Limits and Continuity

Objectives

- Know what left limits, right limits, and limits are.
- Know how to compute simple limits.
- Know what it means for a function to be continuous.
- Know what is the L Hopital′s rule.

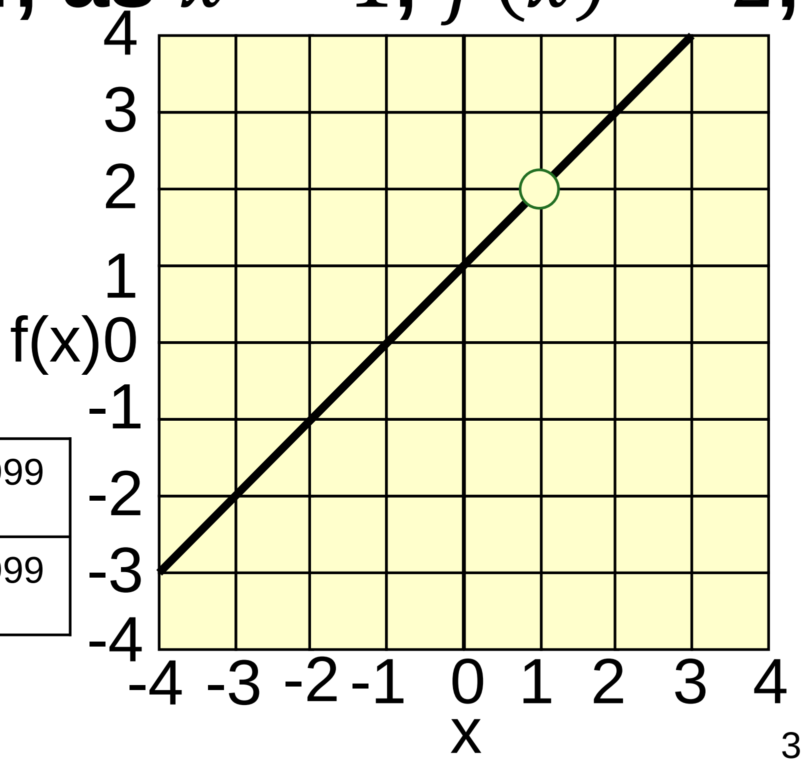
What is a limit?

- **A limit is what happens when you get closer and closer to a point without actually reaching it.**
- **Example: If** $f(x) = 2x$ **then as** $x \to 1$**,** $f(x) \to 2$**.**
- **We write this as** $\lim_{x \to 1} f(x) = 2$**.**

x	0	.9	.99	.999	.9999
f(x)	0	1.8	1.98	1.998	1.9998

Why are limits useful?

- Many functions are not defined at a poir but are well-behaved nearby.
- Example: If $f(x) = \frac{x^2-1}{x-1}$ then $f(1)$ is undefined. However, as $x \to 1$, $f(x) \to 2$, so $\lim_{x \to 1} f(x) = 2$.



x	0	.9	.99	.999	.9999
f(x)	0	1.9	1.99	1.999	1.999

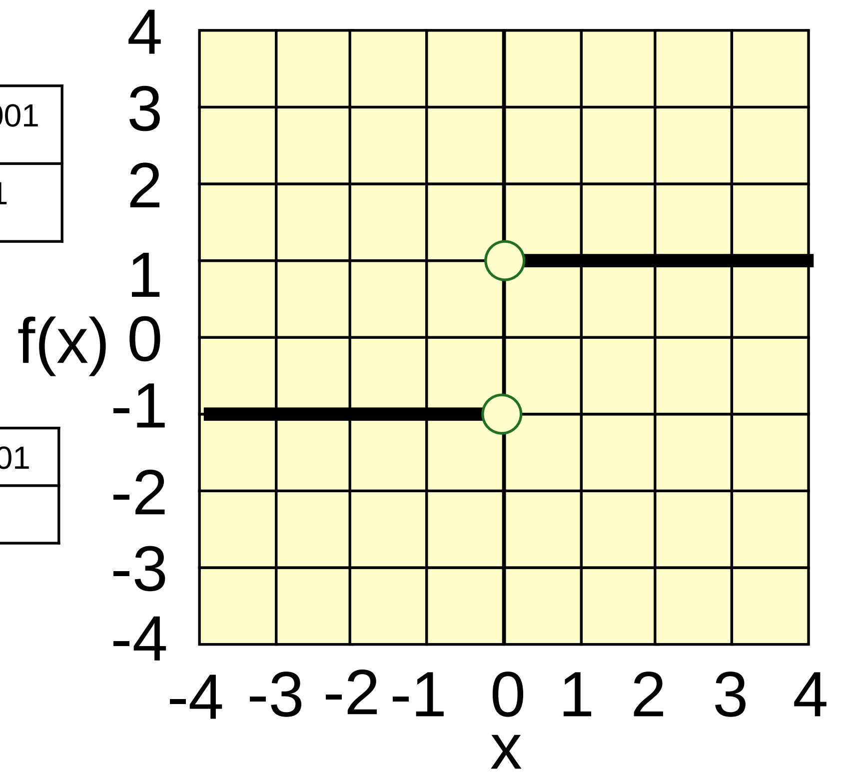
Left Limits and Right Limits

Consider $f(x) = \frac{x}{|x|}$. $f(0)$ is undefined. As $x \to 0^-, f(x) = -1$

x	-1	-.1	-.01	-.001	-.0001
f(x)	-1	-1	-1	-1	-1

As $x \to 0^+, f(x) = 1$

x	1	.1	.01	.001	.0001
f(x)	1	1	1	1	1



We write this as $\lim_{x \to 0^-} f(x) = -1, \lim_{x \to 0^+} f(x) = 1$

Limit Definition Summary

- **We say that** $\lim\limits_{x \to a^-} f(x) = L$ **if** $f(x) \to L$ **as** $x \to a^-$
- **We say that** $\lim\limits_{x \to a^+} f(x) = L$ **if** $f(x) \to L$ **as** $x \to a^+$
- **If** $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$ **(i.e. it doesn't matter which side x approaches a from then we say that** $\lim\limits_{x \to a} f(x) = L$

Absence of Limits

- **Limits can fail to exist in several ways**

1. $\lim_{x\to a^-} f(x)$ **or** $\lim_{x\to a^+} f(x)$ **may not exist.**

- Example: $\sin\left(\frac{1}{x}\right)$ **oscillates rapidly between 0 and 1 as** $x \to 0^+$ **(or** 0^-**). Thus,** $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ **DNE (does not exist)**

- Example: $\frac{1}{x}$ **gets larger and larger as** $x \to 0^+$**. We write this as** $\lim_{x\to 0^+} \frac{1}{x} = \infty$

2. $\lim_{x\to a^-} f(x)$ **and** $\lim_{x\to a^+} f(x)$ **may both exist but have different values. Ex:** $f(x) = \frac{x}{|x|}$ **near** $x = 0$

Computing Limits

- **To compute** $\lim_{x \to a} f(x)$**:**
- **If nothing special happens at** $x = a$**, just compute** $f(a)$**. Example:** $\lim_{x \to 2}(3x - 1) = 5$
- **If plugging in** $x = a$ **gives** $\frac{0}{0}$**, factors can often be cancelled when** $x \neq a$**.**

Example:

$$\lim_{x \to 2}\left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2}\left(\frac{(x-2)(x+2)}{x - 2}\right) = \lim_{x \to 2}(x + 2) = \mathbf{4}$$

Computing Limits Continued

- **Useful trick:** $a - b = (a - b) \cdot \frac{a+b}{a+b} = \frac{a^2 - b^2}{a+b}$
- **Example: What is** $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$**?**

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+1} + 1)} = \lim_{x \to 0} \frac{1}{(\sqrt{x+1} + 1)} = \frac{1}{2}$$

Limits at Infinity

- **We can also consider what happens when $x \to \infty$ or $x \to -\infty$. Example: Consider $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$. As $\mathrm{x} \to \infty$ (or $-\infty$), $f(x) \to 1$. We write this as**
$$\lim_{x\to\infty} \frac{x-1}{x} = 1$$

Computing Limits at $\pm\infty$

- General strategy : figure out the largest terms and ignore everything else
- Example: If $f(x) = \frac{3x^2 - x}{4x^2 + 2x - 5}$, as $x \to \infty$ only the $3x^2$ in the numerator and the $4x^2$ will really matter, so $\lim_{x \to \infty} f(x) = \frac{3}{4}$

Limit Laws

- **If** $\lim_{x \to a} f(x) = \mathrm{L}$ **and** $\lim_{x \to a} g(x) = M$ **then:**
- $\lim_{x \to a} (f(x) + g(x)) = \mathrm{L} + \mathrm{M}$
- $\lim_{x \to a} (f(x) - g(x)) = \mathrm{L} - \mathrm{M}$
- $\lim_{x \to a} (f(x) g(x)) = \mathrm{LM}$
- $\lim_{x \to a} (\frac{f(x)}{g(x)}) = \frac{L}{M}$ (if $M \neq 0$)
- **Etc.**

Continuity

- **Definition:** $f(x)$ **is continuous at a if both** $f(a)$ and $\lim_{x \to a} f(x)$ exist and are equal.
- **Note: Polynomials are always continuous everywhere. Most functions we will be working with are continuous almost everywhere.**

Discontinuous functions

$f(x)$ may fail to be continuous at $x = a$ because:

1. $\lim_{x \to a} f(x)$ or $f(a)$ does not exist.

- **Example: If $f(x) = \lfloor x \rfloor$ then $\lim_{x \to 0} f(x)$ does not exist.**
- **Example: If $f(x) = \frac{x^2-1}{x-1}$ then $f(1)$ is undefined.**

2. $\lim_{x \to a} f(x)$ or $f(a)$ both exist but have different values.

- **Example: If $f(x) = \lceil x \rceil - \lfloor x \rfloor$ then $\lim_{x \to 1} f(x)$ = 1 but $f(1) = 0$**

L Hopital's rule

Johann Bernoulli
1667 - 1748

Consider: $$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Zero divided by zero can not be evaluated, and is an example of **indeterminate form**.

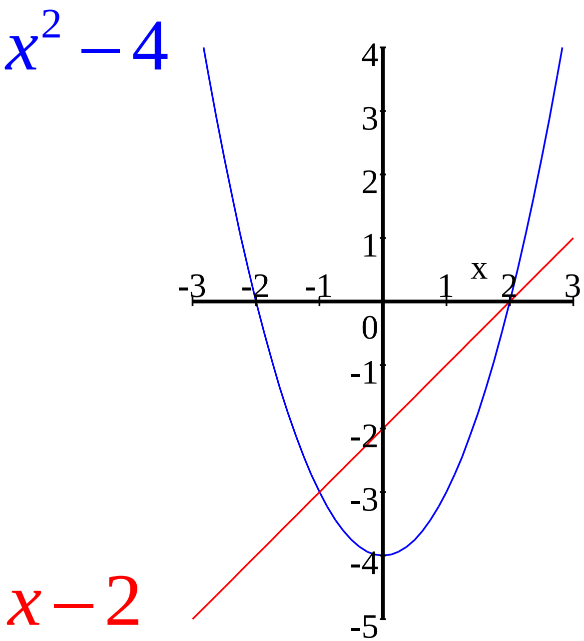
If we try to evaluate this by direct substitution, we get: $\frac{0}{0}$

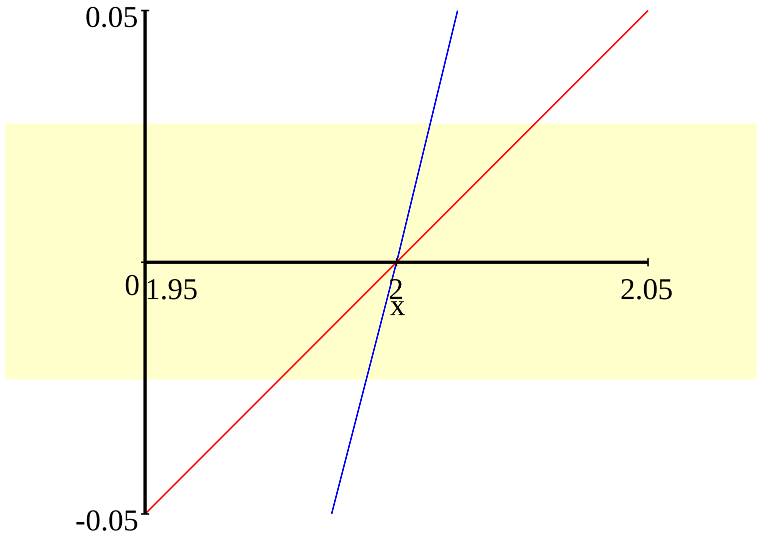
In this case, we can evaluate this limit by factoring and canceling:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2} (x+2) = 4$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} \qquad = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

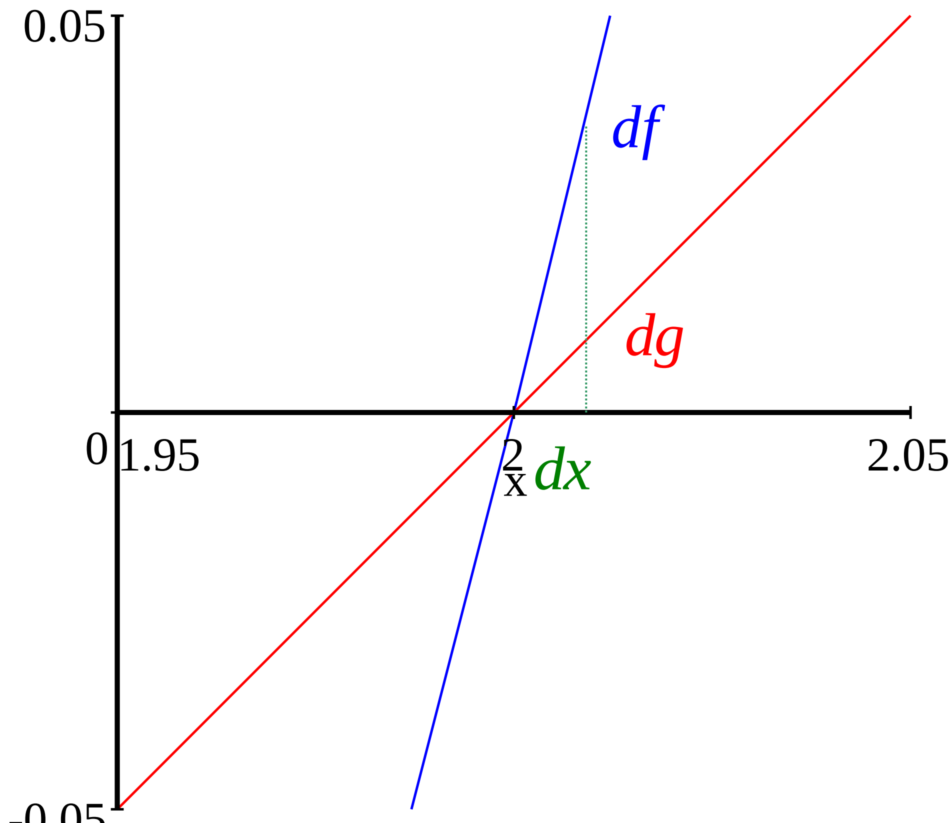
The limit is the ratio of the numerator over the denominator as x approaches 2.





If we zoom in far enough, the curves will appear as straight lines.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2}$$



As $x \to 2$

$\dfrac{f(x)}{g(x)}$ becomes:

$$\frac{df}{dg} = \frac{\dfrac{df}{dx}}{\dfrac{dg}{dx}}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}\left(x^2-4\right)}{\frac{d}{dx}\left(x-2\right)}=\lim_{x\to 2}\frac{2x}{1}\quad=4$$

L’Hôpital’s Rule:

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} \quad = \lim_{x\to 0}\frac{\sin x}{1+2x} \quad = 0$$

If it’s no longer indeterminate, then **STOP**!

~~If we try to continue with L’Hôpital’s rule:~~

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} \quad = \lim_{x\to 0}\frac{\cos x}{2} \quad = \frac{1}{2}$$

which is wrong, wrong, wrong!

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \longleftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \longleftarrow \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \longleftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}=-\frac{1}{8}$$

L'Hôpital's rule can be used to evaluate other indeterminate forms besides $\frac{0}{0}$.

The following are also considered indeterminate:

$$\frac{\infty}{\infty} \qquad \infty \cdot 0 \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

The first one, $\frac{\infty}{\infty}$, can be evaluated just like $\frac{0}{0}$.

<u>The others must be changed to fractions first</u>.

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$ ← This approaches $\infty\cdot 0$

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$ ← This approaches $\frac{0}{0}$

We already know that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$

but if we want to use L’Hôpital’s rule:

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)$$

$$=\cos(0)=1$$

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$ ← This is indeterminate form $\infty-\infty$

If we find a common denominator and subtract, we get:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$ ← Now it is in the form $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$ ← L'Hôpital's rule applied once.

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$ ← Fractions cleared. Still $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$

$$\lim_{x\to 1}\left(\frac{1}{1+1+\ln x}\right)$$

$$\frac{1}{2}$$