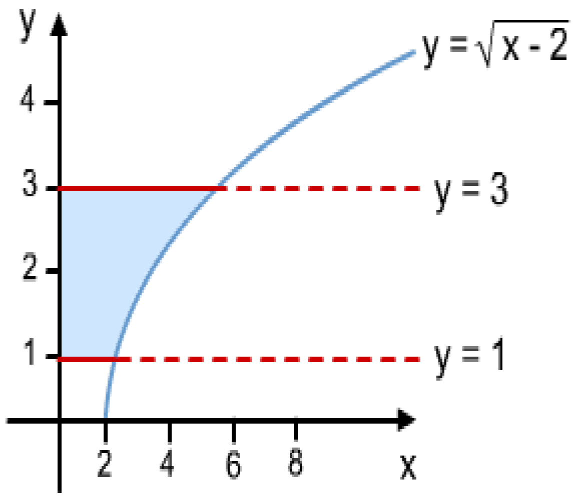
$$Area = \int_{y=1}^{y=3} f(y)\, dy$$

$$= \int_1^3 y^2 + 2\, dy$$

$$= \left[\frac{y^3}{3} + 2y\right]_1^3$$

$$= \left(\frac{3^3}{3} + 6\right) - \left(\frac{1}{3} + 2\right)$$

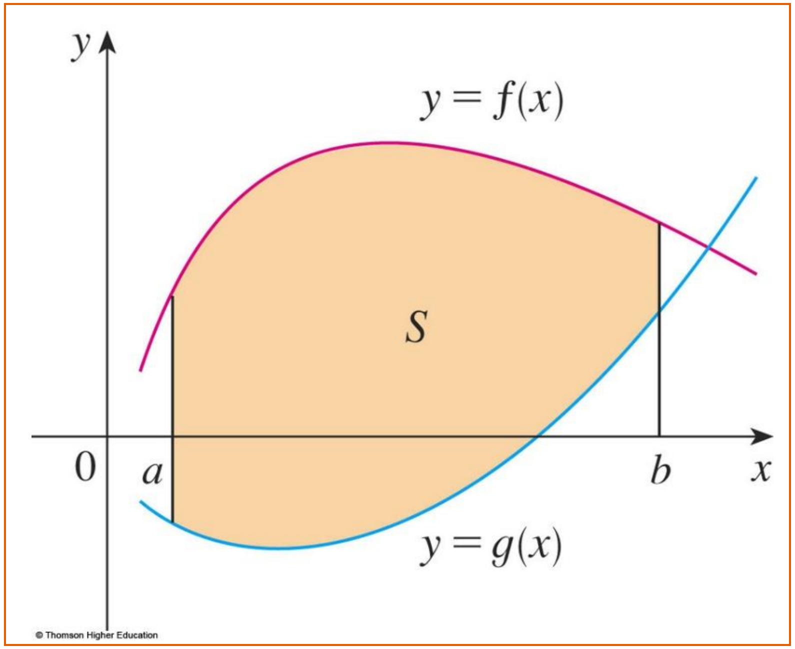
$$= (9+6) - \left(\frac{1}{3} + 2\right) \quad = 12\frac{2}{3}$$



Area = 12 2/3 square units

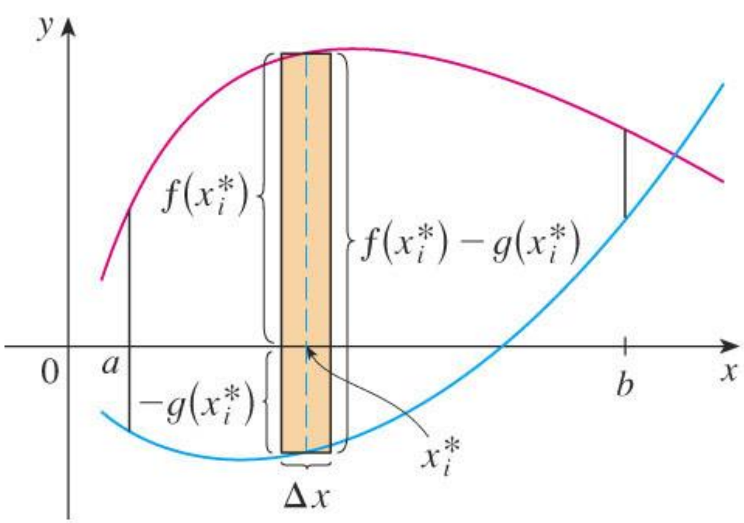
4- Areas between two Curves

- Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.
- Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

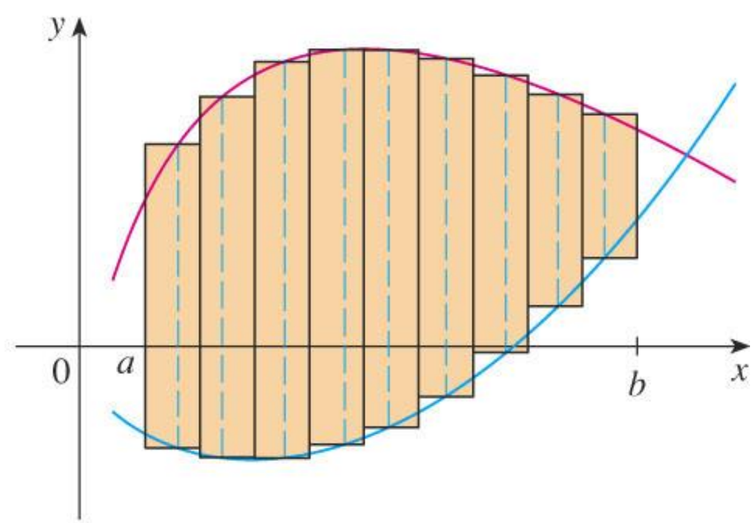


- As we did for areas under curves, we divide *S* into *n* strips of equal width and approximate the *i* th strip by a rectangle with base Δx and height

- $$f(x_i^*) - g(x_i^*)$$



(a) Typical rectangle



(b) Approximating rectangles

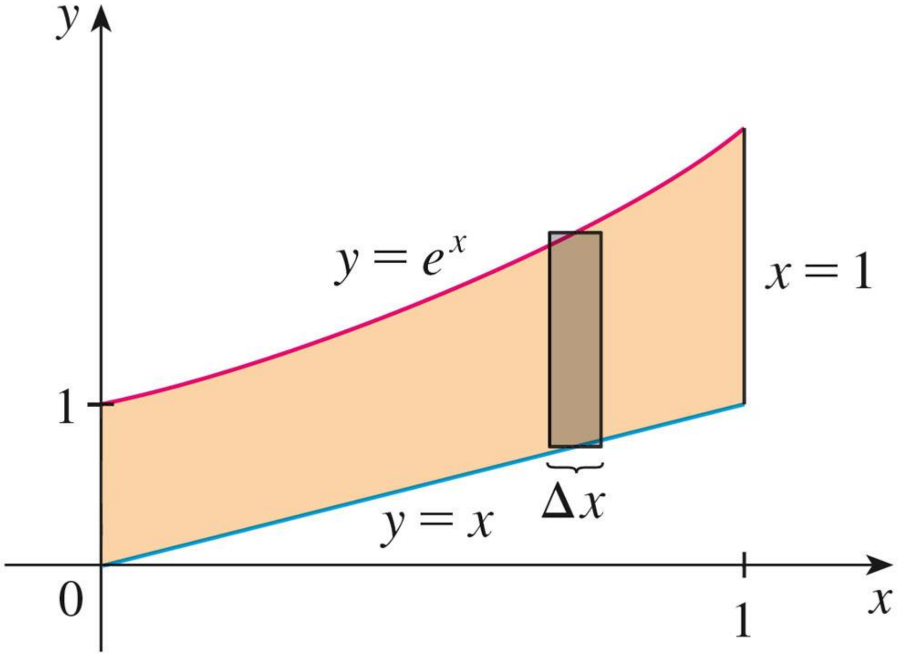
- **Thus, we define the area *A* of the region *S* as the limiting value of the sum of the areas of these approximating rectangles.**

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x$$

- **The limit here is the definite integral of *f* - *g*.**

Example 4

- Find the area of the region bounded above by $y=e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



- **So, we use the area formula with $y=e^x$, $g(x) = x$, $a = 0$, and $b = 1$:**

$$A = \int_0^1 \left(e^x - x\right)\, dx = e^x - \tfrac{1}{2}x^2 \Big]_0^1$$

$$= e - \frac{1}{2} - 1 = e - 1.5$$

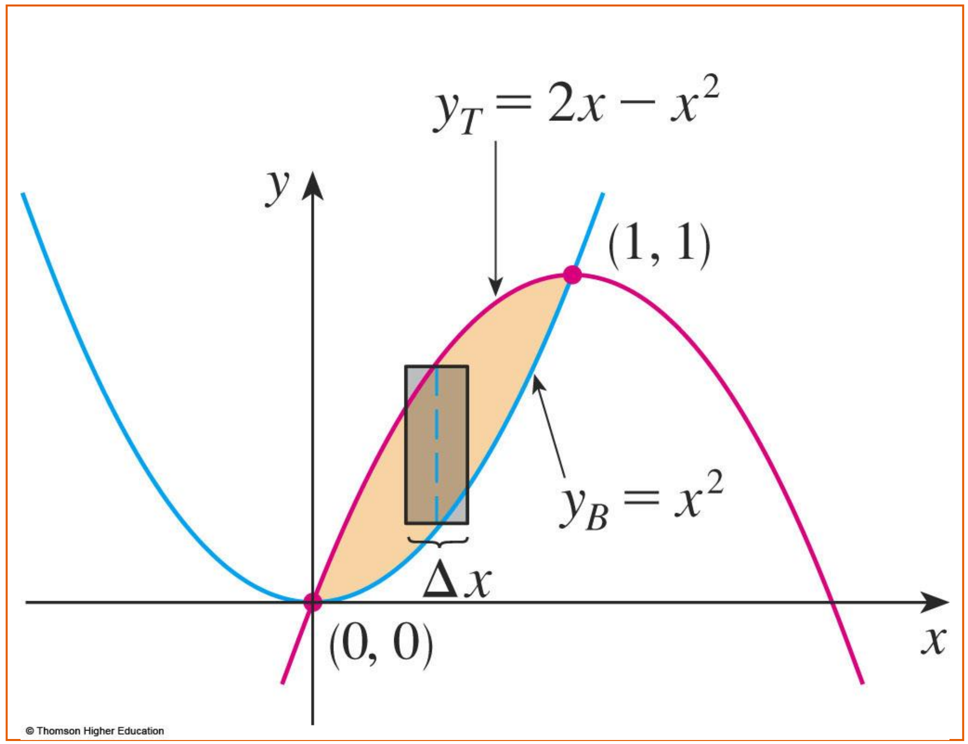
Example 5

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

- **First, we find the points of intersection of the parabolas by solving their equations simultaneously.**
  - **This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$.**
  - **Thus, $2x(x - 1) = 0$, so $x = 0$ or 1.**
  - **The points of intersection are $(0, 0)$ and $(1, 1)$.**

- **From the figure, we see that the top and bottom boundaries are:**

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$



- **The area of a typical rectangle is**

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

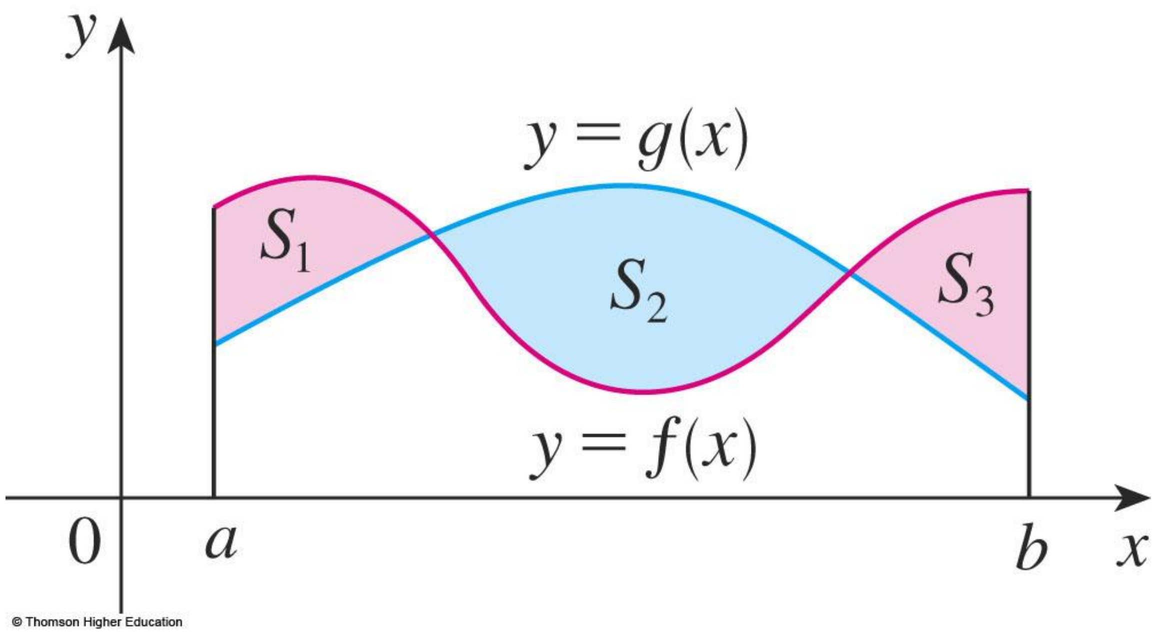
and the region lies between $x = 0$ and $x = 1$.

- **So, the total area is:**

$$A = \int_0^1 \left(2x - 2x^2\right)\,dx = 2\int_0^1 \left(x - x^2\right)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

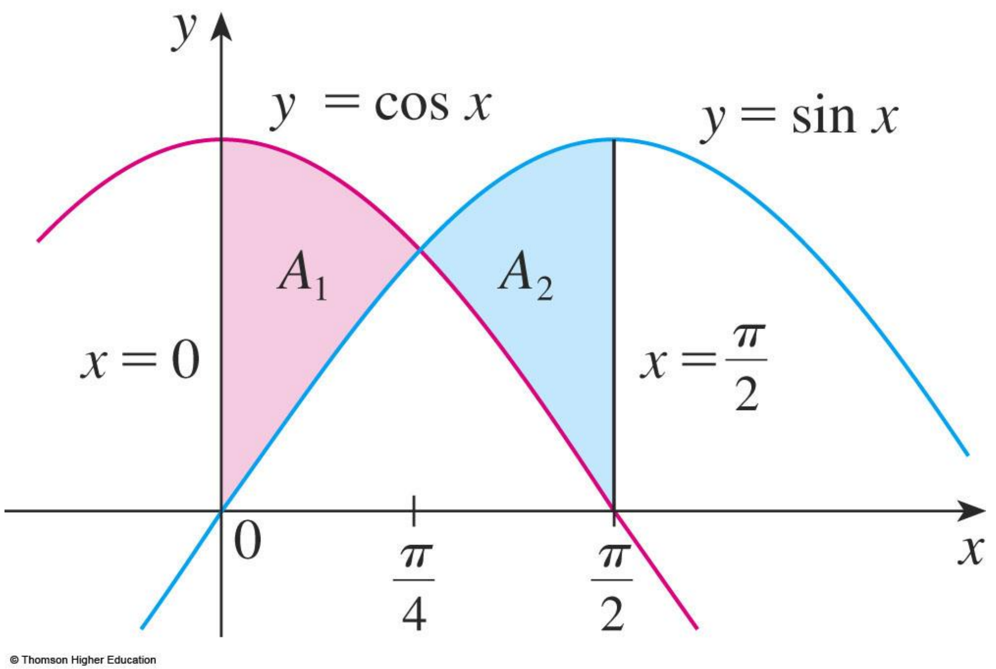
- To find the area between the curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, split the given region S into several regions S_1, S_2, . . . with areas A_1, A_2, . . .



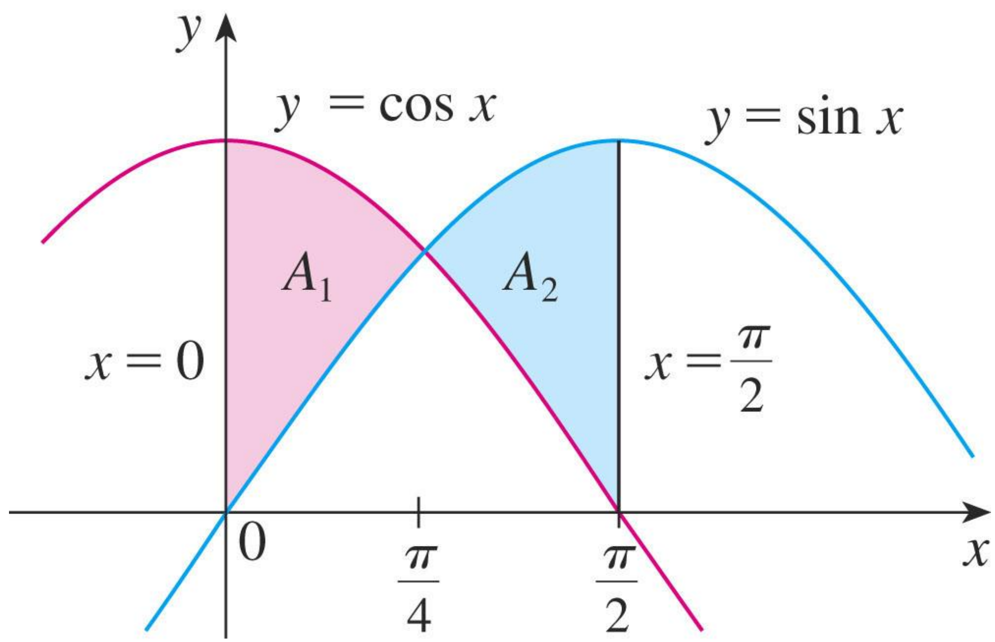
Example 6

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi / 4$ (since $0 \le x \le \pi / 2$).



- **Observe that cos *x* ≥ sin *x* when $0 \le x \le \pi / 4$ but sin *x* ≥ cos *x* when $\pi / 4 \le x \le \pi / 2$.**



So, the required area is:

$$A=\int_0^{\pi/2}\left|\cos x-\sin x\right|\,dx=A_1+A_2$$

$$=\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)\,dx$$

$$=\left[\sin x+\cos x\right]_0^{\pi/4}+\left[-\cos x-\sin x\right]_{\pi/4}^{\pi/2}$$

$$=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-0-1\right)+\left(-0-1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)$$

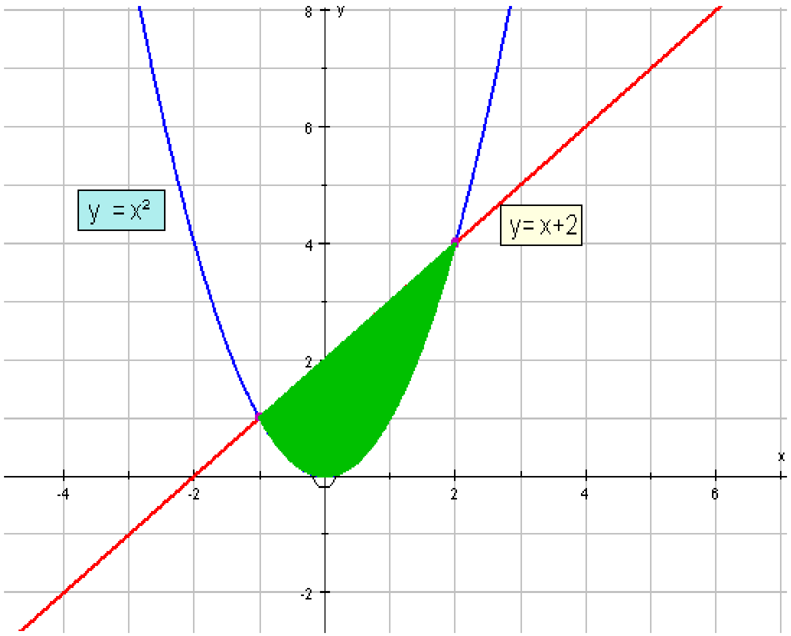
$$=2\sqrt{2}-2$$

Example 7

First find the points of intersection of

curve y=x^2 and line y=x+2

At the points of intersection



$$x^2 = x + 2$$

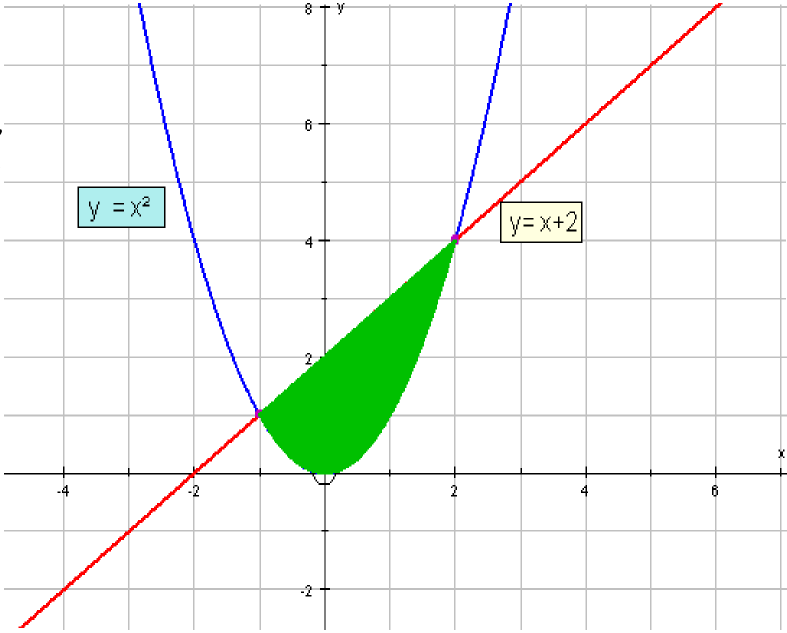
$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \quad or \quad x = -1$$

These will be our limits of integration

$$Area = \int_a^b (f(x) - g(x))\, dx$$



$$A = \int_{-1}^{2} (x+2) - x^2 \; dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$A = \left(\frac{2^2}{2} + 2\times 2 - \frac{2^3}{3} \right) - \left(\frac{(-1)^2}{2} + 2\times(-1) - \frac{(-1)^3}{3} \right)$$

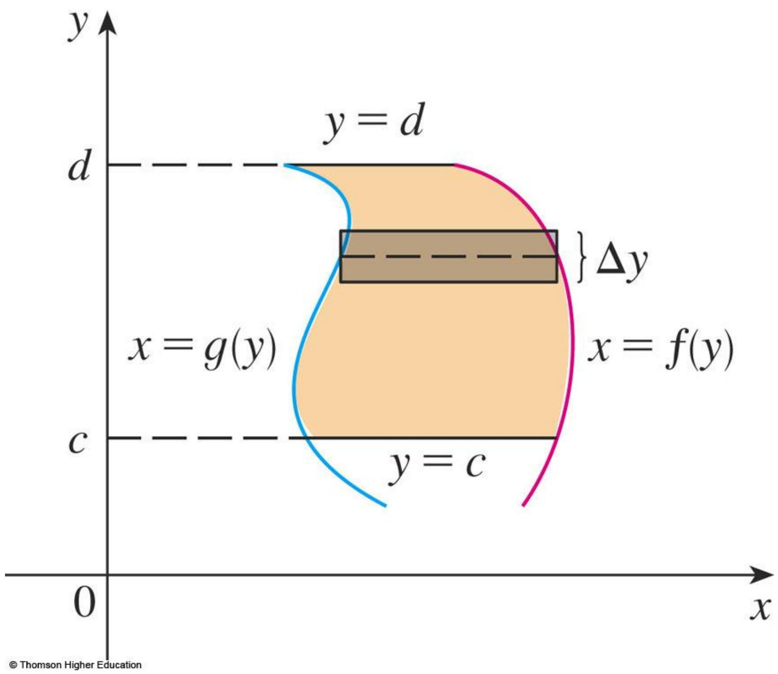
$$= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \left(8 - \frac{1}{2} - 3 \right) = 4\frac{1}{2}$$

Area = 4½ square units

Some regions are best treated by regarding *x* as a function of *y*.

- If a region is bounded by curves with equations *x* = *f*(*y*), *x* = *g*(*y*), *y* = *c*, and *y* = *d*, where *f* and *g* are continuous and *f*(*y*) ≥ *g*(*y*) for *c* ≤ *y* ≤ *d*, then its area is:

$$A = \int_c^d \left[f(y) - g(y) \right] dy$$



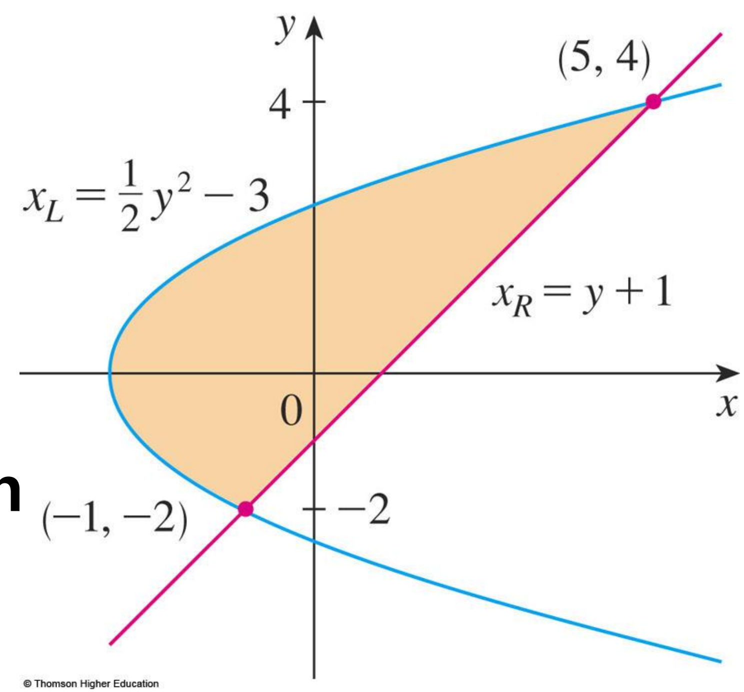
Example 8

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

- **By solving the two equations, we find that the points of intersection are (-1, -2) and (5, 4).**
- **We solve the equation of the parabola for *x*.**
- **From the figure, we notice that the left and right boundary curves are:**

$$x_L = \tfrac{1}{2}y^2 - 3$$

$$x_R = y + 1$$



- **We must integrate between the appropriate *y*-values, *y* = -2 and *y* = 4.**

Thus,

$$A = \int_{-2}^{4} \left(x_R - x_L \right) dy$$

$$= \int_{-2}^{4} \left[\left(y+1 \right) - \left(\tfrac{1}{2} y^2 - 3 \right) \right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2} y^2 + y + 4 \right) dy$$

$$= -\frac{1}{2} \left(\frac{y^3}{3} \right) + \frac{y^2}{2} + 4y \Bigg]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8 \right) = 18$$

E) Volume of solids (Volume of Revolution)

1- Find the Volume of revolution using the disk method

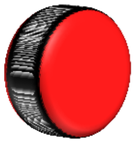
2- Find the volume of revolution using the washer method

3- Find the volume of revolution using the shell method

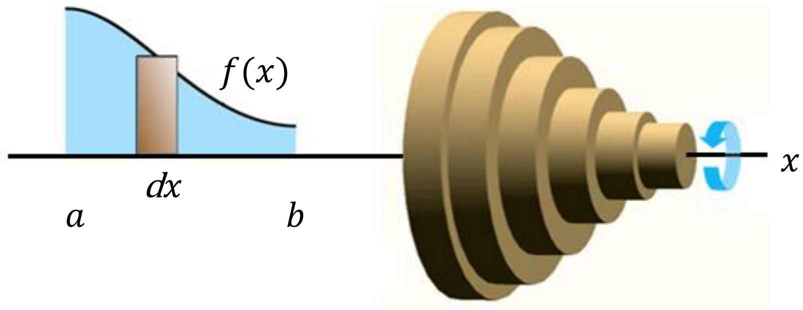
4- Find the volume of a solid with known cross sections

1- Disk Method

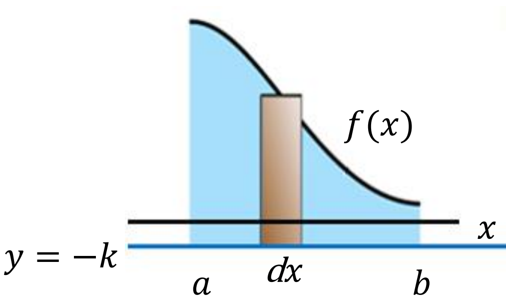
$$V = \int_a^b dV \qquad dV = \pi r^2 dx$$

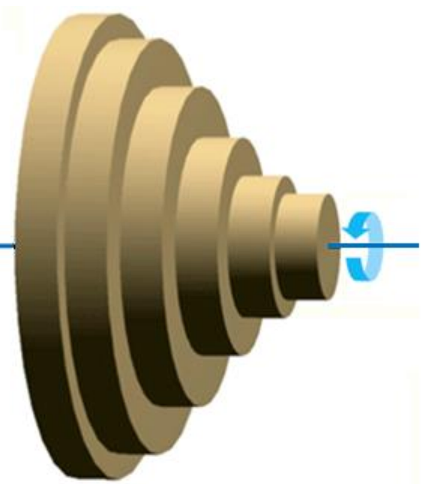


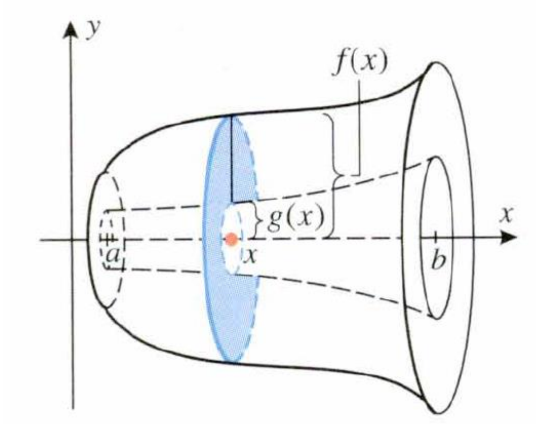
$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi r^2 \Delta x$$



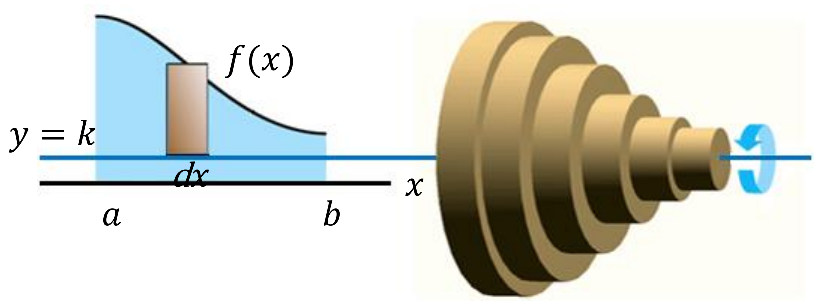
$$dV = \pi[f(x)]^2 dx$$



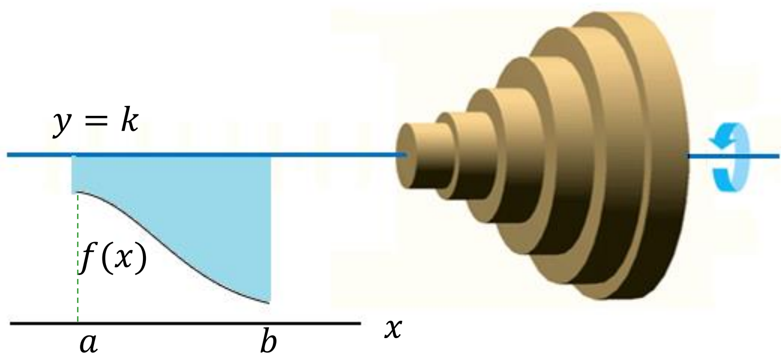




$$dV = \pi[f(x) + k]^2 dx$$

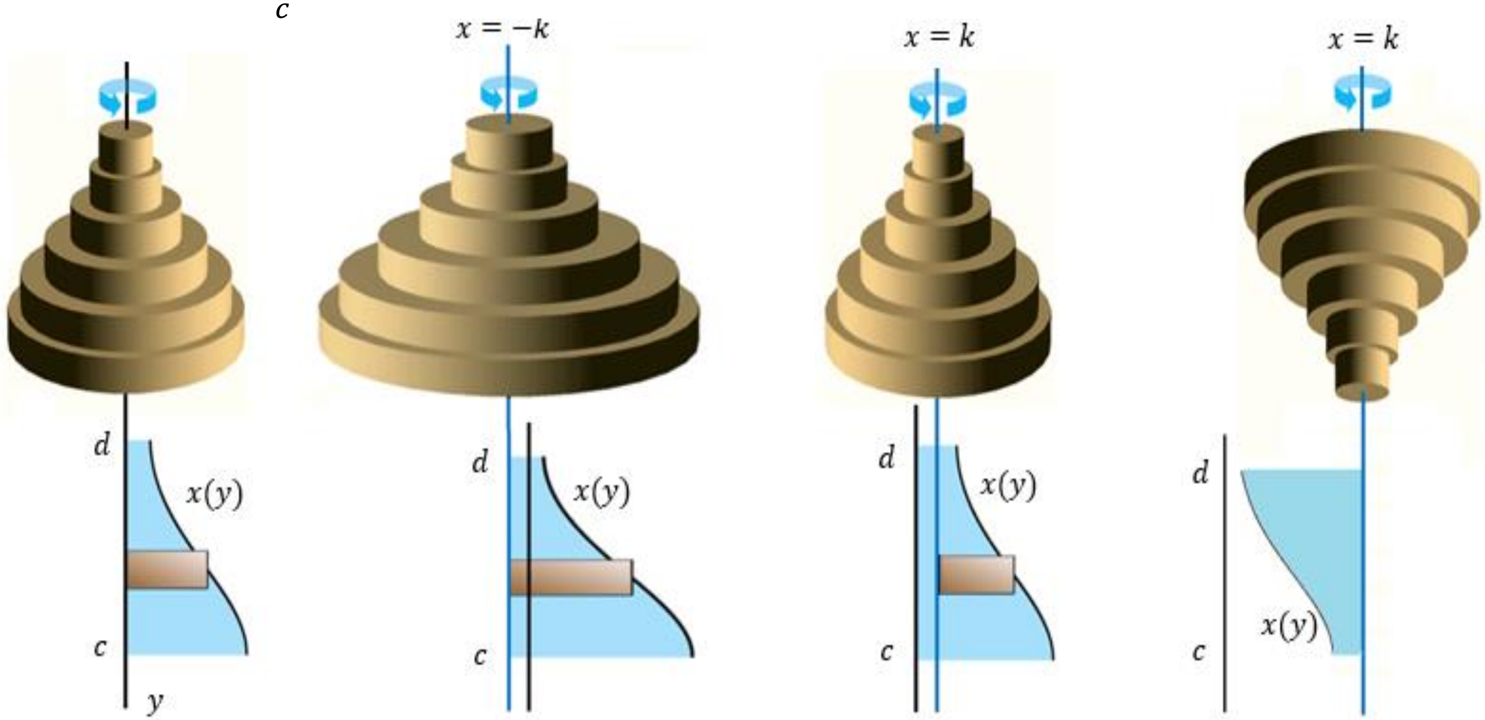


$$dV = \pi[f(x) - k]^2 dx$$



$$dV = \pi[k - f(x)]^2 dx$$

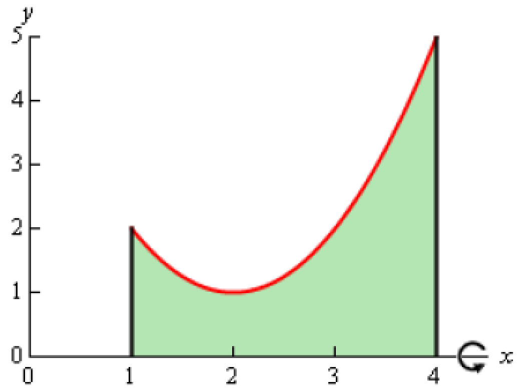
$$V = \int_c^d dV$$

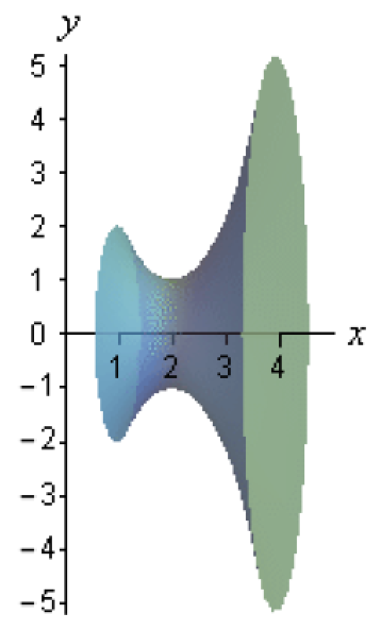


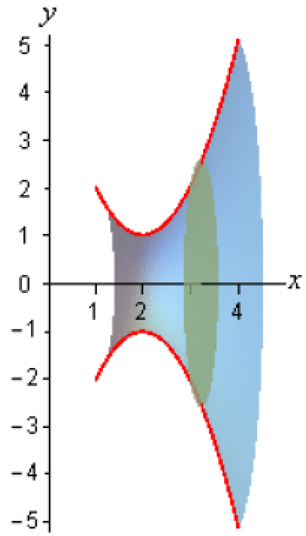
$dV = \pi[x(y)]^2 dy$ $\quad$ $dV = \pi[x(y) + k]^2 dy$ $\quad$ $dV = \pi[x(y) - k]^2 dy$ $\quad$ $dV = \pi[k - x(y)]^2 dy$

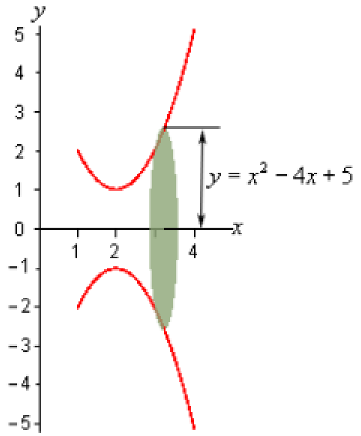
Example:

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1, x = 4$ and the x-axis about the x-axis.









The area of one disk is:

$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

The limit of integration is from x=1 to x=4

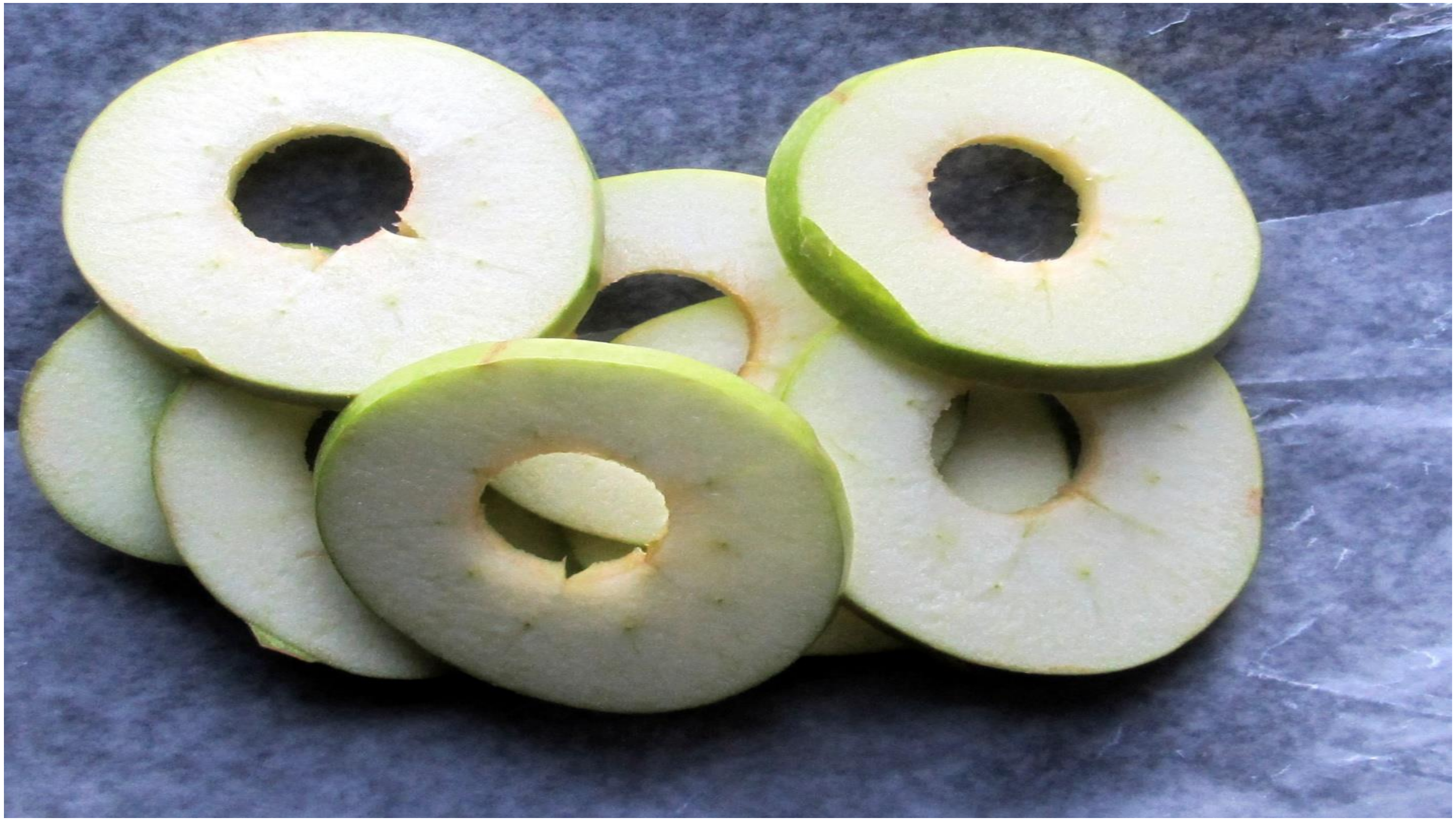
The volume of this solid is then,

$$V = \int_1^4 A(x)dx$$

$$= \pi \int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25 dx$$

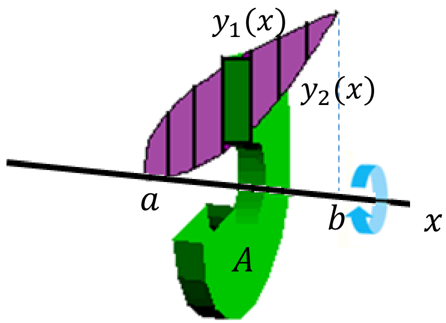
$$= \pi(\frac{x^5}{5} - 2x^4 + 26\frac{x^3}{3} - 20x^2 + 25x)|_1^4 = \frac{78\pi}{5}$$

2-Washer Method

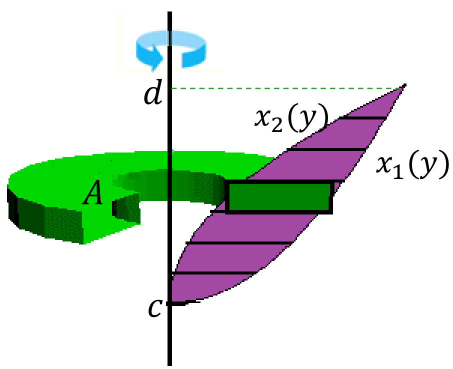


A solid obtained by revolving a region around a line.

$$A = \pi \left\{ \begin{pmatrix} outer \\ radius \end{pmatrix}^2 - \begin{pmatrix} inner \\ radius \end{pmatrix}^2 \right\}$$



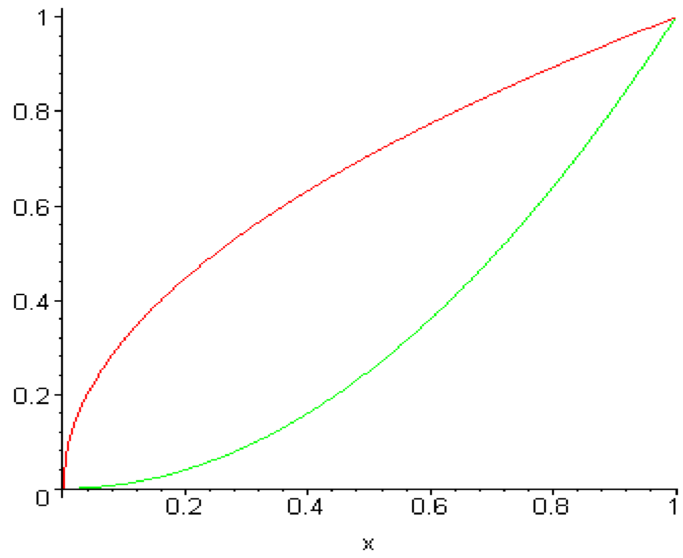
$$dV = \pi\{[y_1(x)]^2 - [y_2(x)]^2\}dx \qquad V = \int_a^b dV \qquad dV = A\,dx$$



$$dV = \pi\{[x_1(y)]^2 - [x_2(y)]^2\}dy \qquad V = \int_c^d dV \qquad dV = A\,dy$$

Example:

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$ and $y = x^2$ over the interval [0, 1] about the x – axis.



$$V = \pi \int_0^1 \{[\sqrt{x}]^2 - [x^2]^2\}dx$$

$$V = \pi \int_0^1 \{x - x^4\}dx$$

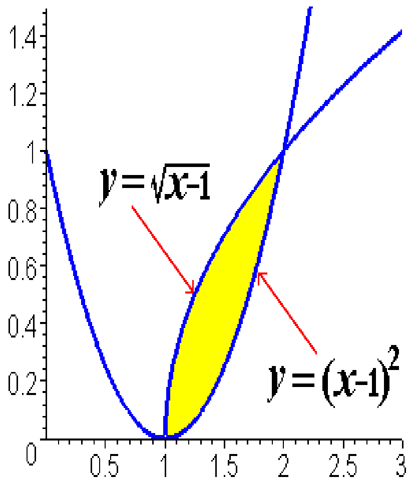
$$V = \pi \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg]_0^1$$

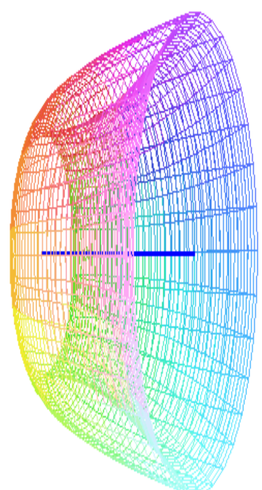
$$V = \frac{3}{10}$$

Example:

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x-1}$ and $y = (x-1)^2$ over the interval [1, 2] about the x – axis.

Find the volume of the solid of revolution formed by rotating the finite region bounded by the graphs of about the *x*-axis.





$$V = \pi\int_1^2\left[\left(\sqrt{x-1}\right)^2-\left((x-1)^2\right)^2\right]dx$$

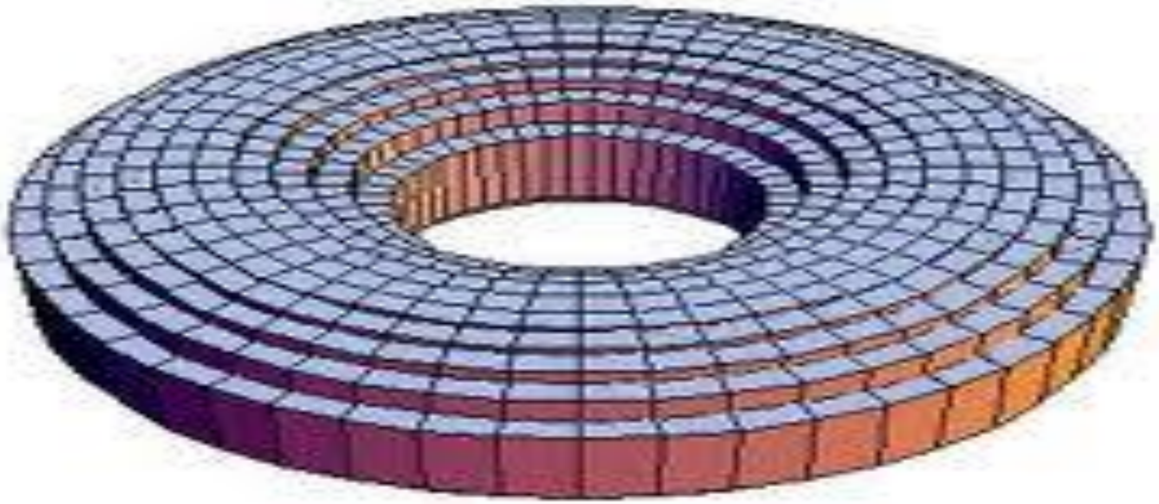
$$= \pi\int_1^2\left(x-1-(x-1)^4\right)dx = \pi\left(\frac{x^2}{2}\Big|_1^2 - x\Big|_1^2 - \int_1^2 (x-1)^4\,dx\right)$$

$$u = x-1$$
$$du = dx$$

$$= \pi\left(\frac{1}{2}\left(2^2-1^2\right)-(2-1)-\int_0^1 u^4\,du\right) = \pi\left(\frac{3}{2}-1-\frac{u^5}{5}\Big|_0^1\right)$$

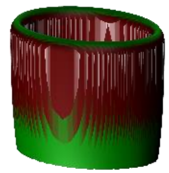
$$= \pi\left(\frac{3}{2}-1-\frac{1}{5}\left(1^5-0^5\right)\right) = \pi\left(\frac{1}{2}-\frac{1}{5}\right) = \boxed{\frac{3\pi}{10}}$$

3-Volumes by Cylindrical Shells

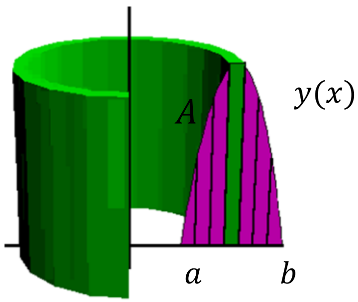


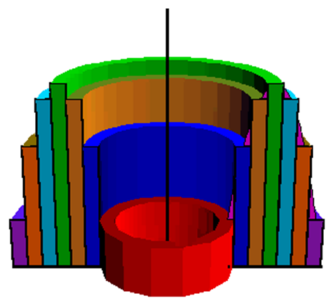
Summing up the volumes of all these infinitely thin shells, we get the total volume of the solid of revolution:

A = $2\pi\, r\, h = 2\pi\, x\, h$



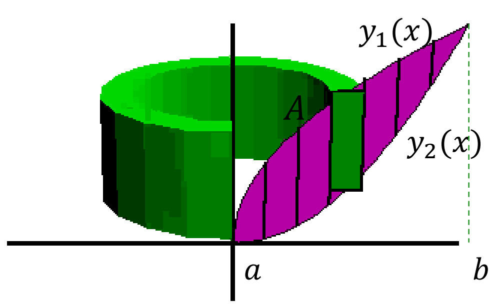
$$dV = A\,dx = 2\pi\, x\, h\, dx \qquad V = \int_a^b dV$$

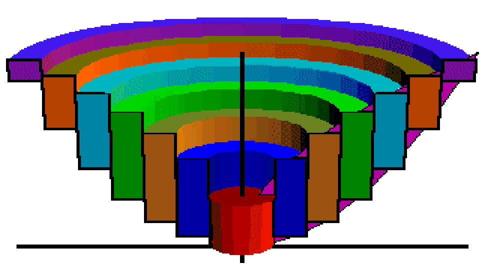




$$dV = A\,dx = 2\pi\, x\, y(x) dx$$

$$V = 2\pi \int_a^b x\, y(x) dx$$



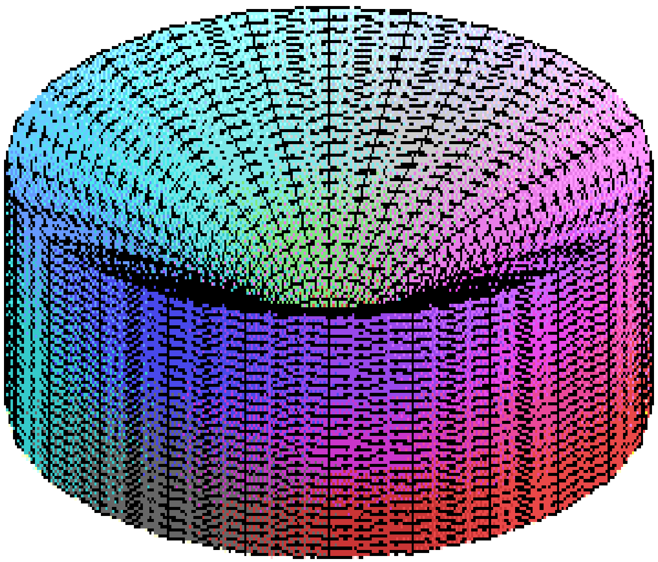


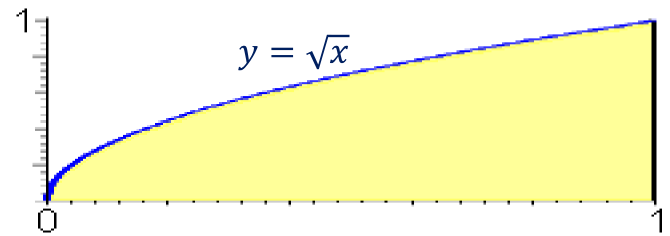
$$dV = A\,dx = 2\pi\, x\,[y_1(x) - y_2(x)]dx$$

$$V = 2\pi \int_a^b x[y_1(x) - y_2(x)]dx$$

Example:

Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis and the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$, about the y - axis.





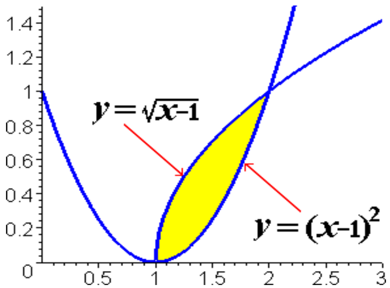
$$V = 2\pi \int_0^1 x\sqrt{x}\,dx$$

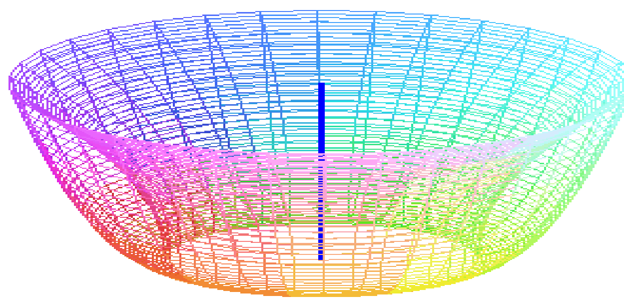
$$V = 2\pi \frac{2}{5} x^{5/2} \Big]_0^1$$

$$V = \frac{4\pi}{5}$$

Example:

Find the volume of the solid of revolution formed by rotating the finite region bounded by the graphs of $y = \sqrt{x-1}$ and $y = (x-1)^2$ about the y-axis.





u = x – 1 x = 1
u=0
x = u + 1 x = 2
u=1
du = dx

$$V = 2\pi \int_1^2 x\left(\sqrt{x-1} - (x-1)^2\right)dx$$

$$V = 2\pi \int_1^2 x\sqrt{x-1}dx - 2\pi \int_1^2 x(x-1)^2 dx$$

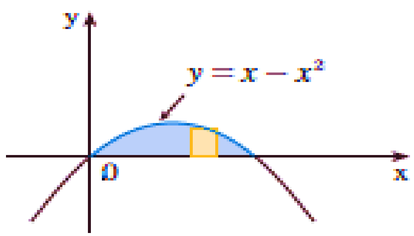
$$V = 2\pi \int_0^1 (u+1)\sqrt{u}dx - 2\pi \int_0^1 (u+1)u^2 dx$$

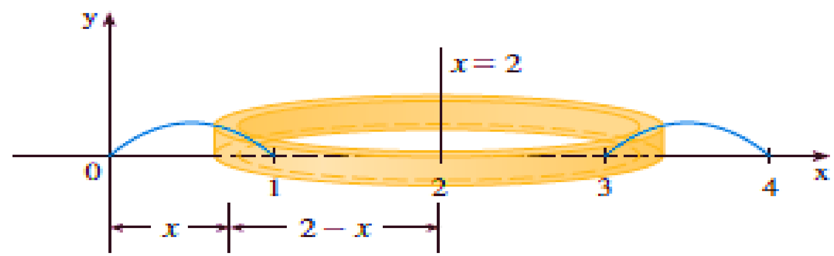
$$V = 2\pi \left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} - \frac{u^4}{4} - \frac{u^3}{3}\right]_0^1 = 2\pi\left[\frac{2}{5} + \frac{2}{3} - \frac{1}{4} - \frac{1}{3}\right]$$

$$V = \frac{29}{30}\pi$$

Example:

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$





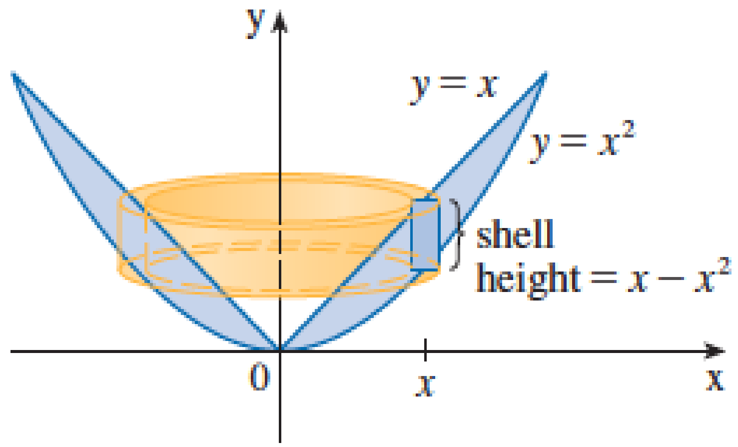
$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx = 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

Example:

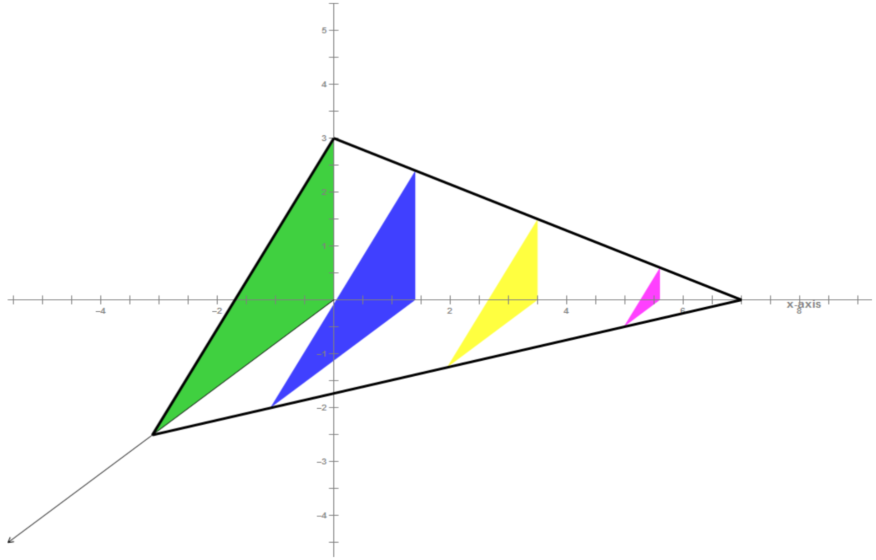
Find the volume of the solid obtained by rotating about the y-axis the region between y=x and y=x^2

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = \quad 2\pi \int_0^1 (x^2 - x^3)\,dx \quad = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{\pi}{6}$$



4-The Volume for Solids with Known Cross Sections (Volume by Slicing)

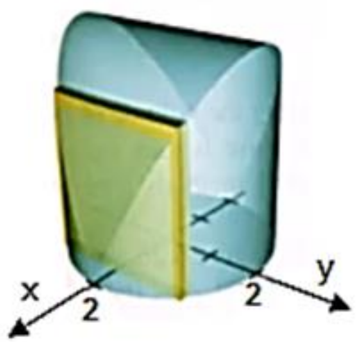


Procedure: volume by slicing

- \- Sketch the solid and a typical cross-section.
- \- find a formula for the area, $A(x)$, of the cross-section.
- \- find limits of integration.
- \- integrate $A(x)$ to get volume.

Example:

Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and where cross sections perpendicular to the x-axis are all squares whose sides lie on the base of the circle.

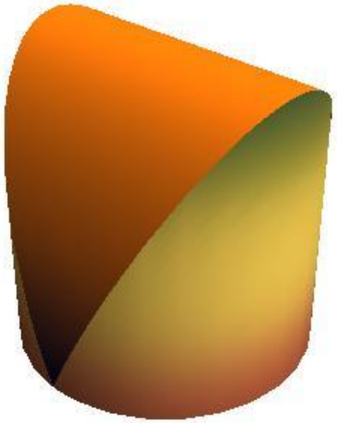


$x^2 + y^2 = 4$ $\quad y = \sqrt{4 - x^2}$

$a = 2\sqrt{4 - x^2}$

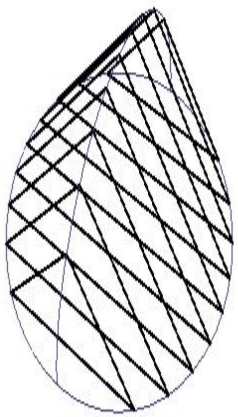
$dV = A\,dx \quad A = a^2$

$$V = 4\int_{-2}^{2}(4 - x^2)dx = \frac{128}{3}$$



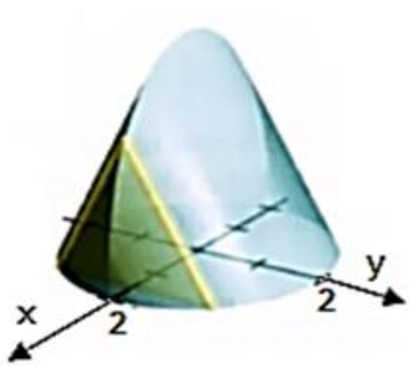
Example:

Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and where cross sections perpendicular to the x-axis are all equilateral triangles whose sides lie on the base of the circle.



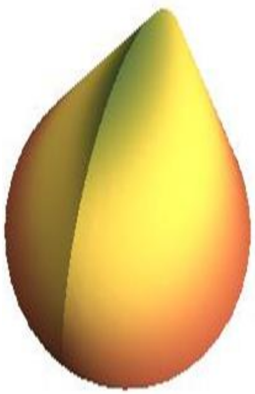
$$dV = A\,dx \qquad A = ?$$

$$x^2 + y^2 = 4 \qquad y = \sqrt{4 - x^2}$$



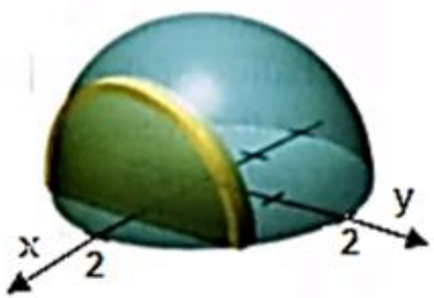
$$A = \frac{1}{2} a \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \frac{\sqrt{3}}{4}\, a^2 = \sqrt{3}(4 - x^2)$$

$$V = \int_{-2}^{2} \sqrt{3}(4 - x^2)dx = \frac{32}{\sqrt{3}} \approx 18.475$$



Example:

Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and where cross sections perpendicular to the x-axis are all semicircles whose sides lie on the base of the circle.



$$A = ?$$

$x^2 + y^2 = 4$ $\qquad y = \sqrt{4 - x^2}$

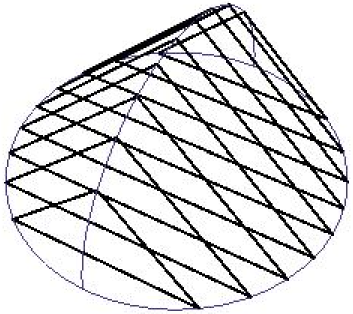
$$A = \frac{1}{2}\,\pi\left(\frac{a}{2}\right)^2 = \frac{1}{8}\pi\,a^2 = \pi\,\frac{4 - x^2}{2}$$

$$dV = A\,dx$$

$$V = \int_{-2}^{2} \pi\,\frac{4 - x^2}{2}\,dx = \frac{16\pi}{3} \approx 16.755$$

Example:

Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and where cross sections perpendicular to the x-axis are all Isosceles right triangles whose sides lie on the base of the circle.



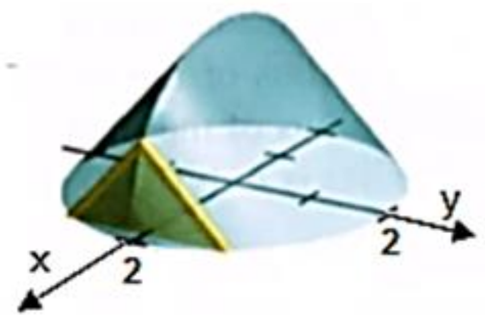
$$A = ?$$

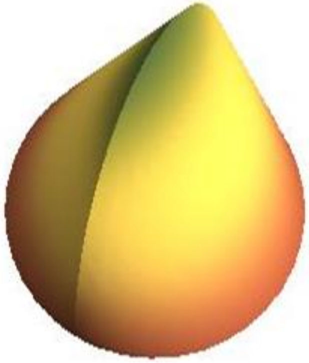
$$x^2 + y^2 = 4 \qquad y = \sqrt{4 - x^2}$$

$$A = \frac{1}{2} a \, \frac{\frac{a}{2}}{\tan \pi/4} = \frac{a^2}{4} = 4 - x^2$$

$$dV = A\,dx$$

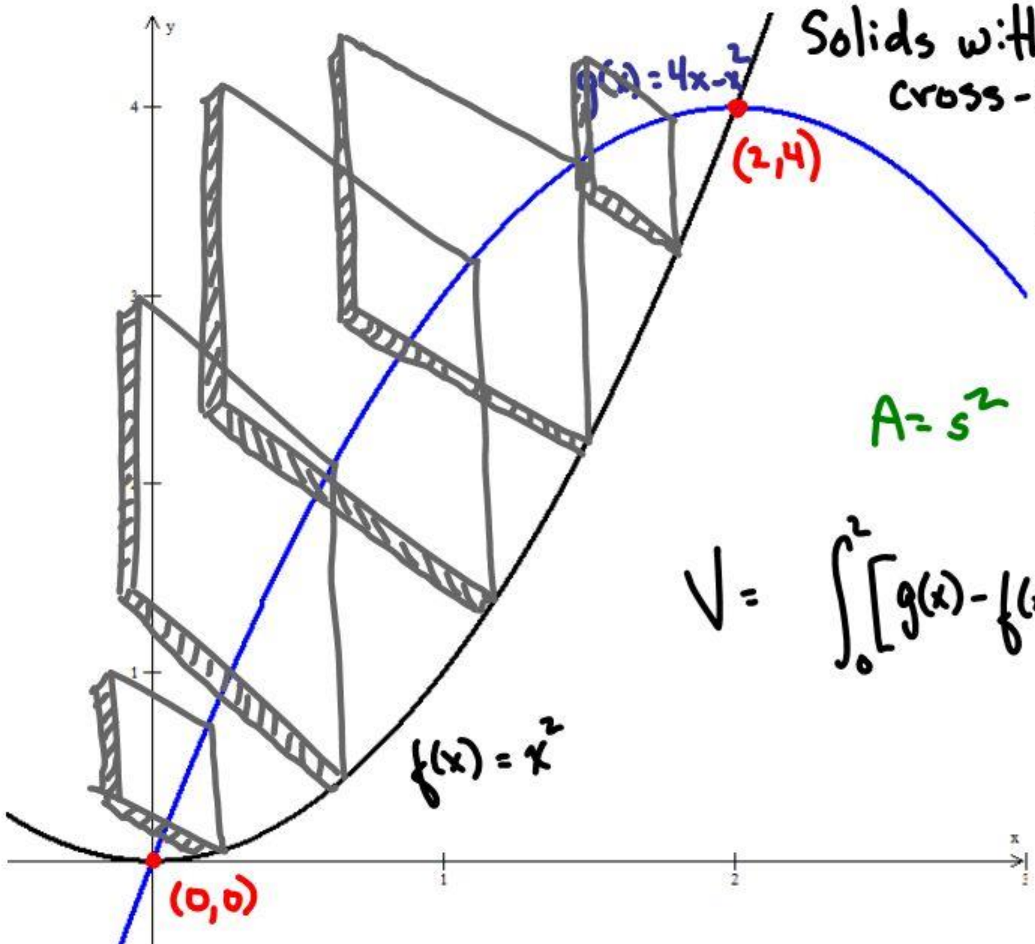
$$V = \int_{-2}^{2} (4 - x^2)\,dx = \frac{32}{3} \approx 10.667$$





Solids with known cross-sections

SQUARES

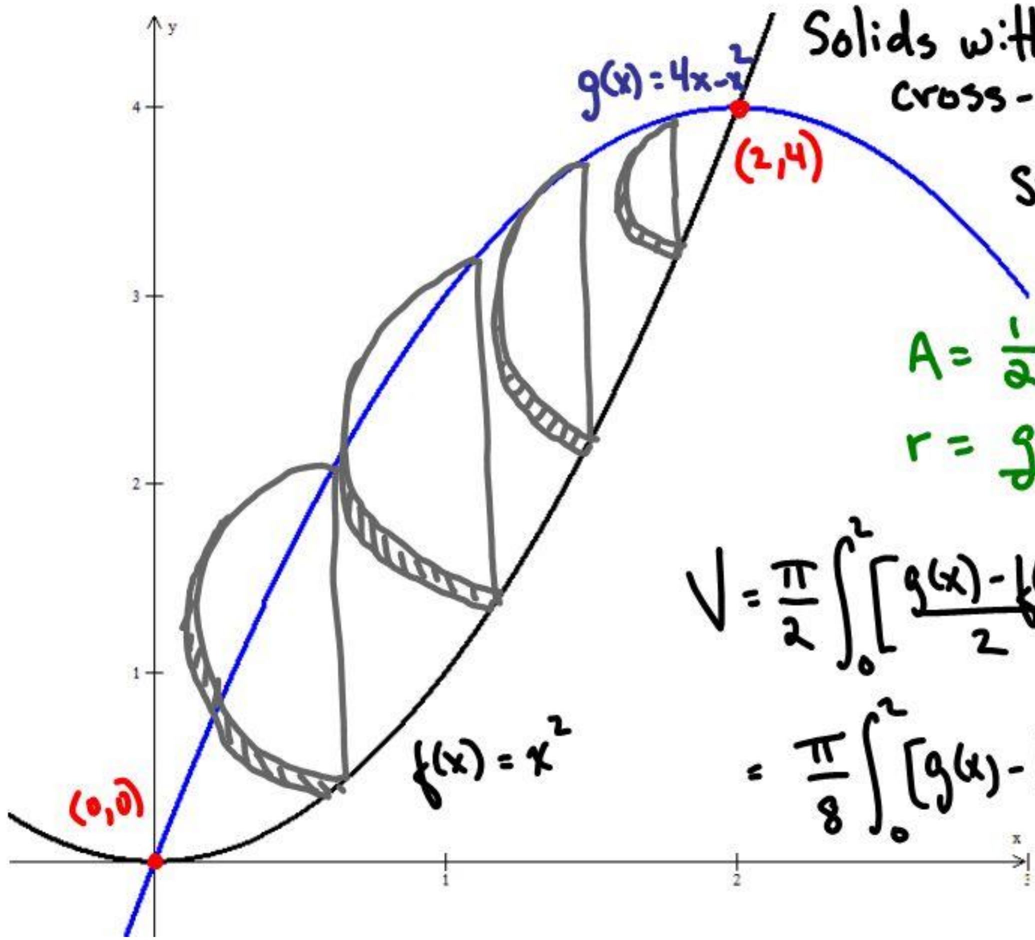


$A = s^2$ $\quad s = g(x) - f(x)$

$$V = \int_0^2 [g(x) - f(x)]^2 \, dx$$

Solids with known cross-sections

SEMI-CIRCLES



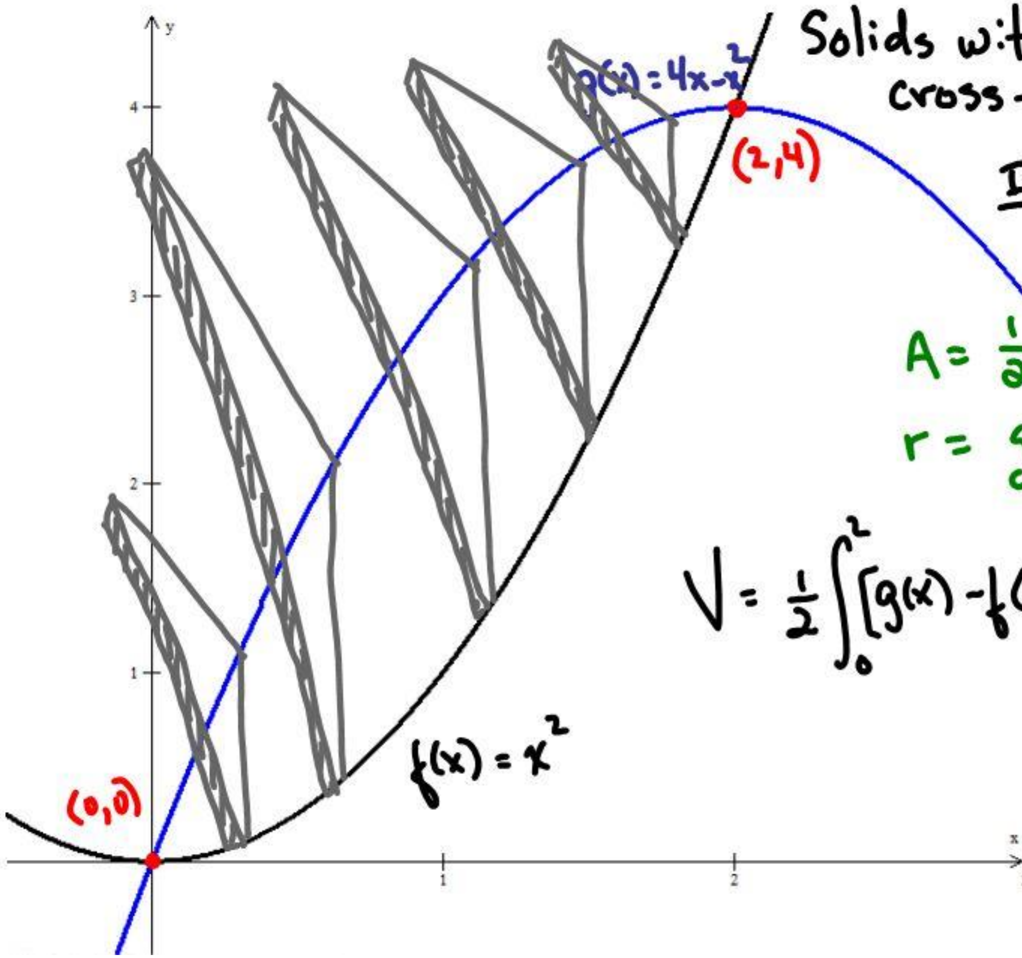
$$A = \frac{1}{2}\pi r^2$$

$$r = \frac{g(x) - f(x)}{2}$$

$$V = \frac{\pi}{2}\int_0^2 \left[\frac{g(x) - f(x)}{2}\right]^2 dx$$

$$= \frac{\pi}{8}\int_0^2 \left[g(x) - f(x)\right]^2 dx$$

Solids with known cross-sections



ISOSCELES RIGHT TRIANGLE

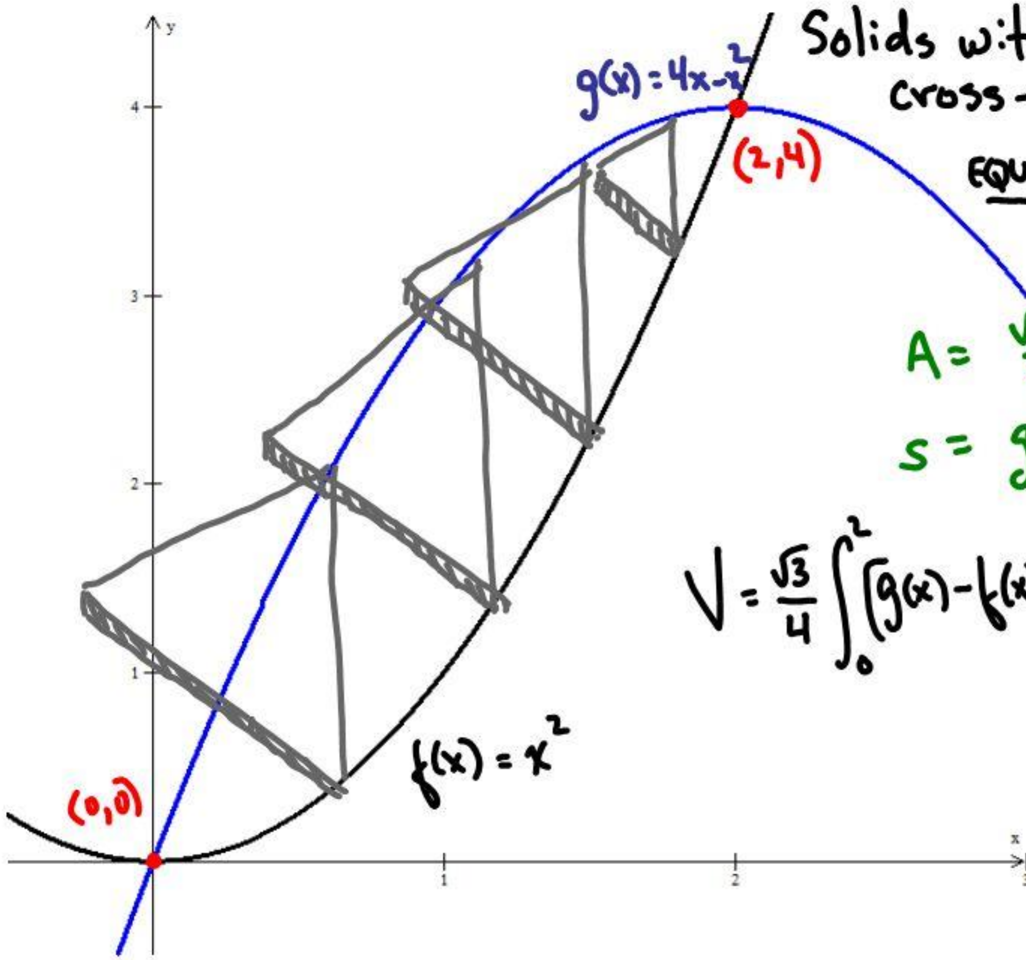
$A = \frac{1}{2} r^2$

$r = g(x) - f(x)$

$$V = \frac{1}{2}\int_0^2 [g(x) - f(x)]^2 \, dx$$

Solids with known cross-sections

EQUILATERAL RIGHT TRIANGLE



$A = \frac{\sqrt{3}}{4} s^2$

$s = g(x) - f(x)$

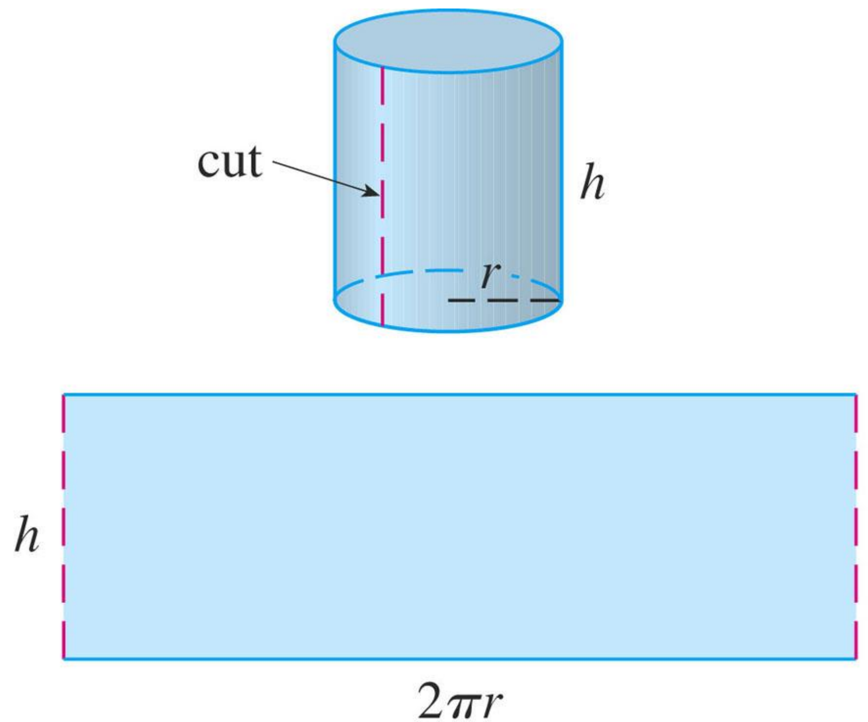
$$V = \frac{\sqrt{3}}{4} \int_0^2 [g(x) - f(x)]^2 \, dx$$

F) Area of a Surface of Revolution

Area of a Surface of Revolution

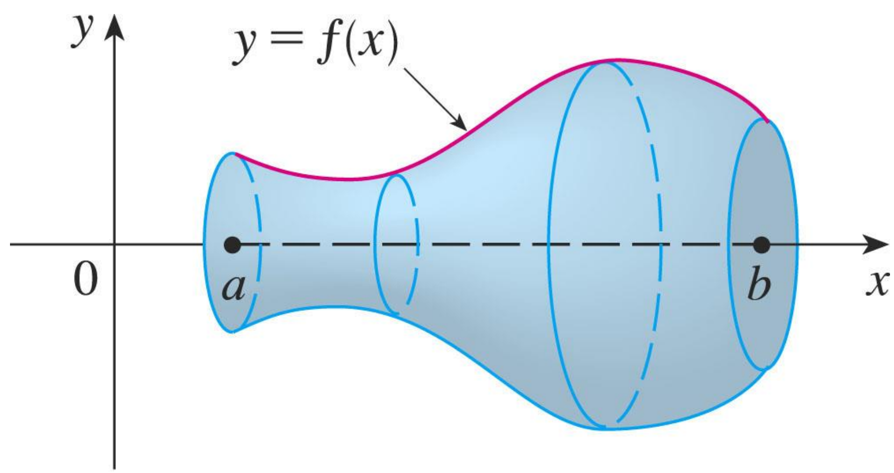
- A surface of revolution is formed when a curve is rotated about a line.
- The lateral surface area of a circular cylinder with radius *r* and height *h* is taken to be:

$$A = 2\pi rh$$



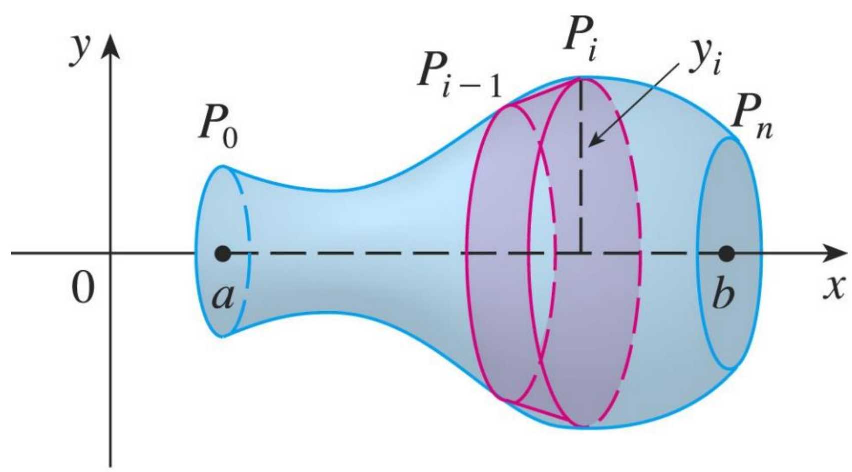
- **Consider the surface shown below.**

- **It is obtained by rotating the curve *y* = *f*(*x*), *a* ≤ *x* ≤ *b*, about the *x*-axis, where *f* is positive and has a continuous derivative.**



(a) Surface of revolution

- **If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve.**
  - **The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis.**



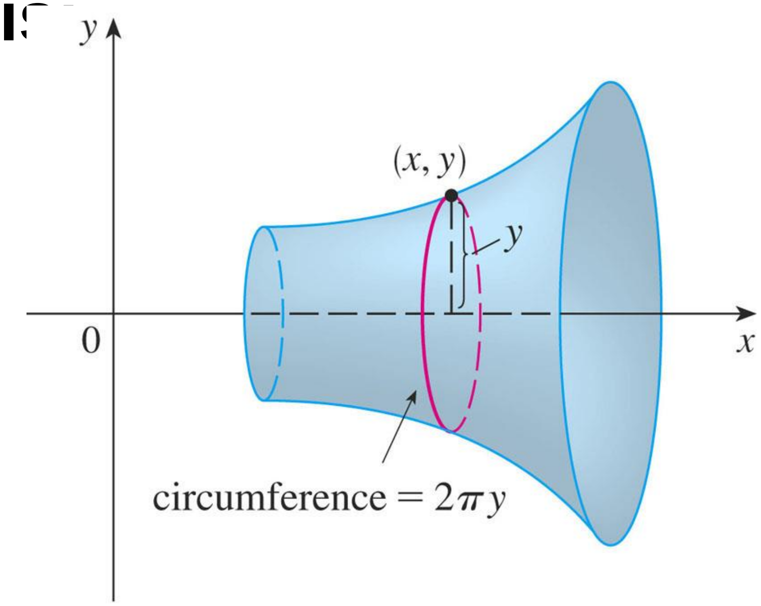
(b) Approximating band

- **The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = ½(y_{i-1} + y_i)$.**

So, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

$$S = \int 2\pi y \, ds$$



(a) Rotation about x-axis: $S = \int 2\pi y \, ds$

$$g(x) = 2\pi f(x)\sqrt{1+[f'(x)]^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$$

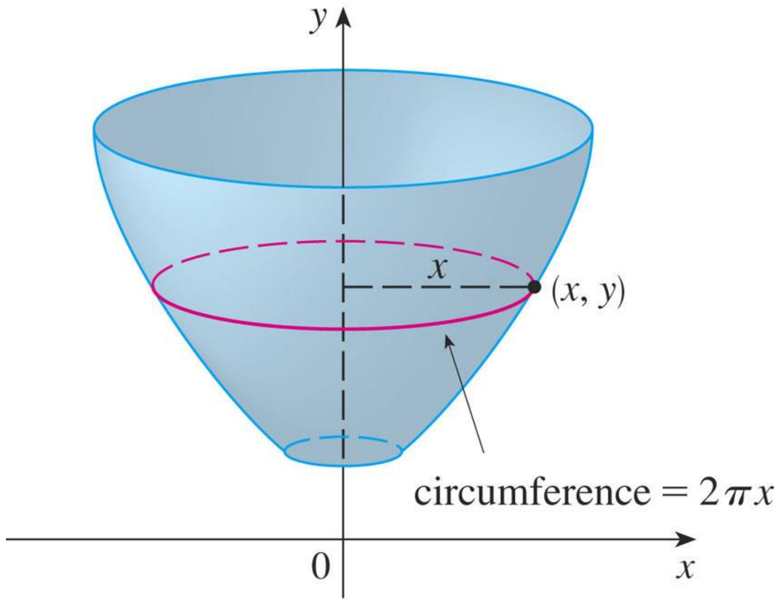
$$= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

- For rotation about the *y*-axis, the formula becomes:

$$S=\int 2\pi x\,ds$$

$$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

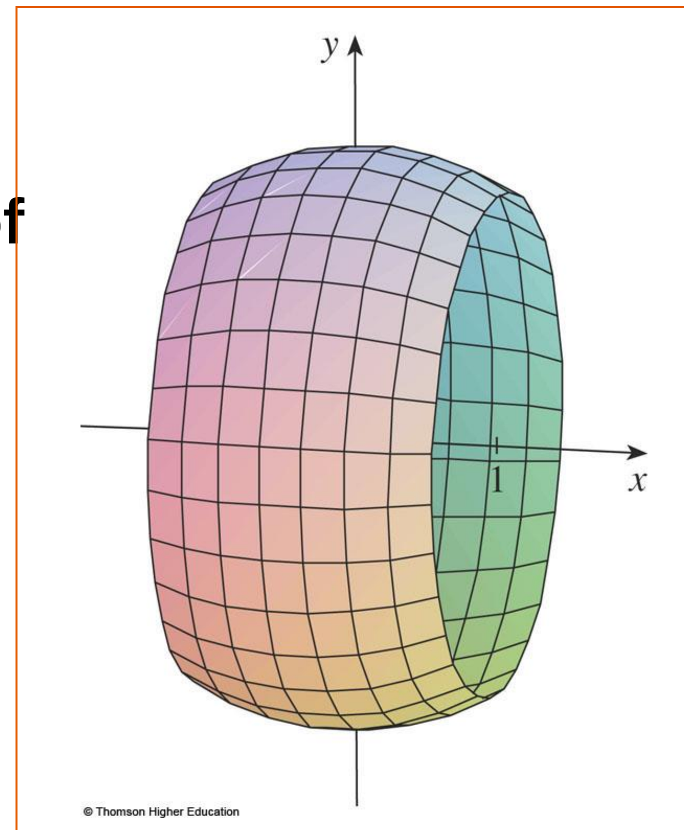


(b) Rotation about *y*-axis: $S=\int 2\pi x\,ds$

Example 1

- The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$.
- Find the area of the surface obtained by rotating this arc about the *x*-axis.

  - The surface is a portion of a sphere of radius 2.



$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{4 - x^2}}$$

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

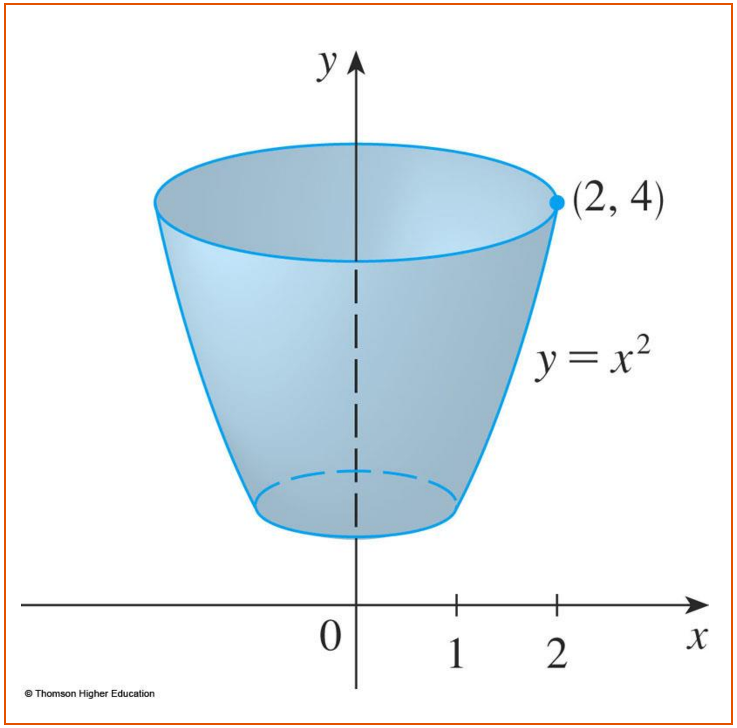
$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \frac{2}{\sqrt{4 - x^2}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1\, dx = 4\pi(2) = 8\pi$$

Example 2

- The arc of the parabola $y = x^2$ from (1, 1) to (2, 4) is rotated about the *y*-axis.

- Find the area of the resulting surface.



- **Using $y = x^2$ and $dy/dx = 2x$,**

$$S = \int 2\pi x\, ds$$

$$= \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x\sqrt{1 + 4x^2}\, dx$$

- **Substituting $u = 1 + 4x^2$, we have $du = 8x\,dx$.**

- **Remembering to change the limits of integration, we have:**

$$S = \frac{\pi}{4}\int_5^{17}\sqrt{u}\,du = \frac{\pi}{4}\left[\tfrac{2}{3}u^{3/2}\right]_5^{17}$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$