Applications of Derivative

Objectives

- Know what the Maximum and minimum are.
- Know what the Increasing and Decreasing are.
- Know what the Concavity & Points of inflection are.
- Know what the Second Derivative Test is.
- Know what the Mean value theorem is.

Maximum and minimum points & Increasing and Decreasing Functions

Derivatives can be used to sketch functions:

First Derivative:

-First derivative indicates slope

-if $y'>0$, function slopes increase

-if $y'<0$, function slopes decrease

-if $y'=0$, function slope is horizontal

-slope may change over time

shape/concavity must be determined

Second Derivative:

-Second derivative indicates concavity

-if y”>0, (concave upward); (minimum point).

-if y”<0, (concave downward); (maximum point).

-if y”=0, (an inflection point occurs)

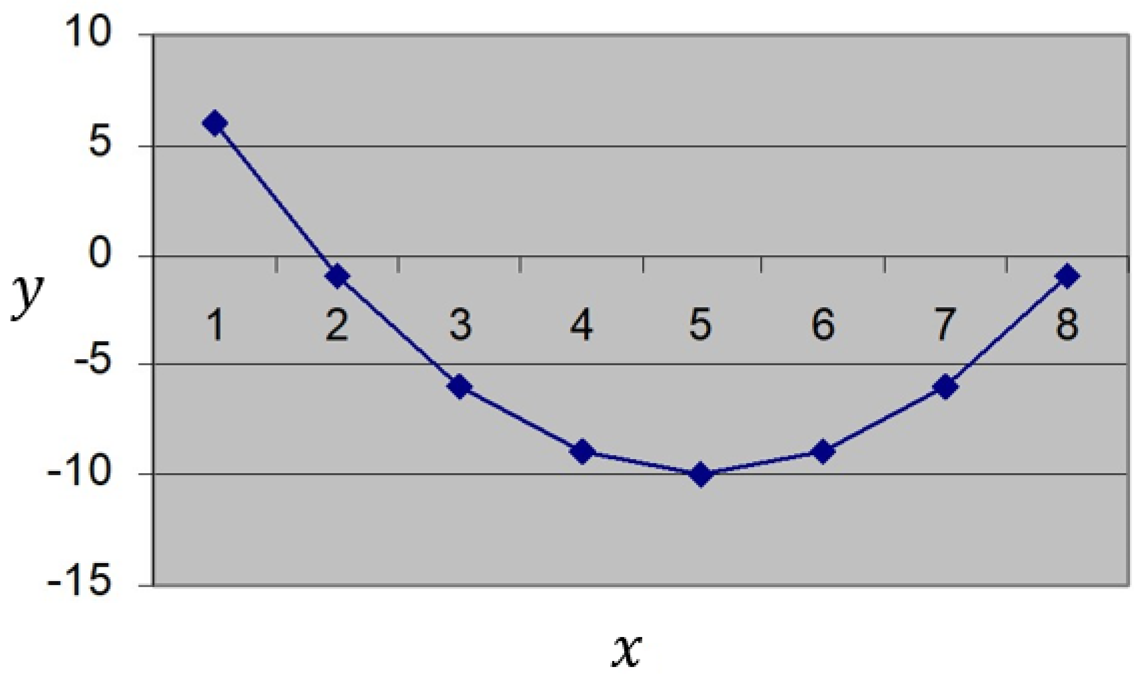
Sample Graphs

y”= 2, graph is concave upward

$$y = x^2 - 10x + 15$$

$$y' = 2x - 10 \quad ; \quad y' = 0 \Rightarrow x = 5$$

$$y'' = 2$$



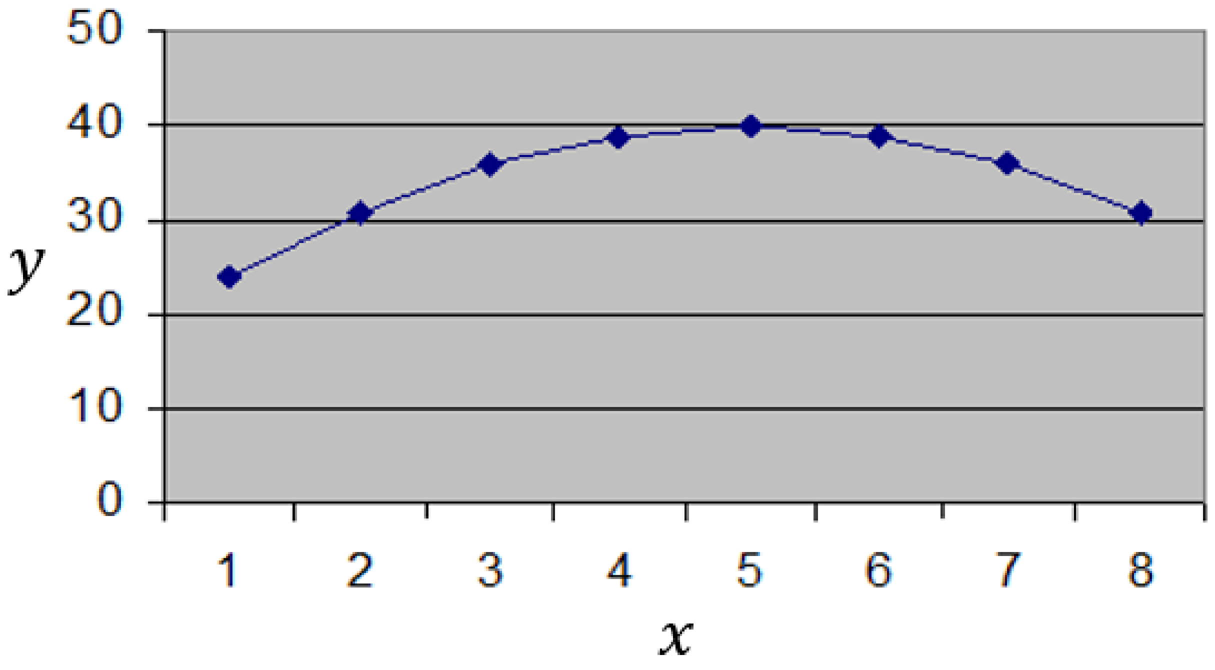
Sample Graphs

y”=-2, graph is concave downward

$$y = -x^2 + 10x + 15$$

$$y' = -2x + 10 \quad ; \quad y' = 0 \Rightarrow x = 5$$

$$y'' = -2$$



Critical Point

An interior point of the domain of a function f where f' is <u>zero</u> or <u>undefined</u>.

Maxima/minima can aid in drawing graphs

<u>Maximum Point:</u>

If 1) $f(a)'=0$, and

2) $f(a)''<0$,

- graph has a maximum point (peak) at $x=a$

<u>Minimum Point:</u>

If 1) $f(a)'=0$, and

2) $f(a)''>0$,

- graph has a minimum point (valley) at $x=a$

Inflection Points:

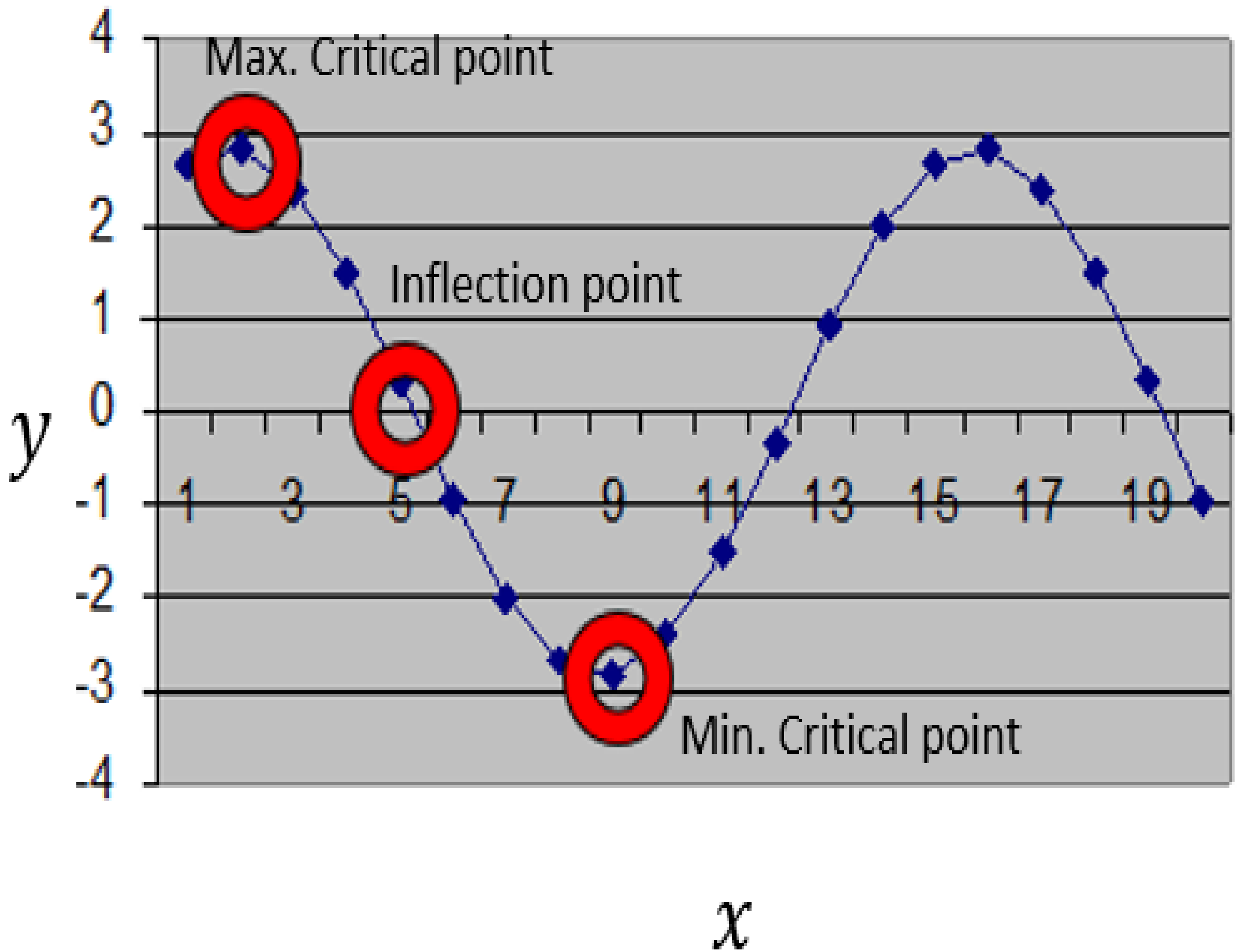
If

1) f(a)”=0, and

2) the graph is not a straight line

-then an inflection point occurs

-(where the graph switches between concave upward and concave downward)


Max. Critical point
Inflection point
Min. Critical point
4
3
2
1
0
-1
-2
-3
-4
y
1
3
5
7
9
11
13
15
17
19
x

Graphing Steps:

i) Evaluate y(x) at intersect with y-axis; (x=0).

ii) Determine where y=0 at intersect with x-axis.

iii) Calculate slope: y’ - and determine where it is positive and negative.

iv) Identify possible maximum and minimum
co-ordinates where y’=0. (Don’t just find the x values).

v) Calculate the second derivative – y'' and use it to determine max/min in iv.

vi) Using the second derivative, determine the curvature (concave upward or concave downward) at other points.

vii) Check for inflection points where y''=0.

Graphing Example 1

$y=(x-5)^2-3$

i) $y(0)=22$,

ii) $y=0$ when

$$(x-5)^2=3$$

$$(x-5) = \pm 3^{1/2}$$

$$x = \pm 3^{1/2}+5$$

$x = 6.7, 3.3$ (x-intercepts)

iii) $y'=2(x-5)$

$y'>0$ when $x>5$

$y'<0$ when $x<5$

$y=(x-5)^2-3$

iv) $y'=0$ when $x=5$

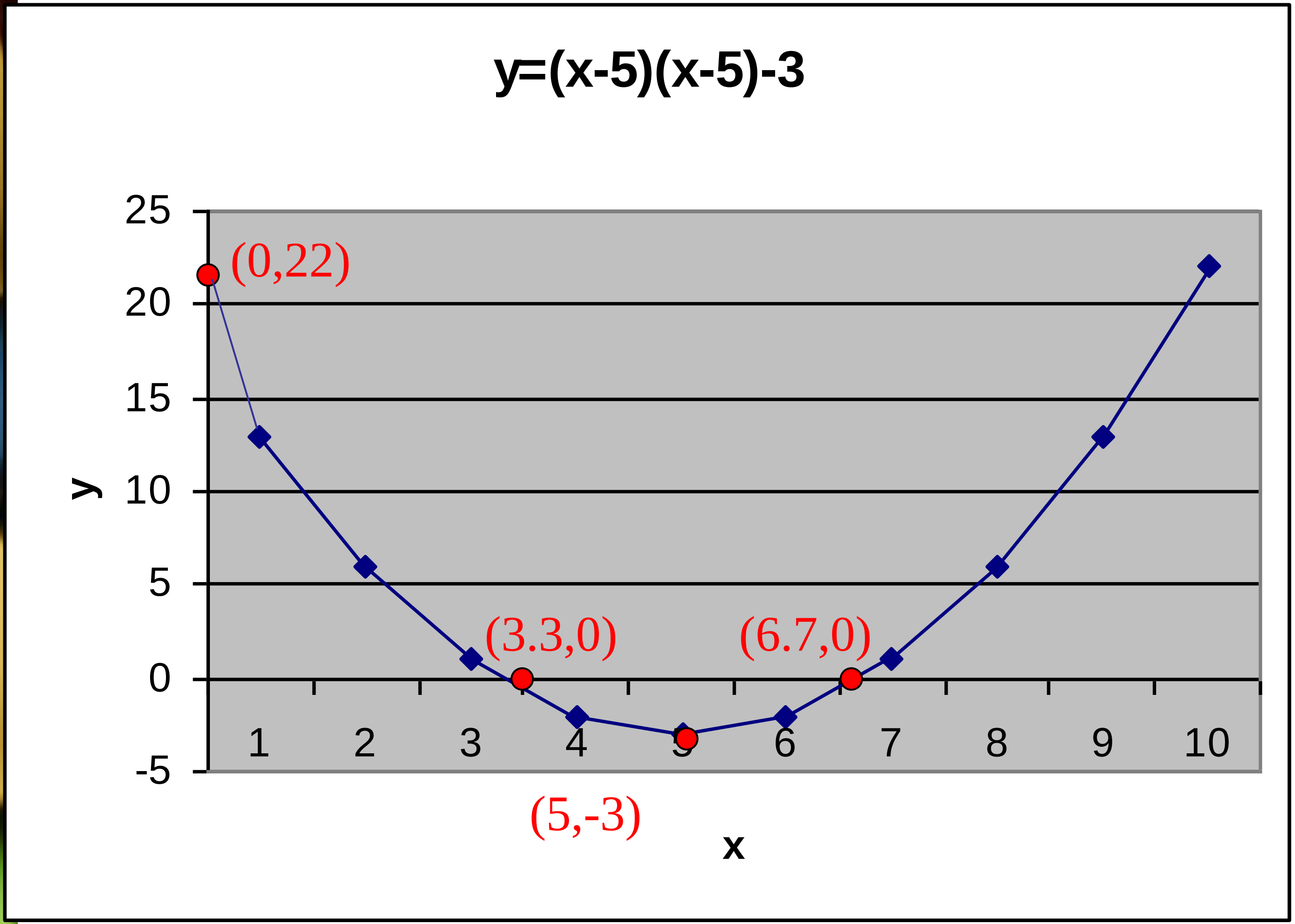
$y(5)=(5-5)^2-3=-3$

$(5,-3)$ is a potential max/min.

v) $y''=2$, $(5,-3)$ is a minimum.

vi) Function is always positive, it is always concave downward.

vii) y'' never equals zero.


y=(x-5)(x-5)-3
25
20
15
10
5
0
-5
y
1
2
3
4
5
6
7
8
9
10
x
(0,22)
(3.3,0)
(6.7,0)
(5,-3)

Graphing Example 2

$y=(x+1)(x-3)=x^2-2x-3$

i) $y(0)=-3$,

ii) $y=0$ when

$$(x+1)(x-3)=0$$

$$x = 3,-1 \text{ (x-intercepts)}$$

iii) $y'=2x-2$

$y'>0$ when $x>1$

$y'<0$ when $x<1$

$y=(x+1)(x-3)=x^2-2x-3$

iv) $y'=0$ when $x=1$

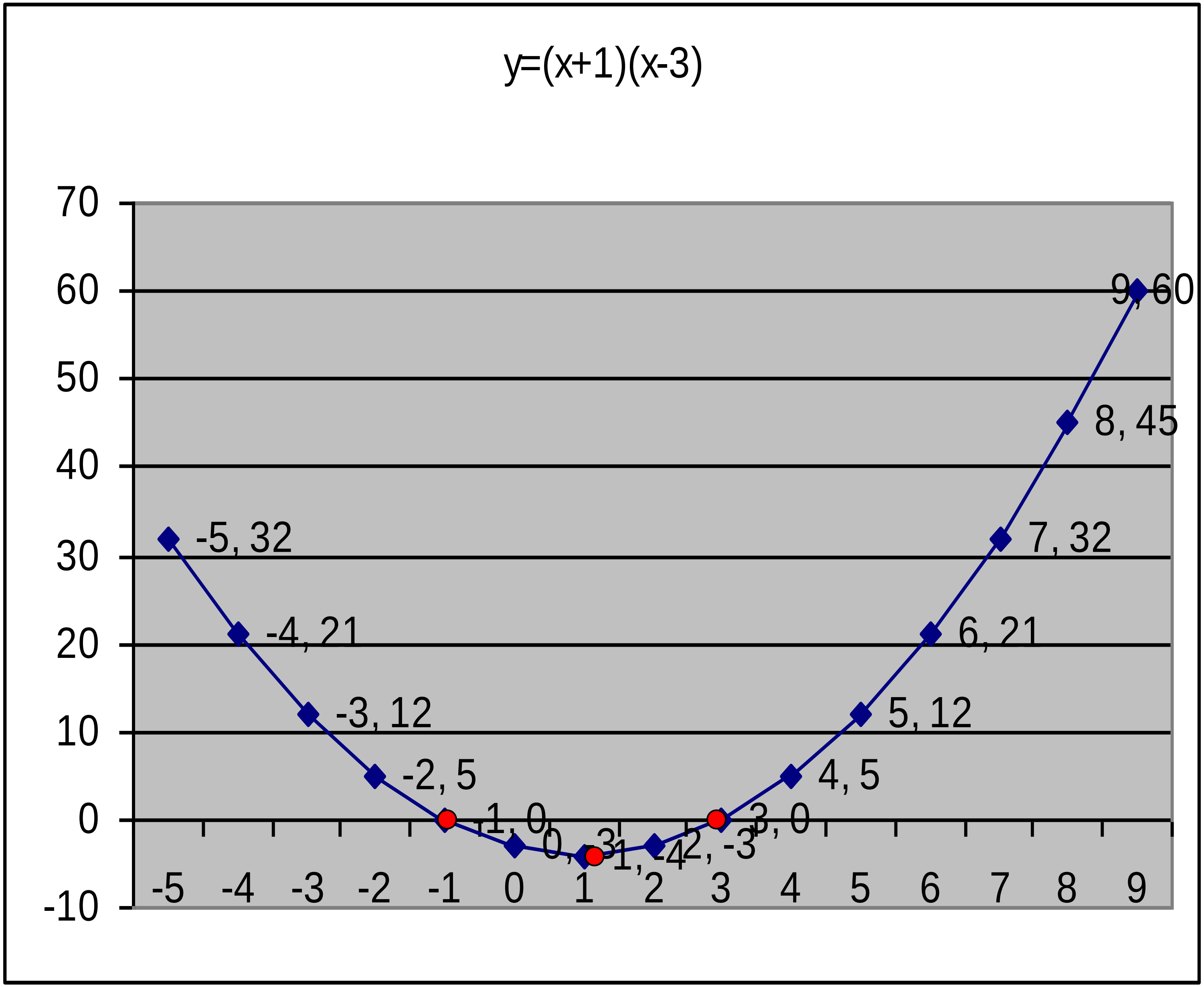
$f(1)=1^2-2(1)-3=-4$

$(1,-4)$ is a potential max/min.

v) $y''=2$, $x=1$ is a minimum.

vi) Function is always positive, it is always concave upward.

vii) y'' never equals zero.

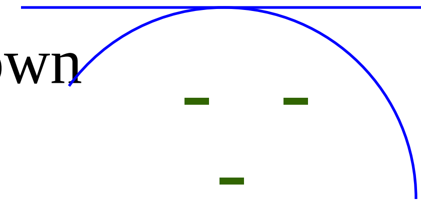


y=(x+1)(x-3)
70
60
50
40
30
20
10
0
-10
-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
-5, 32
-4, 21
-3, 12
-2, 5
-1, 0
0, -3
1, -4
2, -3
3, 0
4, 5
5, 12
6, 21
7, 32
8, 45
9, 60

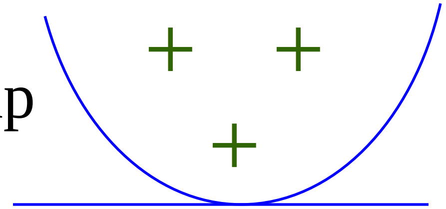
Second derivative test

If there is a critical value at x = c

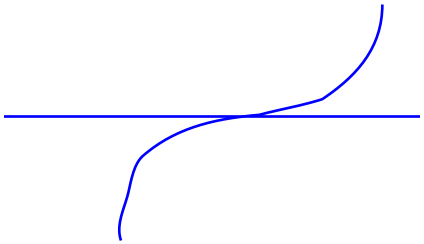
$f''(c) < 0$ A local max at x = c, Concave down



$f''(c) > 0$ A local min at x = c, Concave up



$f''(c) = 0$ inconclusive



Mean Value Theorem

Let f(x) be a function defined on [a, b] such that

(i) it is continuous on [a, b].

(ii) it is differentiable on (a, b).

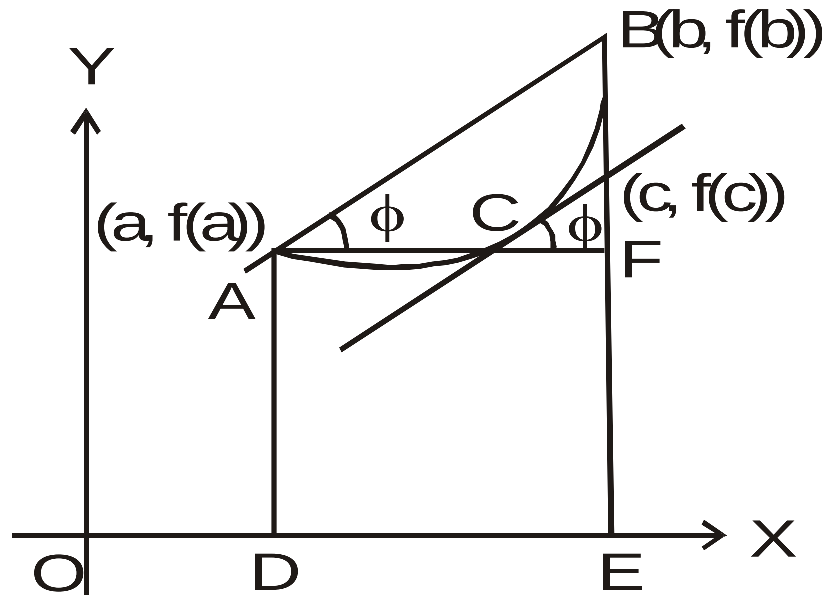
Then, there exists a real number $c \in (a, b)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrical Meaning

From the triangle AFB,

$$\tan\phi = \frac{BF}{AF} \Rightarrow \tan\phi = \frac{f(b)-f(a)}{b-a}$$



By Mean Value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \tan\phi = f'(c)$$

$\Rightarrow$ Slope of the chord AB = Slope of the tangent at $(c, f(c))$

Example- 1

Verify Mean Value theorem for the function $f(x) = x^2 - 3x + 2$ on [-1, 2].

Solution :

(1) The function f(x) being a polynomial function is continuous in [-1, 2].

(2) $f'(x) = 2x - 3$ exists in (-1, 2)

$\therefore$ f(x) is differentiable in (-1, 2)

So, there exists at least one $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow 2c - 3 = \frac{0-6}{2+1}$$

$$\Rightarrow 2c - 3 = -2 \Rightarrow c = \frac{1}{2} \in (-1, 2)$$

Example- 2

Using mean value theorem, find the point on the curve

$$y = x^3 - 3x$$

where tangent is parallel to the chord joining (1, −2) and (2, 1).

Solution :

(1) The function being a polynomial function is continuous on [1, 2].

(2) $y' = 3x^2 - 3$ exists in $(1,\ 2)$

$\therefore$ Function is differentiable in (1, 2).

So, $\exists\, x \in (1,\ 2)$ such that tangent is parallel to chord joining (1, −2) and (2, 1)

$$\therefore f'(x) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow 3\left(x^2 - 1\right) = \frac{2 - (-2)}{1}$$

$$\Rightarrow x^2 - 1 = \frac{4}{3} \Rightarrow x^2 = \frac{7}{3} \Rightarrow x = \pm\sqrt{\frac{7}{3}}$$

$$\therefore y = \left(\frac{7}{3}\right)^{\frac{3}{2}} - 3 \times \left(\pm\sqrt{\frac{7}{3}}\right)$$

$$\therefore y = \mp \frac{2}{3}\sqrt{\frac{7}{3}}$$

$$\therefore \text{The points are } \left(\sqrt{\frac{7}{3}},\ -\frac{2}{3}\sqrt{\frac{7}{3}}\right), \left(-\sqrt{\frac{7}{3}},\ \frac{2}{3}\sqrt{\frac{7}{3}}\right).$$