Derivatives

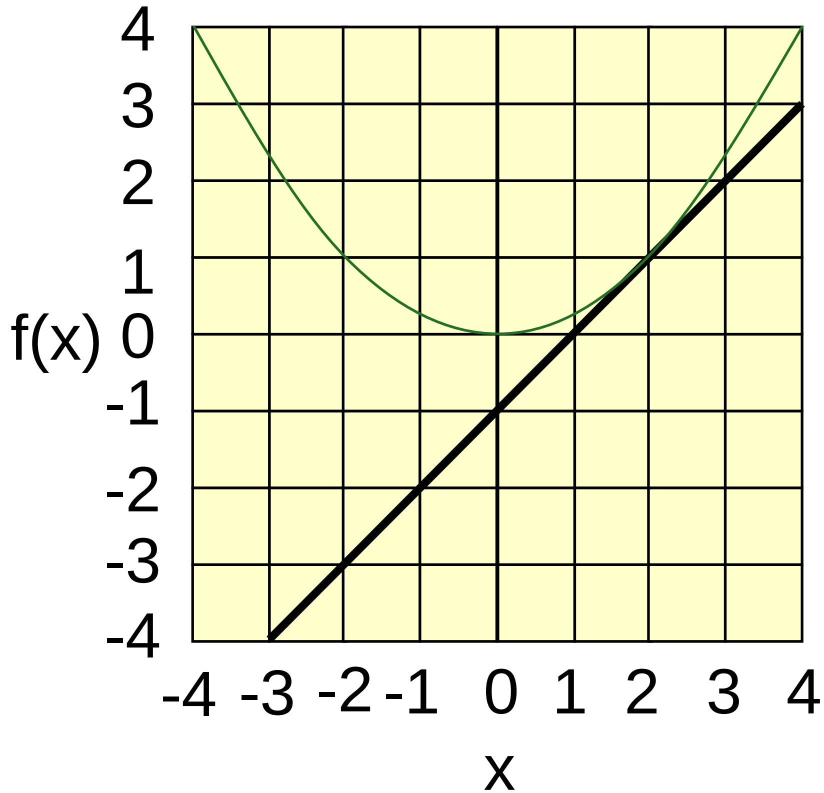
Objectives

- Know what definition of derivative is.
- Know what Power and Sum Rules are.
- Know what Product and Quotient Rules are.
- Know what Chain rule is.
- Know what High-Order derivatives are.
- Know what Implicit differentiation is.

What is a derivative?

- **The derivative $f'(x)$ of a function $f(x)$ says how fast $f(x)$ changes as x changes.**
- **Visually, $f'(x)$ is the slope of $f(x)$ at x.**

Example: If $f(x) = \frac{1}{4}x^2$ then $f'(2) = 1$ because the slope of $f(x)$ at $x = 2$ is 1. We can see this by looking at the tangent line to $f(x)$ at $x = 2$.



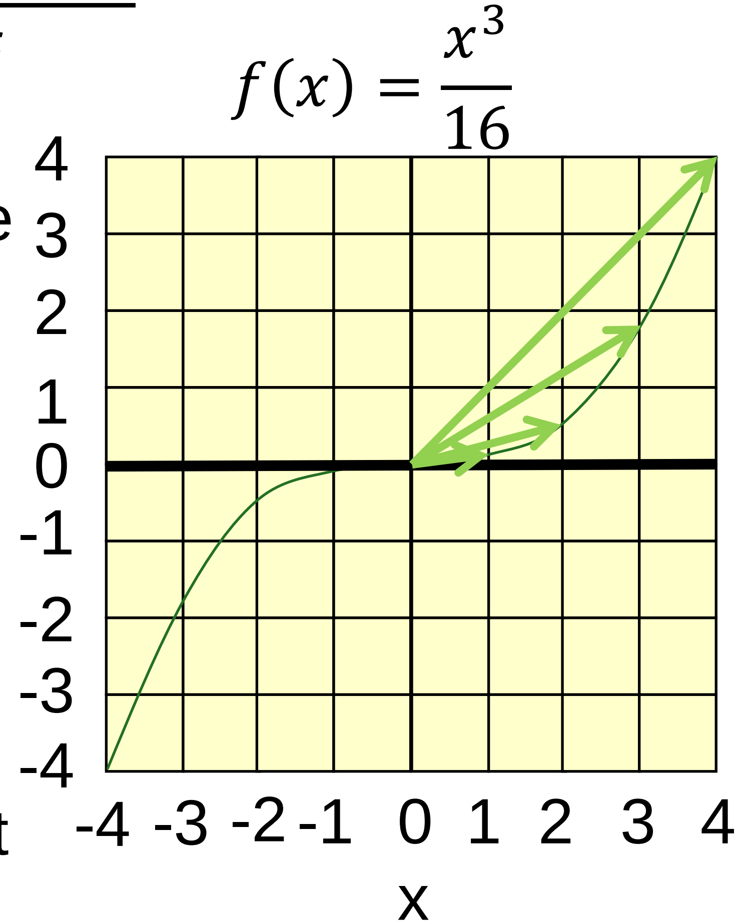
Why are derivatives useful?

- Tells us how quickly something is changing.
- In physics: velocity is the derivative of position and acceleration is the derivative of velocity (with respect to time).
- Optimization: Derivatives are crucial for finding the minimum or maximum of functions.
- And much much more.

Computing derivatives

- **To compute the slope of a line, we take $\frac{\Delta y}{\Delta x}$ (rise/run)**
- **We could try to do the same thing with a function, taking $\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$**

- Unfortunately, the slope of f(x) can change with x, so we get the average slope of f(x) over the interval $[x, x+\Delta x]$ rather than the exact slope of f(x) at x.
- However, if we make Δx smaller and smaller, the slope of f(x) varies less and less in $[x, x+\Delta x]$ and we get a better and better estimate.



Derivative Definition and Examples

- **We accomplish this by taking the limit as** $\Delta x \to 0$**.**
- **Definition:** $\mathrm{f}'(\mathrm{x}) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- **If f'(x) exists then we say that f is differentiable at x**

Example: If $f(x) = 3x + 4$ then

$$\mathrm{f}'(\mathrm{x}) = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)+4-(3x+4)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

Example: If $f(x) = x^2$ then

$$\mathrm{f}'(\mathrm{x}) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Differentiable Implies Continuous

- **Restatement of continuity: f is continuous at x if and only if $f(x)$ exists and $\lim_{\Delta x \to 0} \Delta f = 0$ where $\Delta f = f(x + \Delta x) - f(x)$.**
- **f is differentiable $\Leftrightarrow f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ exists**
- **If f is differentiable at x then**

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x = f'(x) \cdot 0 = 0$$

- **Thus, differentiability implies continuity**
- **Warning: The converse is false. Not all continuous functions are differentiable!**

Power Rule

- **For nonnegative integers n,** $(x+\Delta x)^n = \sum_{j=0}^{n} \binom{n}{j} (\Delta x)^j x^{n-j}$
- **Examples:**
- $(x+\Delta x)^2 = x^2 + 2(\Delta x)x + (\Delta x)^2$
- $(x+\Delta x)^3 = x^3 + 3(\Delta x)x^2 + 3(\Delta x)^2 x + (\Delta x)^3$
- $(x+\Delta x)^4 = x^4 + 4(\Delta x)x^3 + 6(\Delta x)^2 x^2 + 4(\Delta x)^3 x + (\Delta x)^4$
- $\frac{(x+\Delta x)^n - x^n}{\Delta x} = \frac{(x^n + n(\Delta x)x^{n-1} + (\Delta x)^2(\dots)) - x^n}{\Delta x} = nx^{n-1} + (\Delta x)(\dots)$
- $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} nx^{n-1} + (\Delta x)(\dots) = nx^{n-1}$
- **If** $f(x) = x^n$ **then** $f'(x) = nx^{n-1}$
- **This holds for all n, not just nonnegative integers! We'll prove this for rational numbers later using implicit differentiation.**

Derivative of $\sin(x)$

- $\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} = \frac{\sin(x)\cos(\Delta x)+\cos(x)\sin(\Delta x)-\sin(x)}{\Delta x}$
- $\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} = \sin(x)\frac{(\cos(\Delta x)-1)}{\Delta x} + \cos(x)\frac{\sin(\Delta x)}{\Delta x}$
- $\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} = \sin(x)\lim_{\Delta x\to 0}\frac{(\cos(\Delta x)-1)}{\Delta x} + \cos(x)\lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x}$

- **Recall that** $\lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x} = 1$
- **Recall that** $\lim_{\Delta x\to 0}\frac{(\cos(\Delta x)-1)}{\Delta x} = 0$
- $\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$
- **If** $f(x) = \sin(x)$ **then** $f'(x) = \cos(x)$

Derivative of $\cos(x)$

- **Following similar reasoning,**

if $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$

Derivatives of Sums and Differences

- $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d(f-g)}{dx} = \frac{df}{dx} - \frac{dg}{dx}$
- **This seems intuitive, but let's check the first equation to be sure.**
- **Take** $\Delta f = f(x + \Delta x) - f(x)$
- $\frac{d(f+g)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(f+g)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f + \Delta g}{\Delta x} =$
  $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = \frac{df}{dx} + \frac{dg}{dx}$

The Product Rule

- **What is $\frac{d(fg)}{dx}$?**
- **Warning:** $\frac{d(fg)}{dx} \neq \frac{df}{dx} \cdot \frac{dg}{dx}$
- $\Delta(fg) = (f + \Delta f)(g + \Delta g) - fg$
- $\Delta(fg) = f\Delta g + g\Delta f + \Delta f\Delta g$
- $\frac{d(fg)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f\Delta g + g\Delta f + \Delta f\Delta g}{\Delta x}$
- $\frac{d(fg)}{dx} = f \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} + g \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta f\Delta g}{\Delta x}$
- $\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$

The Quotient Rule

- **What is** $\frac{d\left(\frac{f}{g}\right)}{dx}$**?**
- **Warning:** $\frac{d\left(\frac{f}{g}\right)}{dx} \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}$
- $\Delta\left(\frac{f}{g}\right) = \frac{f+\Delta f}{g+\Delta g} - \frac{f}{g} = \frac{fg+g\Delta f-fg-f\Delta g}{g(g+\Delta g)} = \frac{g\Delta f-f\Delta g}{g(g+\Delta g)}$
- $\frac{d\left(\frac{f}{g}\right)}{dx} = \lim_{\Delta x\to 0}\frac{\Delta\left(\frac{f}{g}\right)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{g\Delta f-f\Delta g}{g(g+\Delta g)}}{\Delta x} = \frac{g\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}-f\lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x}}{\lim_{\Delta x\to 0} g(g+\Delta g)}$
- $\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$

The Chain Rule

- **What is** $\frac{d}{dx}(f(u))$ **where** u **is a function of** x**?**
- **Chain rule:** $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$
- **Example: If** $f(x) = \sqrt{1 + x^2}$ **then taking** $u = 1 + x^2$ **and** $f(u) = \sqrt{u}$**,**

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$

Chain Rule:

Consider a simple composite function:

$$y = 6x - 10$$

$$y = 2(3x - 5)$$

If $u = 3x - 5$

then $y = 2u$

$$y = 6x - 10 \qquad y = 2u \qquad u = 3x - 5$$

$$\frac{dy}{dx} = 6 \qquad \frac{dy}{du} = 2 \qquad \frac{du}{dx} = 3$$

$$6 = 2 \cdot 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

and another:

$$y = 5u - 2$$

where $u = 3t$

$$y = 5(3t) - 2 \qquad y = 5u - 2 \qquad u = 3t$$

$$y = 15t - 2$$

$$\frac{dy}{dt} = 15 \qquad \frac{dy}{du} = 5 \qquad \frac{du}{dt} = 3$$

$$15 = 5 \cdot 3$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

and one more:

$$y=(3x+1)^2$$

If $u=3x+1$

then $y=u^2$

$$y=9x^2+6x+1 \qquad y=u^2 \qquad u=3x+1$$

$$\frac{dy}{dx}=18x+6 \qquad \frac{dy}{du}=2u \qquad \frac{du}{dx}=3$$

$$\frac{dy}{du}=2(3x+1)$$

$$\frac{dy}{du}=6x+2$$

$$18x+6=(6x+2)\cdot 3$$

This pattern is called the <u>chain rule</u>.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

If $f \circ g$ is the composite of $y = f(u)$ and $u = g(x)$, then:

$$(f \circ g)' = f'_{\text{at } u=g(x)} \cdot g'_{\text{at } x}$$

$$f'(x) = \cos x \qquad g'(x) = 2x \qquad g(2) = 4 - 4 = 0$$

$$f'(0) \cdot g'(2)$$

$$\cos(0) \cdot (2 \cdot 2)$$

$$1 \cdot 4 = 4$$

We could also do it this way:

$$f(g(x)) = \sin(x^2 - 4)$$

$$y = \sin(x^2 - 4)$$

$$y = \sin u \qquad u = x^2 - 4$$

$$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \cdot 2x$$

$$\frac{dy}{dx} = \cos(x^2 - 4) \cdot 2x$$

$$\frac{dy}{dx} = \cos(2^2 - 4) \cdot 2 \cdot 2$$

$$\frac{dy}{dx} = \cos(0) \cdot 4$$

$$\frac{dy}{dx} = 4$$

Here is a faster way to find the derivative:

$$y = \sin\left(x^2 - 4\right)$$

$$y' = \cos\left(x^2 - 4\right) \cdot \frac{d}{dx}\left(x^2 - 4\right)$$

Differentiate the outside function...

$$y' = \cos\left(x^2 - 4\right) \cdot 2x$$

…then the inside function

$$\text{At } \ x = 2, \ \ y' = 4$$

Another example:

$$\frac{d}{dx}\cos^2(3x)$$

$$\frac{d}{dx}\left[\cos(3x)\right]^2$$

It looks like we need to use the chain rule again!

$$2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

derivative of the outside function

derivative of the inside function

Another example:

$$\frac{d}{dx}\cos^2(3x)$$

$$\frac{d}{dx}\left[\cos(3x)\right]^2$$

$$2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

$$2\cos(3x)\cdot -\sin(3x)\cdot\frac{d}{dx}(3x)$$

← The chain rule can be used more than once.

$$-2\cos(3x)\cdot\sin(3x)\cdot 3$$

(That’s what makes the “chain” in the “chain rule”!)

$$-6\cos(3x)\sin(3x)$$

Derivative formulas include the chain rule!

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx} \qquad \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

etcetera…

Every derivative problem could be thought of as a chain-rule problem:

$$\frac{d}{dx}x^2 = \underbrace{2x}\;\underbrace{\frac{d}{dx}x} = 2x \cdot 1 = 2x$$

derivative of outside function

derivative of inside function

The derivative of x is one.

Higher order derivatives

Do you remember your different notations for derivatives?

$f'(x)$ $\qquad y'$ $\qquad \dfrac{dy}{dx}$

Well these are the same notations for higher power derivatives! Any guesses on what each means?

$f''(x)$ — *the second derivative of* f

y''' — *the third derivative*

$\dfrac{d^2y}{dx^2}$ — *the second derivative*

Example

Find the fourth derivative of f(x) = $x^4 - 2x^3$

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x$$

$$f'''(x) = 24x - 12$$

$$f''''(x) = 24$$

Implicit Differentiation

- Consider an equation involving both x and y:

$$x^2 - y^2 = 49$$

- This equation implicitly defines a function in x
- It could be defined explicitly

$$y = \sqrt{x^2 - 49} \qquad (where\ |x| \geq 7)$$

Differentiate

- **Differentiate both sides of the equation**
  - **each term**
  - **one at a time**
  - **use the chain rule for terms containing y**
- **For** $x^2 - y^2 = 49$ **we get**
- **Now solve for dy/dx** $2x - 2y\dfrac{dy}{dx} = 0$

Differentiate

- Then $2x - 2y\frac{dy}{dx} = 0$ gives us

$$\frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

- We can replace the y in the results with the explicit value of y as needed
- This gives us the slope on the curve for any legal value of x

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 49}}$$

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx by dividing both sides of the equation by the left-hand factor that does not contain dy/dx.

Slope of a Tangent Line

- Given $x^3 + y^3 = y + 21$
  find the slope of the tangent at (3,-2)
- $3x^2 + 3y^2y' = y'$
- Solve for y'

$$y' = \frac{3x^2}{1-3y^2}$$

- Substitute x = 3, y = -2

$$slope = \frac{27}{-11}$$

Second Derivative

- **Given $x^2 - y^2 = 49$**

- **y' =??**

$$y' = \frac{x}{y}$$

- **y" =**

Substitute

$$\frac{d^2y}{dx^2} = \frac{y - x \cdot y'}{y^2}$$