Functions

Objectives

- Know what the Domain and Range of function are.
- Know what Implicit functions is.
- Know how to graph the function.
- Know what the Symmetry of Function is.
- Know what the Transformation of Function is.

Domain and Range

- **To specify a function f you must:**

a) give a rule which tells you how to compute the value *f*(x) of the function for a given real number x.

b) say for which real numbers x the rule may be applied.

- **The set of numbers for which a function is defined is called *domain of the function*.**
- **The set of all possible numbers *f*(x) as x runs over the domain is called the *range of the function*.**

Example: Find the domain and range of $f(x) = {}^{1}/_{x^2}$

Solution:

- The expression ${}^{1}/_{x^2}$ can be computed for all real numbers *x* except *x* = 0 since this leads to division by zero. Hence *the domain* of the function *f(x)* = ${}^{1}/_{x^2}$ is:

"all real numbers except 0" = *{x | x ≠ 0}= (- ∞,0) U (0,∞)*

- If $y = {}^{1}/_{x^2}$ then we must have $x^2 = {}^{1}/_{y}$, so first of all, since we have to divide by y, y can't be zero. Furthermore, ${}^{1}/_{y} = x^2$ says that y must be positive. On the other hand, if y > 0 then $y = {}^{1}/_{x^2}$ has a solution (in fact two solutions), namely $x = \mp {}^{1}/_{\sqrt{y}}$

This shows that *the range* of f is:

"all positive real numbers" = *{x | x > 0}= (0,∞)*

- For instance, one can define a function *f* by putting *f*(x) =$\sqrt{x}$ for all x ≥ 0. Here the rule defining *f* is take the square root of whatever number you're given, and the function *f* will accept all nonnegative real numbers.
- The rule which species a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & if\ x \geq 0 \\ x^2 & if\ x < 0 \end{cases}$$ domain of g = all real numbers.

- Functions which are defined by different formulas on different intervals are sometimes called *piecewise defined functions*.

Implicit functions

- **Implicit functions:** For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an “implicit function”.

- **Example**: One can dene a function *f* by saying that for each *x* the value of *f(x)* is the solution *y* of the equation

$$x^2 + 2y - 3 = 0$$

In this example you can solve the equation for *y*,

$$y = \frac{3 - x^2}{2}$$

Thus we see that the function we have defined is:

$$f(x) = \frac{3 - x^2}{2}$$

Here we have two definitions of the same function, namely:

i. *y = f(x)* is defined by $x^2 + 2y - 3 = 0$.

ii. *y = f(x)* is defined by $f(x) = \frac{3-x^2}{2}$.

- The first definition is the implicit definition, the second is explicit. You see that with an \implicit function"
- it isn't the function itself, but rather the way it was defined that's implicit.

Type of functions:

1) **Linear functions:** $y = mx + b$

2) **Power functions:** $y = x^a$

3) **Polynomial Functions:** $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$

4) **Trigonometric functions: sin, cos, tan, sec, csc & cot.**

5) **Exponential functions:** $y = a^x$

6) **Logarithmic functions:** $y = \log_a x$

7) **Rational functions: it is the ratio of two polynomial,** $f(x) = \frac{g(x)}{q(x)}$

Symmetry of Functions

- ***Even and odd functions:***

- **If *f(-x)= f(x)* *then, the function is even. It is symmetry about the y-axis.***

- **If *f(-x)= -f(x)* *then, the function is odd. It is symmetry about the origin.***

Example: check the symmetry of function:

$$\boldsymbol{f(x) = x}$$

Solution:

$$\boldsymbol{f(-x) = -x}$$
$$\boldsymbol{f(-x) \neq f(x)}$$
$$\boldsymbol{f(-x) = -f(x)}$$

So the function is odd, it has symmetry about the origin.

A function is <u>even</u> if $f(-x) = f(x)$ for every number x in the domain.

If you plug a **-x** into the function and you get the original function back again, the function is even.

$$\text{Given } f(x) = 3x^2 - 10$$

$$f(-x) = 3(-x)^2 - 10$$
$$= 3x^2 - 10$$

Same!

Even Function

A function is <u>even</u> if $f(-x) = f(x)$ for every number x in the domain.

If you plug a **−x** into the function and you get the original function back again, the function is even.

$$f(x) = \boxed{5x^4 - 2x^2 + 1}$$

YES

$$f(-x) = 5(-x)^4 - 2(-x)^2 + 1 = \boxed{5x^4 - 2x^2 + 1}$$

$$f(x) = \boxed{2x^3 - x}$$

Is this function even?

NO

$$f(-x) = 2(-x)^3 - (-x) = \boxed{-2x^3 + x}$$

A function is <u>odd</u> if $f(-x) = -f(x)$ for every number x in the domain.

If you plug a **−x** into the function and you get the negative of the function back (all terms change signs), the function is odd.

$$\text{Given } f(x) = 4x^3 - 5x$$

$$f(-x) = 4(-x)^3 - 5(-x)$$
$$= -4x^3 + 5x$$

ALL signs of the terms changed!

Odd Function

A function is <u>odd</u> if $f(-x) = -f(x)$ for every number x in the domain.

If you plug a **−x** into the function and you get the negative of the function back (all terms change signs), the function is odd.

$$f(x) = \boxed{5x^4 - 2x^2 + 1}$$

Is this function odd?

NO

$$f(-x) = 5(-x)^4 - 2(-x)^2 + 1 = \boxed{5x^4 - 2x^2 + 1}$$

$$f(x) = \boxed{2x^3 - x}$$

Is this function odd?

YES

$$f(-x) = 2(-x)^3 - (-x) = \boxed{-2x^3 + x}$$

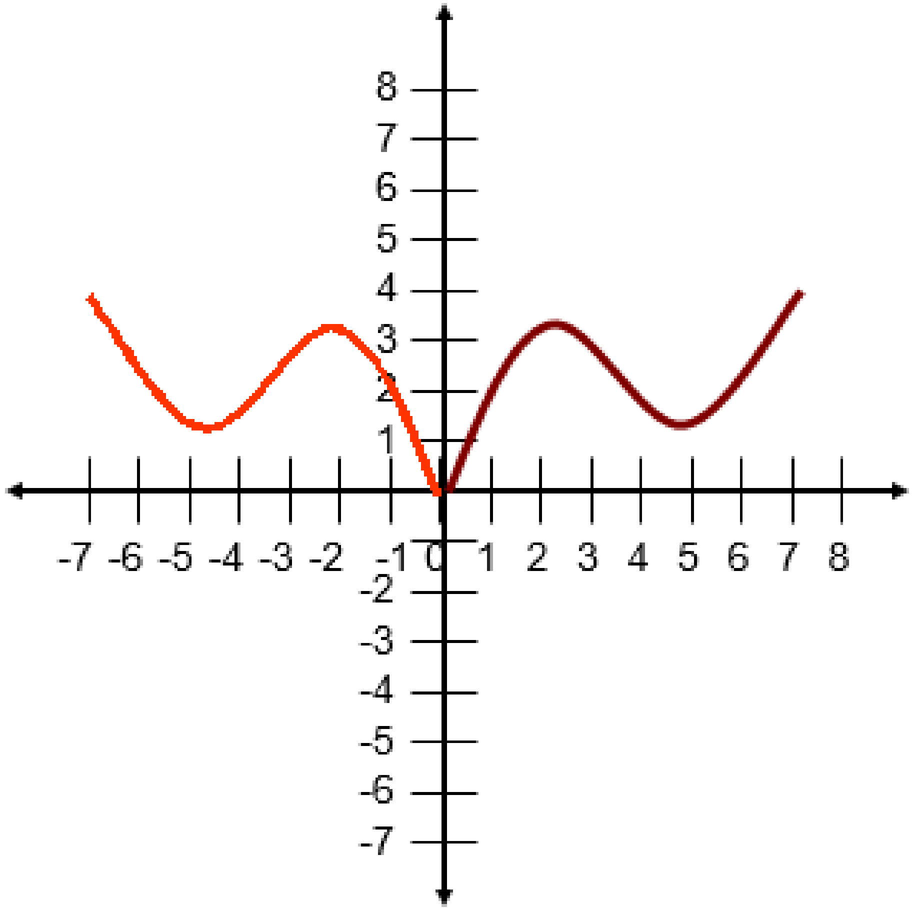
Odd, Even, or Neither?

$$f(x) = 4x^3 + x^2 - 7$$

$$f(-x) = 4(-x)^3 + (-x)^2 - 7$$
$$= -4x^3 + x^2 - 7$$

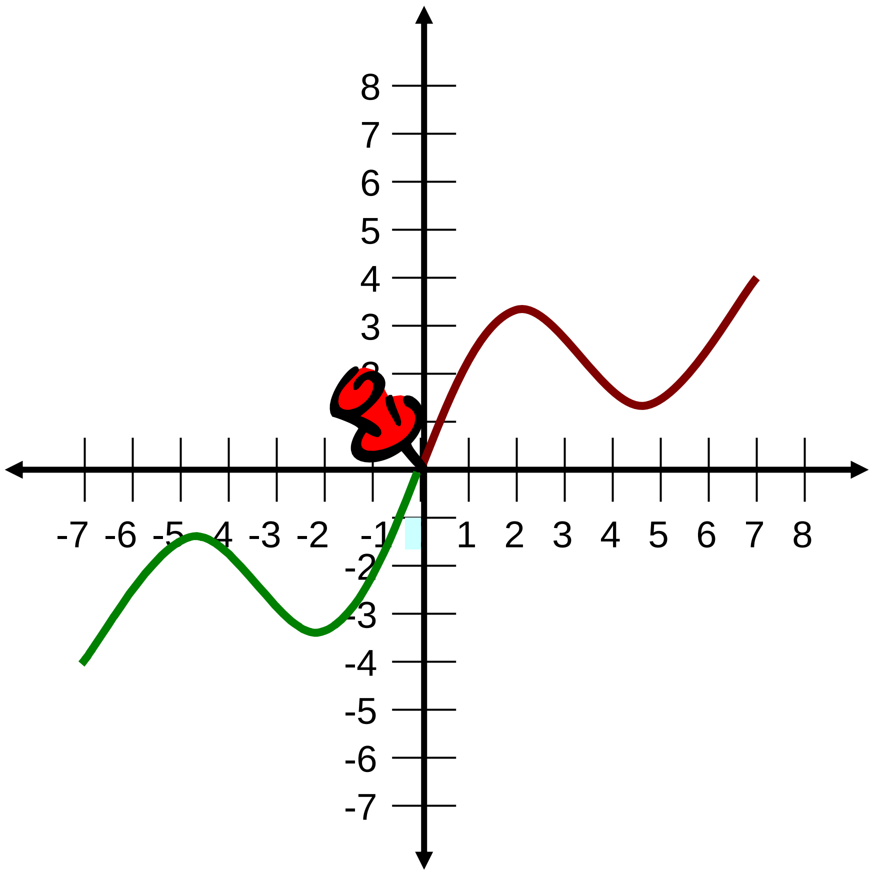
Neither Odd or Even

Even functions have y-axis Symmetry



For an even function: for every point (x, y) on the graph, the point (-x, y) is also on the graph.

Odd functions have origin Symmetry



For an odd function: for every point (x, y) on the graph, the point (-x, -y) is also on the graph.

Transformation of Function

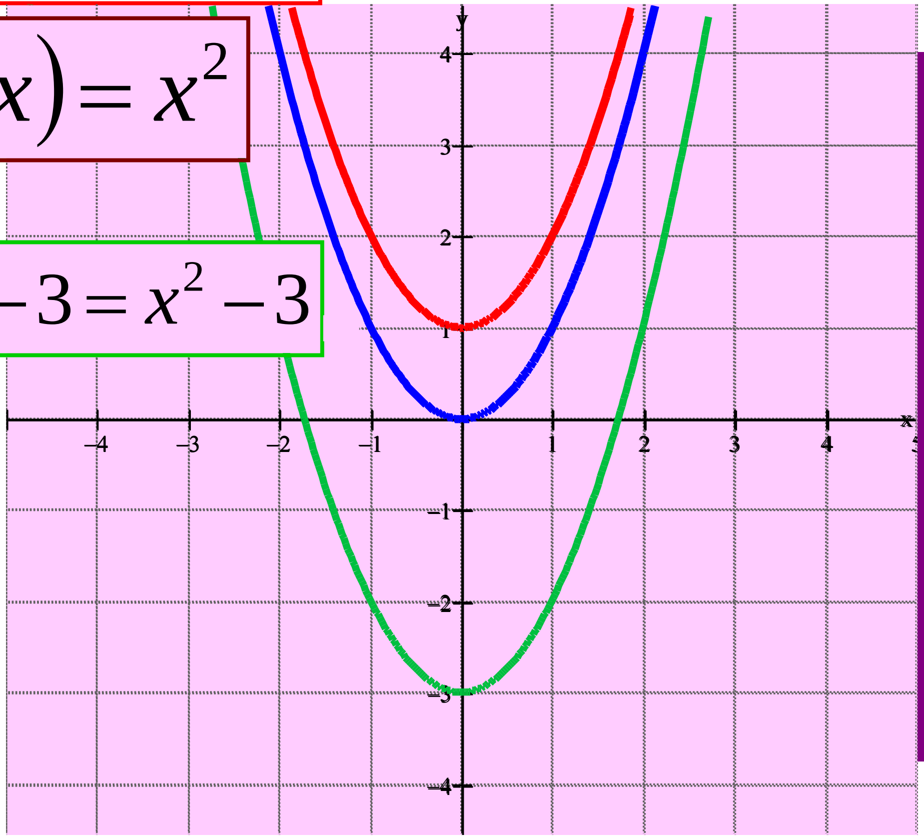
We will be looking at simple functions and seeing how various modifications to the functions transform them.

VERTICAL TRANSLATIONS

$$f(x)+1=x^2+1$$

$$f(x)=x^2$$

$$f(x)-3=x^2-3$$



As you can see, a number added or subtracted from a function will cause a vertical shift or translation in the function.

Above is the graph of $f(x)=x^2$

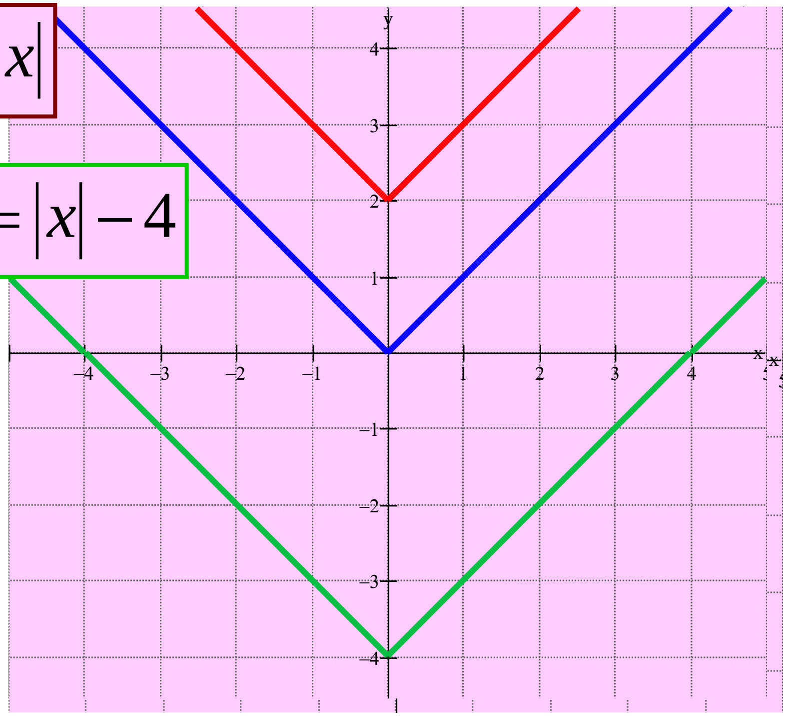
What would $f(x)$ - 3 look like? (This would mean taking all the function values and subtracting 3 from them).

VERTICAL TRANSLATIONS

$f(x)+2=|x|+2$

$f(x)=|x|$

$f(x)-4=|x|-4$



So the graph *f*(*x*) + *k*, where *k* is any real number is the graph of *f*(*x*) but vertically shifted by *k*. If *k is* positive it will shift up. If *k* is negative it will shift down

Above is the graph of $f(x)=|x|$

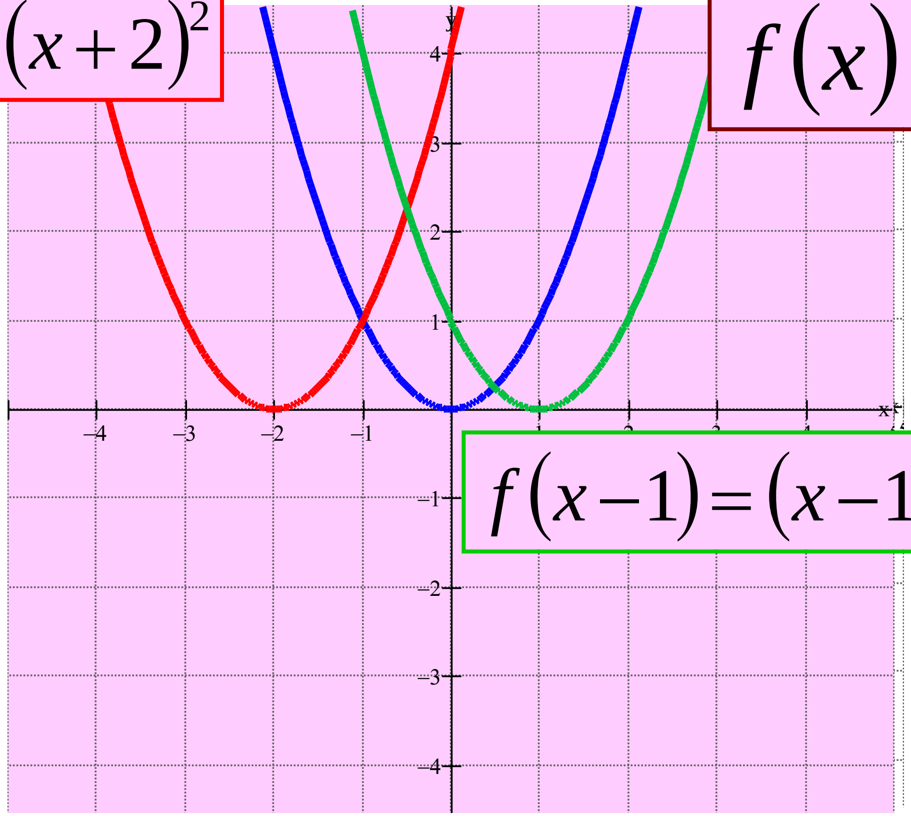
What would *f*(*x*) + 2 look like?

What would *f*(*x*) - 4 look like?

HORIZONTAL TRANSLATIONS

$$f(x+2)=(x+2)^2$$

$$f(x)=x^2$$



$$f(x-1)=(x-1)^2$$

As you can see, a number added or subtracted from the *x* will cause a horizontal shift or translation in the function but opposite way of the sign of the number.

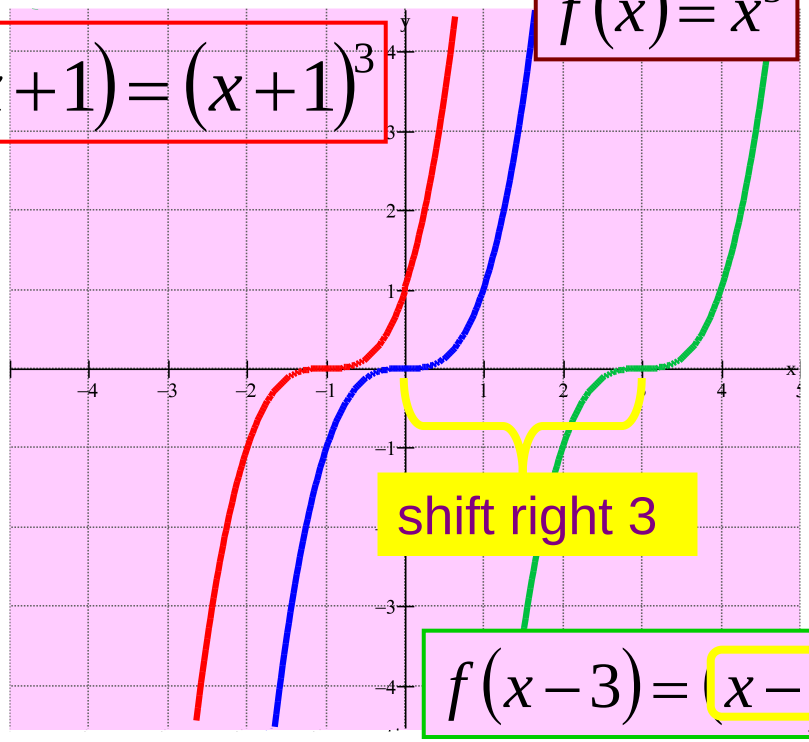
Above is the graph of $f(x)=x^2$

What would *f*(*x*-1) look like? (This would mean taking all the *x* values and subtracting 1 from them before putting them in the function).

HORIZONTAL TRANSLATIONS

$f(x+1)=(x+1)^3$

$f(x)=x^3$



$f(x-3)=\boxed{x-3}^3$

So the graph ***f*(*x*-*h*), where *h* is any real number is the graph of *f*(*x*) but horizontally shifted by *h*.**

Notice the negative.

(If you set the stuff in parenthesis = 0 & solve it will tell you how to shift along *x* axis).

$x-3=0 \quad x=3$

So shift along the *x*-axis by 3

Above is the graph of $f(x)=x^3$

What would $f(x+1)$ look like?

What would $f(x-3)$ look like?

Summary of Transformations So Far

Do reflections and dilations BEFORE vertical and horizontal translations

If $a > 1$, then vertical dilation or stretch by a factor of a

If $0 < a < 1$, then vertical dilation or compression by a factor of a

If $a < 0$, then reflection about the x-axis
(as well as being dilated by a factor of a)

vertical translation of k

$$a\ f(x-h)+k$$

$f(-x)$ reflection about y-axis

horizontal translation of h
(opposite sign of number with the x)

ASYMPTOTES

Horizontal
Vertical
Slant

Definition of an asymptote

- An asymptote is a straight line which acts as a boundary for the graph of a function.
- When a function has an asymptote (and not all functions have them) the function gets closer and closer to the asymptote as the input value to the function approaches either a specific value *a* or positive or negative infinity.
- The functions most likely to have asymptotes are rational functions

Vertical Asymptotes

Vertical asymptotes occur when the following condition is met:

> **The denominator of the *simplified* rational function is equal to 0. Remember, the simplified rational function has cancelled any factors common to both the numerator and denominator.**

Finding Vertical Asymptotes Example 1

Given the function $$f(x)=\frac{2-5x}{2+2x}$$

The first step is to cancel any factors common to both numerator and denominator. In this case there are none.

The second step is to see where the denominator of the simplified function equals 0.

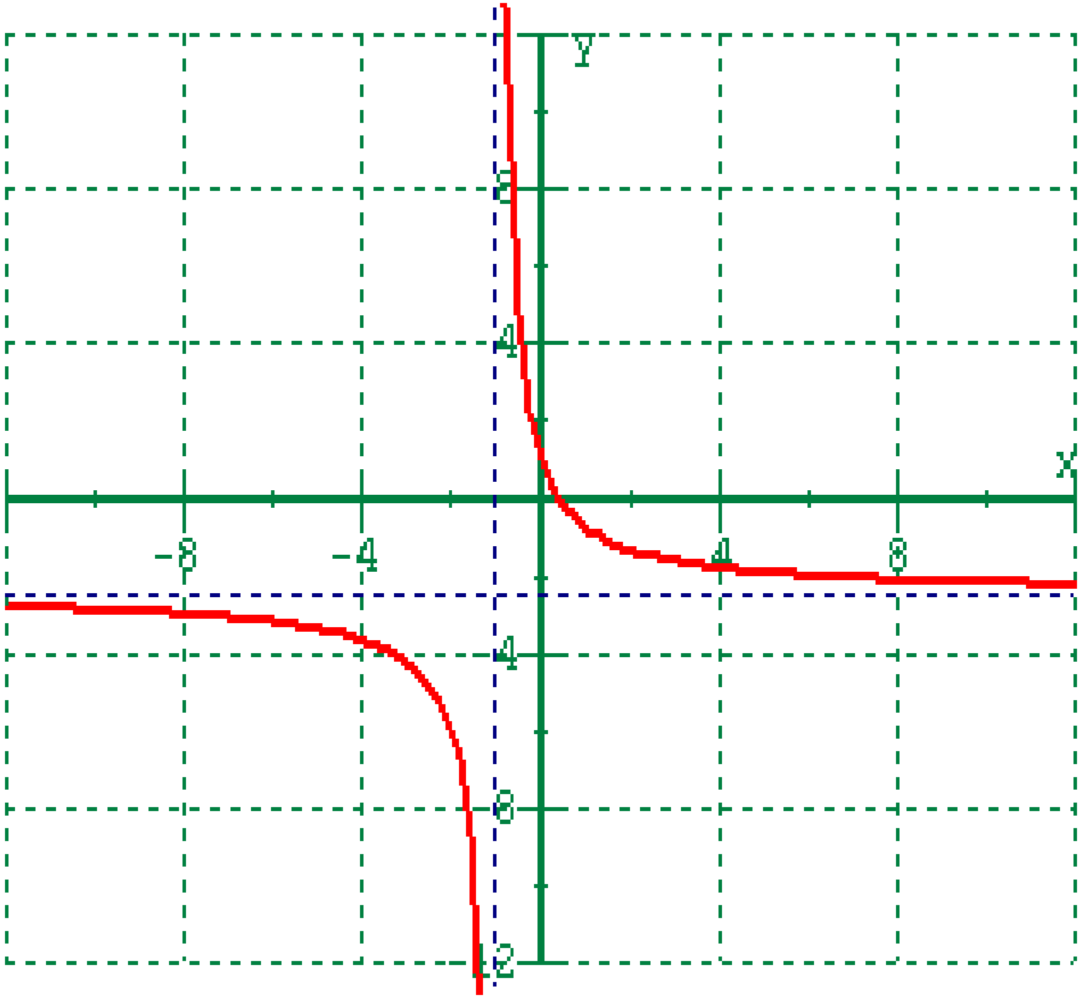
$$2+2x=0$$
$$2(1+x)=0$$
$$1+x=0$$
$$x=-1$$

Finding Vertical Asymptotes Example 1 Con't.

The vertical line $x = -1$ is the only vertical asymptote for the function. As the input value x to this function gets closer and closer to -1 the function itself looks and acts more and more like the vertical line

$x = -1$.

Graph of Example 1



The vertical dotted line at $x = -1$ is the vertical asymptote.

Finding Vertical Asymptotes Example 2

If

$$f\left(x\right)=\frac{2x^2+10x+12}{x^2-9}$$

First simplify the function. Factor both numerator and denominator and cancel any common factors.

$$\frac{2x^2+10x+12}{x^2-9}=\frac{\cancel{\left(x+3\right)}\left(2x+4\right)}{\cancel{\left(x+3\right)}\left(x-3\right)}$$

$$=\frac{2x+4}{x-3}$$

Finding Vertical Asymptotes Example 2 Con’t.

The asymptote(s) occur where the simplified denominator equals 0.

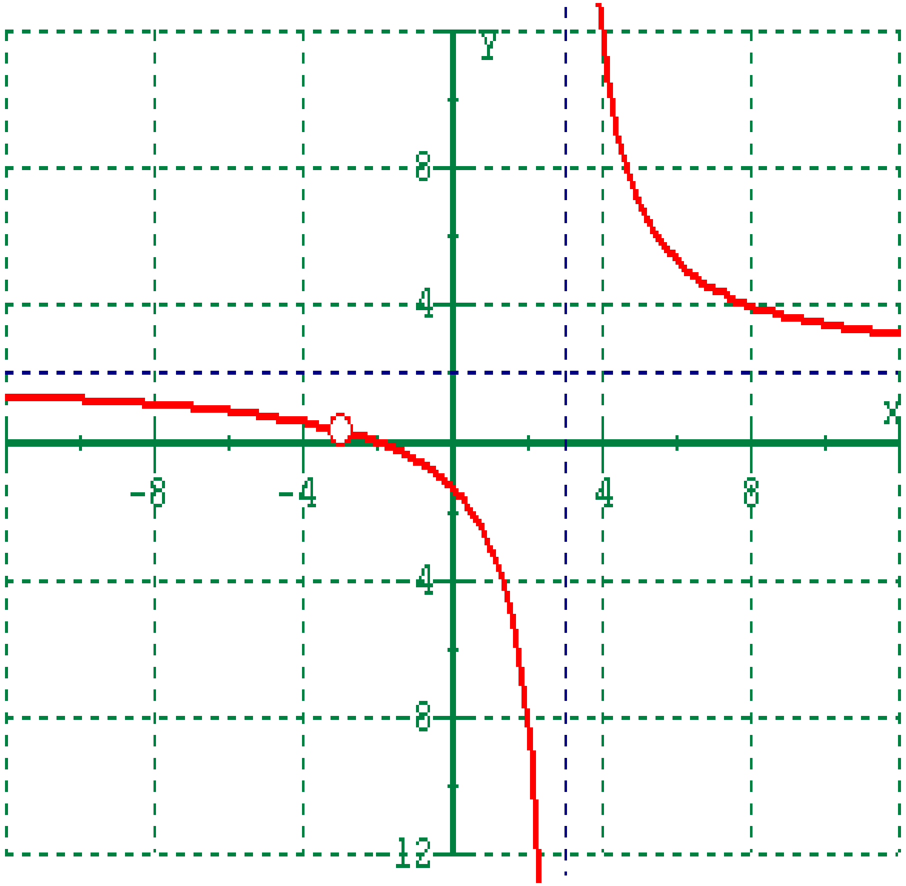
$$x - 3 = 0$$

The vertical line x =3 is the only vertical asymptote for this function.

$$x = 3$$

As the input value x to this function gets closer and closer to 3 the function itself looks more and more like the vertical line x =3.

Graph of Example 2



The vertical dotted line at $x = 3$ is the vertical asymptote

Finding Vertical Asymptotes Example 3

If $$g(x)=\frac{x-5}{x^2-x-6}$$

Factor both the numerator and denominator and cancel any common factors.
In this case there are no common factors to cancel.

$$\frac{x-5}{x^2-x-6}=\frac{x-5}{(x+2)(x-3)}$$

Finding Vertical Asymptotes Example 3 Con't.

The denominator equals zero whenever either

$$x + 2 = 0$$

$$x = -2$$

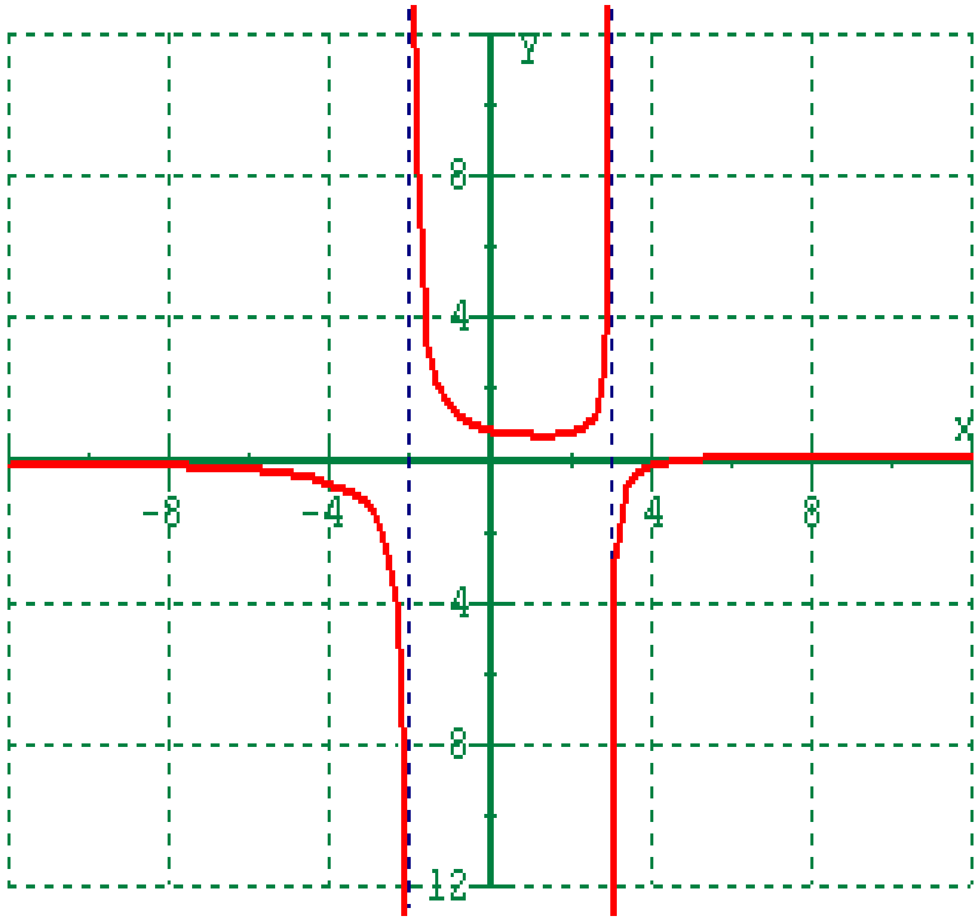
or

$$x - 3 = 0$$

$$x = 3$$

This function has two vertical asymptotes, one at $x = -2$ and the other at $x = 3$

Graph of Example 3



The two vertical dotted lines at x = -2 and x = 3 are the vertical asymptotes

Horizontal Asymptotes

Horizontal asymptotes occur when *either one* of the following conditions is met (you should notice that both conditions *cannot* be true for the same function).

The degree of the numerator is *less than* the degree of the denominator. In this case the asymptote is the horizontal line $y = 0$.

The degree of the numerator is *equal to* the degree of the denominator. In this case the asymptote is the horizontal line $y = {}^{a}/_{b}$ where a is the leading coefficient in the numerator and b is the leading coefficient in the denominator.

When the degree of the numerator is *greater than* the degree of the denominator there is no horizontal asymptote

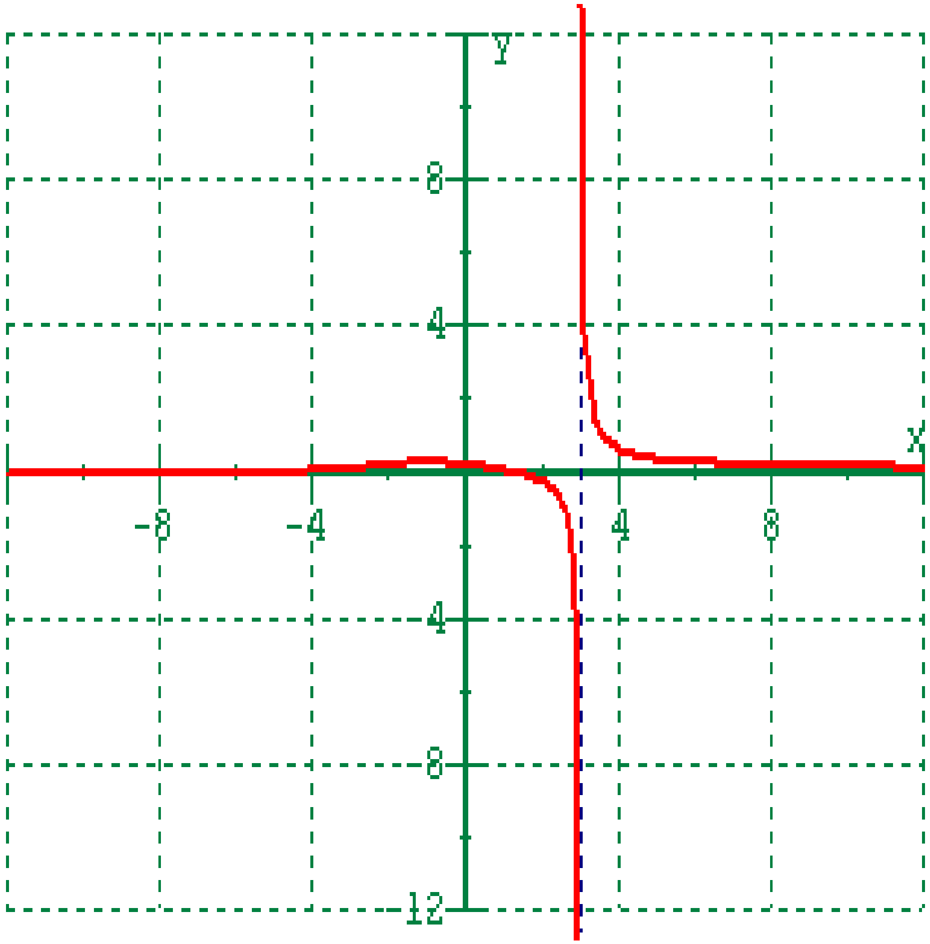
Finding Horizontal Asymptotes
Example 4

If $$f\left(x\right)=\frac{x^2+3x-5}{x^3-27}$$

then there is a horizontal asymptote at the line y=0 because the degree of the numerator (2) is *less than* the degree of the denominator (3). This means that as x gets larger and larger in both the positive and negative directions ($x \rightarrow \infty$ and $x \rightarrow -\infty$)

the function itself looks more and more like the horizontal line $y = 0$

Graph of Example 4



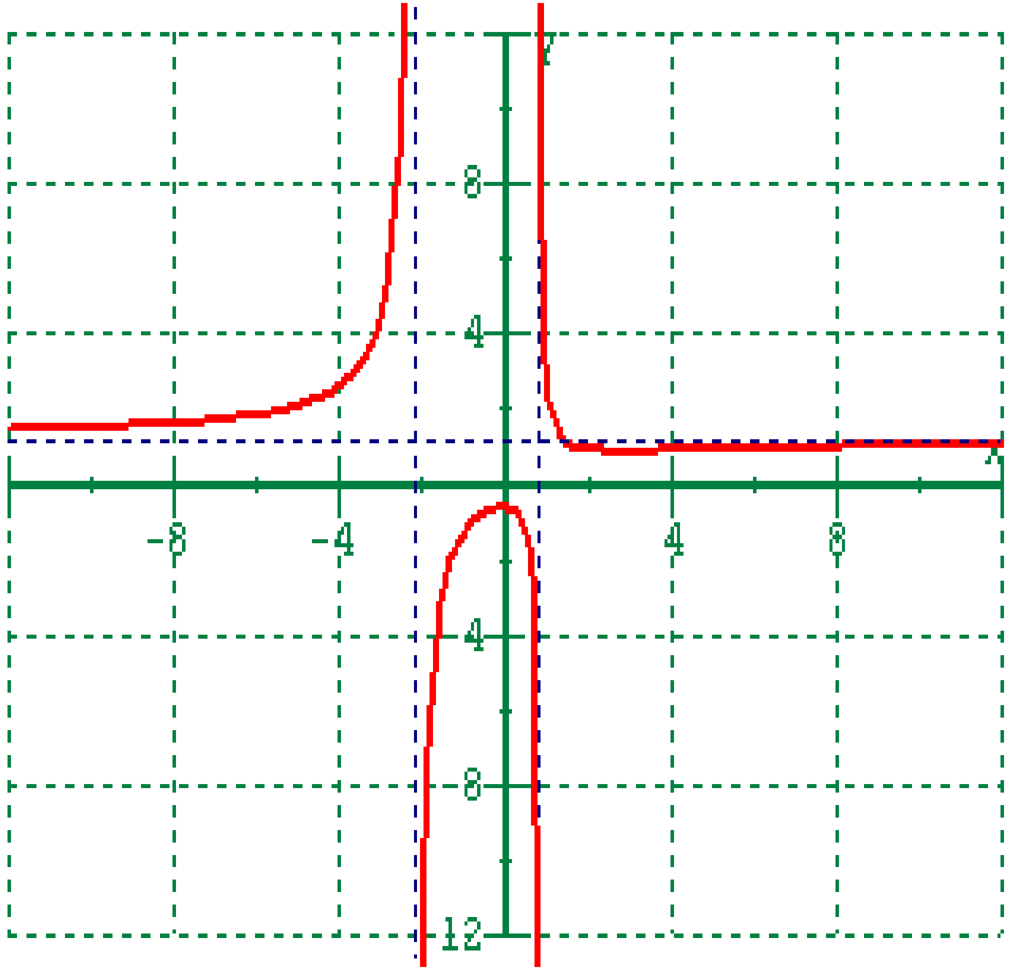
The horizontal line $y = 0$ is the horizontal asymptote.

Finding Horizontal Asymptotes Example 5

If $$g(x)=\frac{6x^2-3x+5}{5x^2+7x-9}$$

then because the degree of the numerator (2) is *equal to* the degree of the denominator (2) there is a horizontal asymptote at the line $y={}^{6}/_{5}$. Note, 6 is the leading coefficient of the numerator and 5 is the leading coefficient of the denominator. As $x\to\infty$ and as $x\to-\infty$ $g(x)$ looks more and more like the line $y={}^{6}/_{5}$

Graph of Example 5



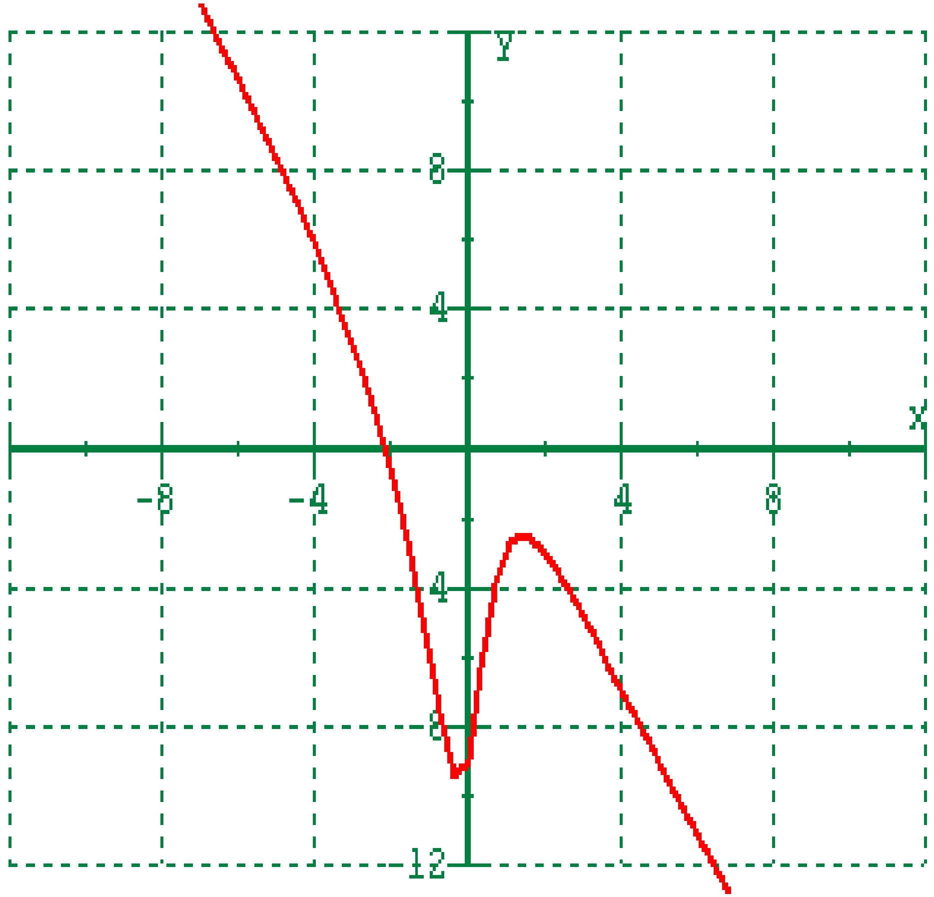
The horizontal dotted line at $y = {}^{6}/_{5}$ is the horizontal asymptote.

Finding Horizontal Asymptotes Example 6

If $$f(x)=\frac{-2x^3+5x-9}{x^2+1}$$

There are *no* horizontal asymptotes because the degree of the numerator is *greater than* the degree of the denominator.

Graph of Example 6



Slant Asymptotes

- Slant asymptotes occur when the degree of the numerator is ***exactly one bigger*** than the degree of the denominator. In this case a slanted line (not horizontal and not vertical) is the function's asymptote.
- To find the equation of the asymptote we need to use long division – dividing the numerator by the denominator.

Finding a Slant Asymptote
Example 7

- **If** $$f(x)=\frac{x^3+2x^2+5x-9}{x^2-x+1}$$
- **There will be a slant asymptote because the degree of the numerator (3) is one bigger than the degree of the denominator (2).**
- **Using long division, divide the numerator by the denominator.**

Finding a Slant Asymptote
Example 7 Con't.

$$
\begin{array}{r}
x+3 \\
x^2-x+1\overline{\smash{\big)}\,x^3+2x^2+5x-9} \\
\underline{-\left(x^3-x^2+x\right)} \\
3x^2+4x-9 \\
\underline{-\left(3x^2-3x+3\right)} \\
7x-12
\end{array}
$$

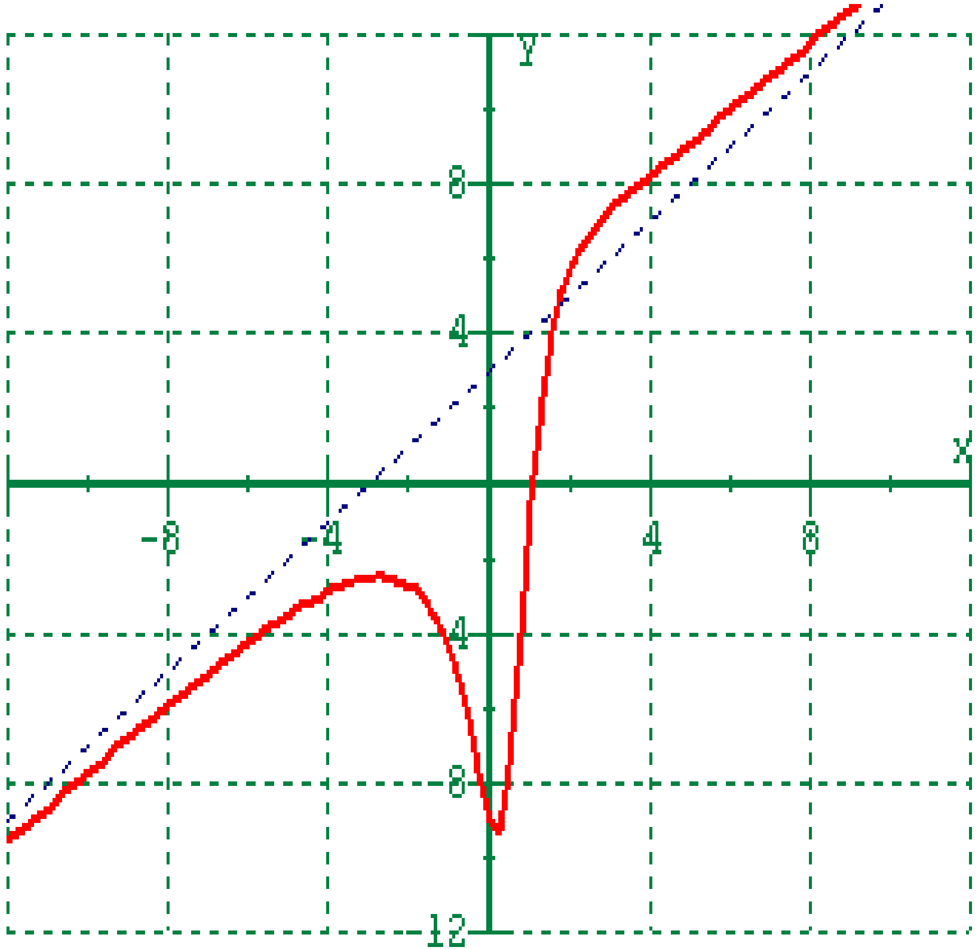
Finding a Slant Asymptote Example 7 Con't.

We can ignore the remainder $7x-12$

The answer we are looking for is the quotient $x+3$

and the equation of the slant asymptote is $y=x+3$

Graph of Example 7



The slanted line $y = x + 3$ is the slant asymptote

Problems

Find the vertical asymptotes, horizontal asymptotes, slant asymptotes for each of the following functions.

$$f\left(x\right)=\frac{x^2+2x-15}{x^2+7x+10}$$

Vertical: x = -2
Horizontal : $y = 1$
Slant: none

$$g\left(x\right)=\frac{2x^2+5x-7}{x-3}$$

Vertical: $x = 3$
Horizontal : none
Slant: $y = 2x + 11$