Algebraic Preliminaries

Objectives

- Know what the Numbers are.
- Know what Set notation is.
- Know what the Inequalities are.
- Know what the Absolute value is.

Numbers

- **What is a number?**

There are Different kinds of numbers. The simplest numbers are the *positive integers*

1, 2, 3, 4,….

the number *zero*

0

and the *negative integers*

……,-1,-2,-3,-4

Together these form the integers or whole numbers.

- **there are the numbers you get by dividing one whole number by another (nonzero) whole number.**

These are the so called *fractions* or *rational numbers* such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, \ldots\ldots\ldots$$

Or

$$\ldots\ldots\ldots, -\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{3}{4}, -\frac{4}{3}$$

One can represent certain fractions as *decimal fractions*

$$\frac{279}{25} = \frac{1116}{100} = 11.16$$

Not all fractions can be represented as decimal fractions. For instance, expanding 1/3 into a decimal fraction leads to an *unending decimal fraction*

0.3333333333333………

- ***A real number*** **is specified by a possibly unending decimal expansion. For instance,**

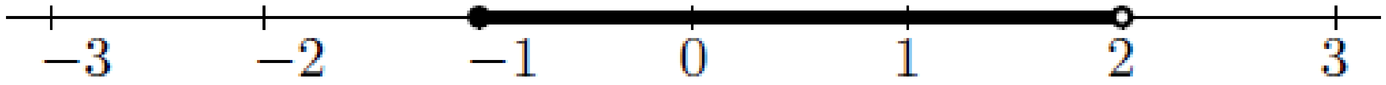
$\sqrt{2}$=1.414 213 562 373 095 048 801 688 724 209 698 078 569…….

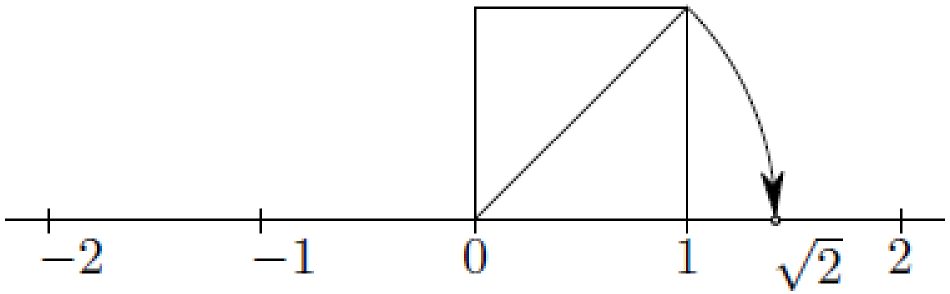
- ***An imagined number*** **is a number whose square is -1, no real number has this property since the square of any real number is positive.**

- ***The real number line and intervals***

It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the origin. We also decide which direction we call (left) and hence which we call (right).

- To plot any real number x one marks o a distance x from the origin, to the right if $x > 0$, to the left if $x < 0$.
- The distance along the number line between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.





- Almost every equation involving variables x, y, etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a set. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

The collection of all real numbers between two given real numbers form an interval. The following notation is used

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \le x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \le b$.
- $[a, b]$ is the set of all real numbers x which satisfy $a \le x \le b$.
- $(-\infty, b]$ is the set of all real numbers x which satisfy $x \le b$.

Set notation

- ***Set notation*** **is a common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation**

A = {x | x satisfies this or that condition}

- Most of the time we will use upper case letters to denote sets. (A,B,C,D, . . .).

For instance, the interval (a, b) can be described as

$$(a, b) = \{x \mid a < x < b\}$$

The set

$$B = \{x \mid x^2 - 1 > 0\}$$

The set consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval (- ∞,-1) and the interval (1, ∞).

Or the set

$$E = \{x \mid x^2 - 1 = 0\}$$

which consists of the solutions of the equation $x^2 - 1 = 0$ (There are three of them) $x=1$ and $x=-1$.

- If A and B are two sets then the union of A and B is the set which contains all numbers that belong either to A or to B. The following notation is used:

$$A \cup B = x \mid x \text{ belongs to } A \text{ or to } B \text{ or both}$$

- Similarly, the intersection of two sets A and B is the set of numbers which belong to both sets. This notation is used:

$$A \cap B = x \mid x \text{ belongs to both } A \text{ and } B$$

Exercises

- Draw the following sets of real numbers. Each of these sets is the union of one or more intervals. Find those intervals. Which of thee sets are finite?

- $A = \{x \mid x^2 - 3x + 2 \leq 0\}$
- $B = \{x \mid x^2 - 3x + 2 \geq 0\}$
- $C = \{x \mid x^2 - 3x > 3\}$
- $D = \{x \mid x^2 - 5 > 2x\}$
- $E = \{t \mid t^2 - 3t + 2 \leq 0\}$
- $F = (0, 1) \cup (5, 7]$
- $G = \{x \mid \sin x = 0.5\}$
- $H = \{\Phi \mid \cos \Phi > 0\}$

Inequalities

- Inequalities: **if a, b & c are real numbers, then**

- $a < b$ → $a+c < b+c$
- $a < b$ → $a-c < b-c$
- $a < b$ & $c > 0$ → $a*c < b*c$
- $a < b$ & $c < 0$ → $a*c > b*c$
- $a < b$ → $-a > -b$
- $a > 0$ → $1/a > 0$
- **If a & b are both positive or negative numbers, then**

$a < b$ → $1/a > 1/b$

Example: Solve the following inequalities and show their solution set.

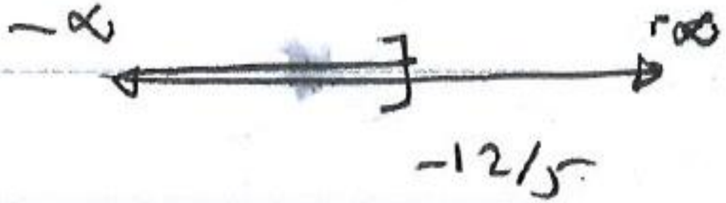
1- $3 + 7x \leq 2x - 9$

Sol. $3 + 7x - 3 \leq 2x - 9 - 3 \Rightarrow 7x \leq 2x - 12$

$\Rightarrow 7x - 2x \leq 2x - 12 - 2x \Rightarrow 5x \leq -12$

$\Rightarrow x \leq -12/5$

The solution set is $(-\infty, -12/5]$



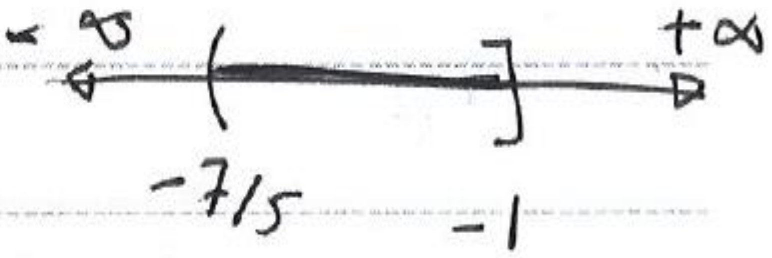
2. $7 \le 2-5x < 9$

Sol. $7-2 \le 2-5x-2 < 9-2 \Rightarrow 5 \le -5x < 7$

$\Rightarrow 5/(-5) \le -5x/(-5) < 7/(-5)$

$\Rightarrow -1 \ge x > -7/5 \Rightarrow -1 \le x \le -7/5$

The solution set is $(-7/5, -1]$ $\quad -7/5 < x \le -1$



Absolute value:

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

- Absolute value properties,

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. $|a + b| \leq |a| + |b|$

If a is any positive number, then

1. $|x| = a \quad \text{if and only if} \quad x = \mp a$
2. $|x| < a \quad \text{if and only if} \quad -a < x < a$
3. $|x| > a \quad \text{if and only if} \quad x > a \text{ or } x < -a$
4. $|x| \leq a \quad \text{if and only if} \quad -a \leq x \leq a$
5. $|x| \geq a \quad \text{if and only if} \quad x \geq a \text{ or } x \leq -a$

Examples :

1. Solve the equation $|2x-3|=1$

Sol.: $2x-3=\pm 1 \Rightarrow$ either $2x-3=1 \Rightarrow 2x=4 \Rightarrow x=2$

or $2x-3=-1 \Rightarrow 2x=2 \Rightarrow x=1$

$\therefore$ The solutions are $x=1$ and $x=2$.

2. Solve the inequality $|2x-3|\leq 1$

Sol. $|2x-3|\leq 1 \Rightarrow -1\leq 2x-3\leq 1$

$\Rightarrow 2\leq 2x\leq 4 \Rightarrow 1\leq x\leq 2$

$\therefore$ The solution set is the closed interval $[1,2]$.