Chapter One

Vector Analysis

CE311

Third Year Class

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1.1 Scalars and Vectors.

A scalar is a quantity that has only magnitude. Quantities such as time, mass, distance, temperature, entropy, electric potential and population are scalars. Symbolically, a scalar is represented by either lower or upper case letters.

A vector is described by two quantities: a magnitude and a direction in space at any point and for any given time. Therefore, vectors may be space and time dependent. Vector quantities include velocity, force, displacement and electric field intensity.

Graphically, a vector is represented by directed line segment in the direction of the vector with its length proportional to its magnitude. Symbolically, a vector is represented by placing a bar over the letter symbol used for a given quantity, such as $\bar{A}\ and\ \bar{B}$, or by a letter in boldface type such as $\boldsymbol{A}$ and $\boldsymbol{B}$.

1.2 Vector Addition and Subtraction.

Two vectors $\bar{A}\ and\ \bar{B}$ can be added (subtracted) together to give another vector $\bar{C}$ ($\bar{D}$); i.e., $\bar{C} = \bar{A} + \bar{B}$; $\bar{D} = \bar{A} - \bar{B} = \bar{A} + (-\bar{B})$.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head to tail rule as portrayed in **Fig. 1.1** and **1.2**, respectively.

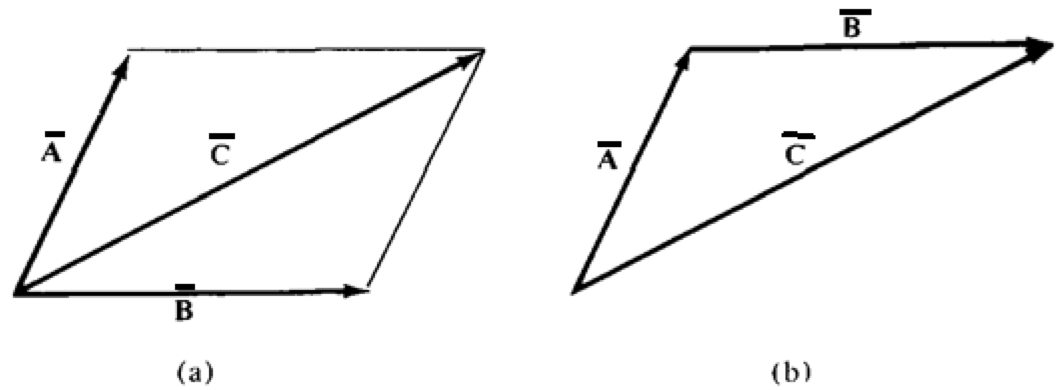


Fig. 1.1 Vector addition $\bar{C} = \bar{A} + \bar{B}$: (a) parallelogram rule, (b) head to tail rule

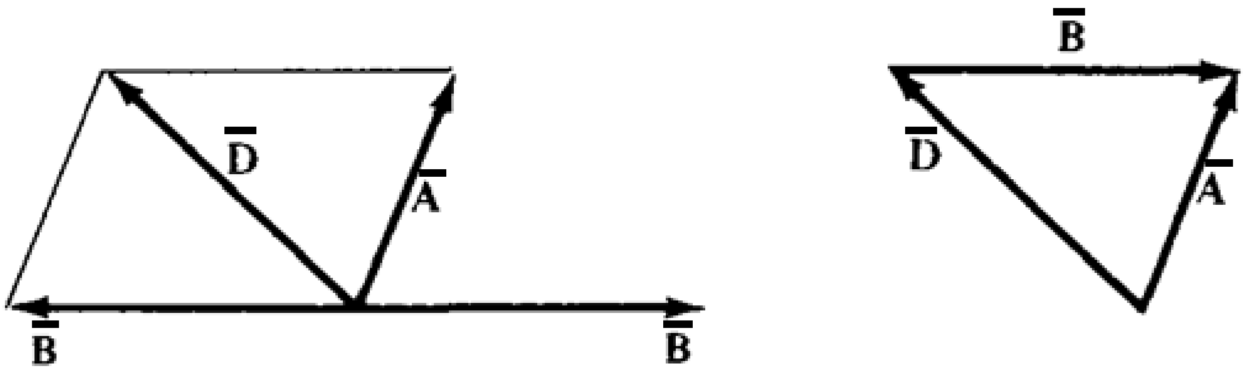


Fig. 1.2 Vector subtraction $\bar{D} = \bar{A} - \bar{B}$: (a) parallelogram rule, (b) head-to-tail rule.

The three basic laws of algebra obeyed by any given vectors $\bar{A}, \bar{B}$ *and* $\bar{C}$ are summarized as follows:

Law	**Addition**	**Multiplication**
Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$k\,\bar{A} = \bar{A}\ k$
Associative	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$k\,(L\bar{A}) = (kL)\,\bar{A}$
Distributive	$k\,(\bar{A} + \bar{B}) = k\bar{A} + k\bar{B}$	

Where k and L are scalars.

1.3 Products of Vectors

The multiplication of two vectors is called a product. two types of products based on the result obtained from the product. The first type is the scalar product. This is a product of two vectors which results in a scalar. The second is a vector product of two vectors, which results in a vector.

1.3.1 The Dot Product:

The dot product of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A}\,.\,\bar{B}$, is defined geometrically as the product of the magnitude of $\bar{A}$ and $\bar{B}$ and the cosine of the smaller angle between them.

Thus:

$\bar{A}\,.\,\bar{B} = |\bar{A}||\bar{B}|\,\cos\theta_{AB}$

If $\bar{A} = A_x\,\bar{a}x + A_y\,\,\bar{a}y + A_z\,\bar{a}z$ and $\bar{B} = B_x\,\bar{a}x + B_y\,\,\bar{a}y + B_z\,\bar{a}z$, then:

$\bar{A}\,.\bar{B} = A_x\,B_x + A_y\,B_y + A_z\,B_z$

Notes:

1- $\bar{A}\,.\,\bar{B} = \bar{B}\,.\,\bar{A}$ (Commutative Law)

2- $\bar{A}\,.(\bar{B} + \bar{C}) = \bar{A}.\bar{B} + \bar{A}.\bar{C}$ (Distributive Law)

3- $\bar{A}\,.\,\bar{A} = |\bar{A}|^2$

4- $\bar{a}x.\bar{a}y = \bar{a}y.\bar{a}z = \bar{a}x.\bar{a}z = 0$ and $\bar{a}x.\bar{a}x = \bar{a}y.\bar{a}y = \bar{a}z.\bar{a}z = 1$

A direct application of dot product is its use in determining the projection (or Component) of a vector in a given direction. The projection can be scalar or vector. Given a vector $\bar{A}$, we define the scalar projection A_B of $\bar{A}$ along $\bar{B}$ as [see **Fig. 1.3a**]

$A_B = |\bar{A}| \cos\theta_{AB} = |\bar{A}||\bar{a}_B| \cos\theta_{AB}$

Or $A_B = \bar{A} \, . \, \bar{a}_B$

The vector projection $\bar{A}_B$ of $\bar{A}$ along $\bar{B}$ is simply the scalar projection A_B multiplied by a unit vector along $\bar{B}$; is:

$\bar{A}_B = A_B \, \bar{a}_B = (\bar{A} \, . \, \bar{a}_B) \, \bar{a}_B$

Both the scalar and vector projections of $\bar{A}$ are illustrated in **Fig. 1.3**.

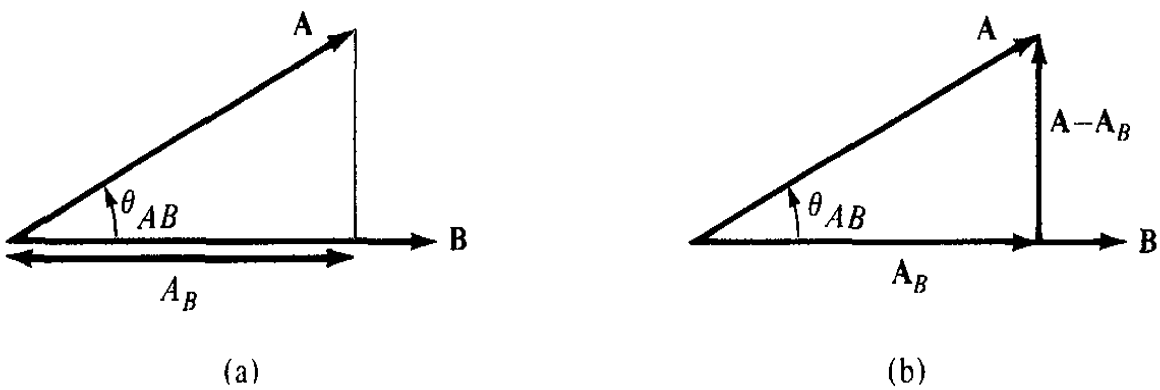


Fig. 1.3 Components of $\bar{A}$ along $\bar{B}$: **(a)** scalar component A_B ; **(b)** vector component $\bar{A}_B$.

Example 5: -

Given vectors $\bar{A} = 3\bar{a}x + 4\bar{a}y + \bar{a}z$ and $\bar{B} = 2\bar{a}y - 5\bar{a}z$. Find: (a) $\bar{A} \, . \, \bar{B}$; (b) θ_{AB} ; (c) The scalar component of $\bar{A}$ along $\bar{B}$; (d) The vector projection of $\bar{A}$ along $\bar{B}$.

Solution:

(a) $\bar{A} \, . \, \bar{B} = (3\bar{a}x + 4\bar{a}y + \bar{a}z).(2\bar{a}y - 5\bar{a}z) = 3(0) + 4(2) + 1(-5) = 3$

(b) $|\bar{A}| = \sqrt{9 + 16 + 1} = \sqrt{26}$ and $|\bar{B}| = \sqrt{0 + 4 + 25} = \sqrt{29}$

$$\bar{A}\,.\,\bar{B} = |\bar{A}||\bar{B}|\,\cos\theta_{AB} \implies \cos\theta_{AB} = \frac{\bar{A}\,.\,\bar{B}}{|\bar{A}||\bar{B}|} = \frac{3}{\sqrt{26}\sqrt{29}} = 0.1092$$

$$\therefore\ \theta_{AB} = \cos^{-1}(0.1092) = 83.73°$$

(c) $A_B = \bar{A}\,.\,\bar{a}_B = \dfrac{\bar{A}\,.\,\bar{B}}{|\bar{B}|} = \dfrac{3}{\sqrt{29}} = 0.557$

(d) $\bar{A}_B = (\bar{A}\,.\,\bar{a}_B)\,\bar{a}_B = 0.557\,\bar{a}_B = 0.557\,\dfrac{\bar{B}}{|\bar{B}|} = \dfrac{0.557\,(2\bar{a}x - 5\bar{a}z)}{\sqrt{29}}$

$$\bar{A}_B = 0.207\bar{a}x - 0.517\bar{a}z$$

H.W 5:

Decompose the vector $\bar{A} = -2\bar{a}x + 3\bar{a}y + 5\bar{a}z$ on to vectors parallel and perpendicular to the vector $\bar{B} = \bar{a}x - 2\bar{a}y - 2\bar{a}z$.

Ans.: $-2\bar{a}x + 4\bar{a}y + 4\bar{a}z\ ;\ -\bar{a}y + \bar{a}z$

1.3.2 The Cross Product:

The cross product of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A} \times \bar{B}$, is a vector quantity whose magnitude is the area of the parallelepiped formed by $\bar{A}$ and $\bar{B}$ (see **Fig. 1.4**) and is in the direction of advanced of right-handed screw as $\bar{A}$ is turned in to $\bar{B}$.

Thus: $\bar{A} \times \bar{B} = |\bar{A}||\bar{B}|\,\sin\theta_{AB}\ \bar{a}n$

Where $\bar{a}n$ is a unit vector normal to the plane containing $\bar{A}$ and $\bar{B}$. The direction of $\bar{a}n$ is taken as the direction of the right thumb when the fingers of the right hand rotate from $\bar{A}$ to $\bar{B}$ as shown in **Fig. 1.5a**. Alternatively, the direction of $\bar{a}n$ is taken as that of the advance of a right-handed screw as $\bar{A}$ is turned into $\bar{B}$ as shown in **Fig. 1.5b**. If $\bar{A} = A_x\,\bar{a}x + A_y\;\bar{a}y + A_z\,\bar{a}z$ and $\bar{B} = B_x\,\bar{a}x + B_y\;\bar{a}y + B_z\,\bar{a}z$, then:

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}x & \bar{a}y & \bar{a}z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(A_yB_z - B_yA_z\right)\bar{a}x - (A_xB_z\text{-}A_zB_x)\;\bar{a}y + \left(A_xB_y - A_yB_x\right)\bar{a}z$$

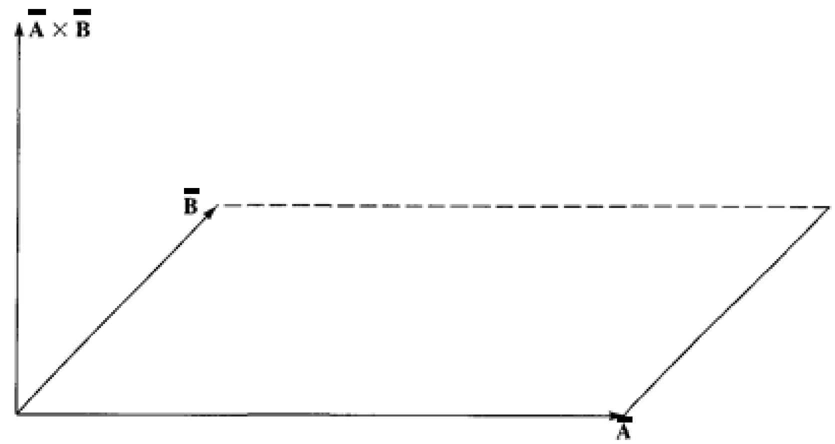


Fig. 1.4 The cross product of $\bar{A}$ and $\bar{B}$ is a vector with magnitude equal to the area of the parallelogram and direction as indicated.

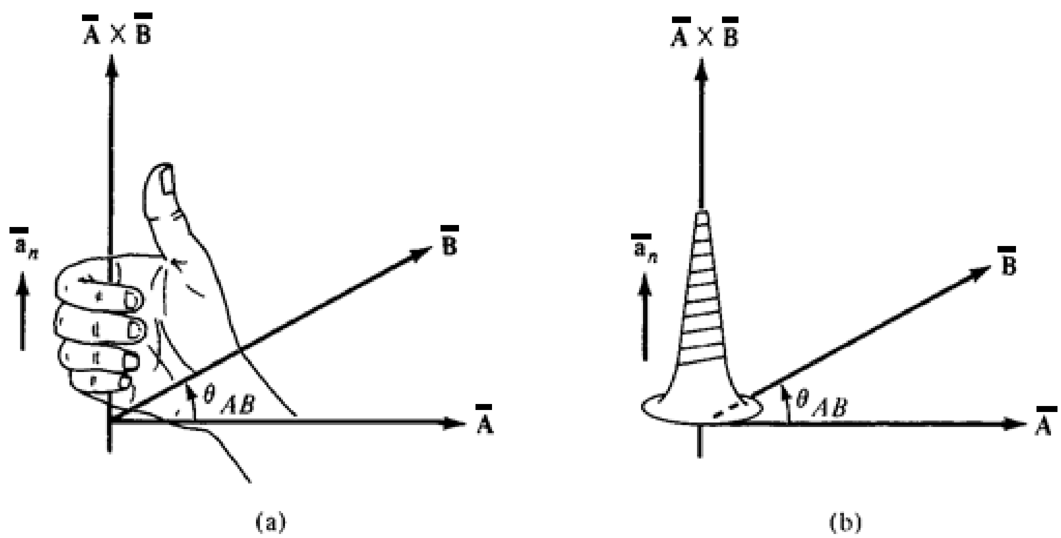


Fig. 1.5: Direction of $\bar{A} \times \bar{B}$ and $\bar{a}n$ using: **(a)** right-hand rule, **(b)** right-handed screw rule.

Notes:

1- $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$ (it is not commutative)

$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ (it is anti-commutative)

2- $\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C}$ (It is not associative)

3- $\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$ (It is distributive)

4- $\bar{A} \times \bar{A} = 0$

5- $\bar{a}x \times \bar{a}y = \bar{a}z$; $\bar{a}y \times \bar{a}z = \bar{a}x$; $\bar{a}x \times \bar{a}z = \bar{a}y$

6- $\bar{a}x \times \bar{a}x = \bar{a}y \times \bar{a}y = \bar{a}z \times \bar{a}z = 0$

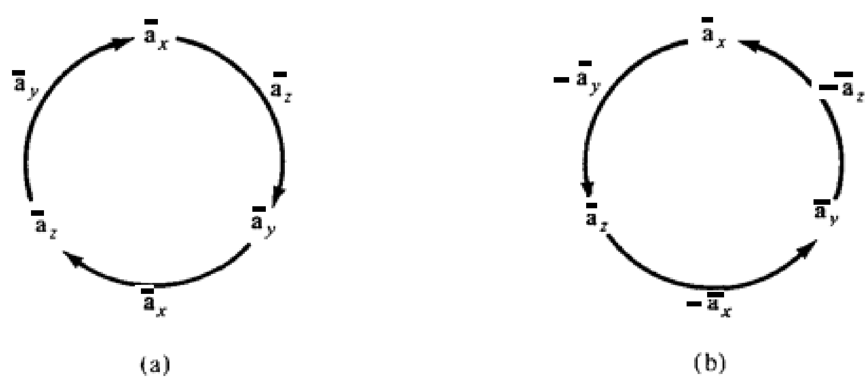


Fig. 1.6 Cross product using cyclic permutation: **(a)** moving clockwise leads to positive results: **(b)** moving counterclockwise leads to negative results.

Example 6: -

Points $P_1(1,2,3)$, $P_2(-5,2,0)$ and $P_3(2,7,-3)$ form a triangle in space. Calculate (a) The area of the triangle; (b) The unit vector perpendicular to the plane containing the triangle.

Solution:

$\bar{r}_{p1} = \bar{a}x + 2\bar{a}y + 3\bar{a}z$; $\bar{r}_{p2} = -5\bar{a}x + 2\bar{a}y$ and $\bar{r}_{p3} = 2\bar{a}x + 7\bar{a}y - 3\bar{a}z$

(a) $\bar{r}_{p1p2} = \bar{r}_{p2} - \bar{r}_{p1} = -6\bar{a}x - 3\bar{a}z$ and $\bar{r}_{p1p3} = \bar{r}_{p3} - \bar{r}_{p1} = \bar{a}x + 5\bar{a}y - 6\bar{a}z$

$$\bar{r}_{p1p2} \times \bar{r}_{p1p3} = \begin{vmatrix} \bar{a}x & \bar{a}y & \bar{a}z \\ -6 & 0 & -3 \\ 1 & 5 & -6 \end{vmatrix} = (0+15)\bar{a}x - (36+3)\,\bar{a}y + (-30-0)\bar{a}z$$

$\bar{r}_{p1p2} \times \bar{r}_{p1p3} = 15\bar{a}x - 39\bar{a}y - 30\bar{a}z$

Area of the triangle = $\frac{1}{2}\left|\bar{r}_{p1p2} \times \bar{r}_{p1p3}\right| = \frac{1}{2}\sqrt{15^2 + 39^2 + 30^2} = 25.72$

(b) $\bar{a}_n = \mp \dfrac{\bar{r}_{p1p2} \times \bar{r}_{p1p3}}{\left|\bar{r}_{p1p2} \times \bar{r}_{p1p3}\right|} = \mp \dfrac{15\bar{a}x - 39\bar{a}y - 30\bar{a}z}{51.44}$

$\therefore\ \bar{a}_n = \mp(0.291\bar{a}x - 0.758\bar{a}y - 0.583\bar{a}z$

Example 7: -

The vertices of triangle are located at $P_1(4,1,-3)$, $P_2(-2,5,4)$ and $P_3(0,1,6)$. Find the three angles of the triangle.

Solution:

$\bar{r}_{p1} = 4\bar{a}x + \bar{a}y - 3\bar{a}z$; $\bar{r}_{p2} = -2\bar{a}x + 5\bar{a}y + 4\bar{a}z$ and $\bar{r}_{p3} = \bar{a}y + 6\bar{a}z$

Let $\bar{A} = \bar{r}_{p1p2} = \bar{r}_{p2} - \bar{r}_{p1} = -6\bar{a}x + 4\bar{a}y + 7\bar{a}$

$\bar{B} = \bar{r}_{p2p3} = \bar{r}_{p3} - \bar{r}_{p2} = 2\bar{a}x - 4\bar{a}y + 2\bar{a}z$

$\bar{C} = \bar{r}_{p3p1} = \bar{r}_{p1} - \bar{r}_{p3} = 4\bar{a}x - 9\bar{a}z$

Note that $\bar{A} + \bar{B} + \bar{C} = 0$

$$\bar{A}.\bar{B} = |\bar{A}||\bar{B}|\cos\alpha_1 \qquad \Rightarrow \quad \cos\alpha_1 = \frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|} = \frac{-12-16-14}{\sqrt{101}\sqrt{24}}$$

$$\therefore \ \alpha_1 = \cos^{-1}\frac{-14}{\sqrt{101}\sqrt{24}} = 106.52° \ \Rightarrow \ \theta_1 = 180 - \alpha_1 = 73.48°$$

$$\bar{B}.\bar{C} = |\bar{B}||\bar{C}|\cos\alpha_2 \qquad \Rightarrow \quad \cos\alpha_2 = \frac{\bar{B}.\bar{C}}{|\bar{B}||\bar{C}|} = \frac{8+0-18}{\sqrt{24}\sqrt{97}}$$

$$\therefore \ \alpha_2 = \cos^{-1}\frac{-10}{\sqrt{24}\sqrt{97}} = 101.96° \Rightarrow \ \theta_2 = 180 - \alpha_2 = 78.04°$$

$$\bar{C}.\bar{A} = |\bar{C}||\bar{A}|\cos\alpha_3 \qquad \Rightarrow \quad \cos\alpha_3 = \frac{\bar{C}.\bar{A}}{|\bar{C}||\bar{A}|} = \frac{-24+0-63}{\sqrt{97}\sqrt{101}}$$

$$\therefore \ \alpha_3 = \cos^{-1}\frac{-87}{\sqrt{97}\sqrt{101}} = 151.52° \ \Rightarrow \ \theta_3 = 180 - \alpha_3 = 28.48°$$

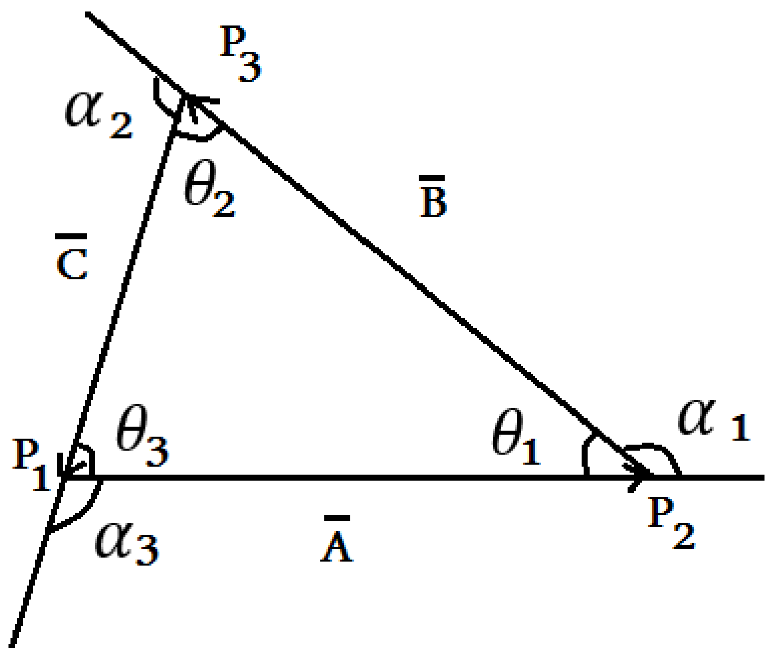


Fig. 1.13 for Example 7.

Note that $\theta_1 + \theta_2 + \theta_3 = 180°$

H.W 6: Show that vectors $\overline{A} = -5\bar{a}x - 3\bar{a}y - 3\bar{a}z$, $\overline{B} = \bar{a}x + 3\bar{a}y + 4\bar{a}z$ and $\overline{C} = 4\bar{a}x - \bar{a}z$ form the sides of a triangle. Is this a right angle triangle? Calculate the area of the triangle.

Ans.: Yes; 10.5

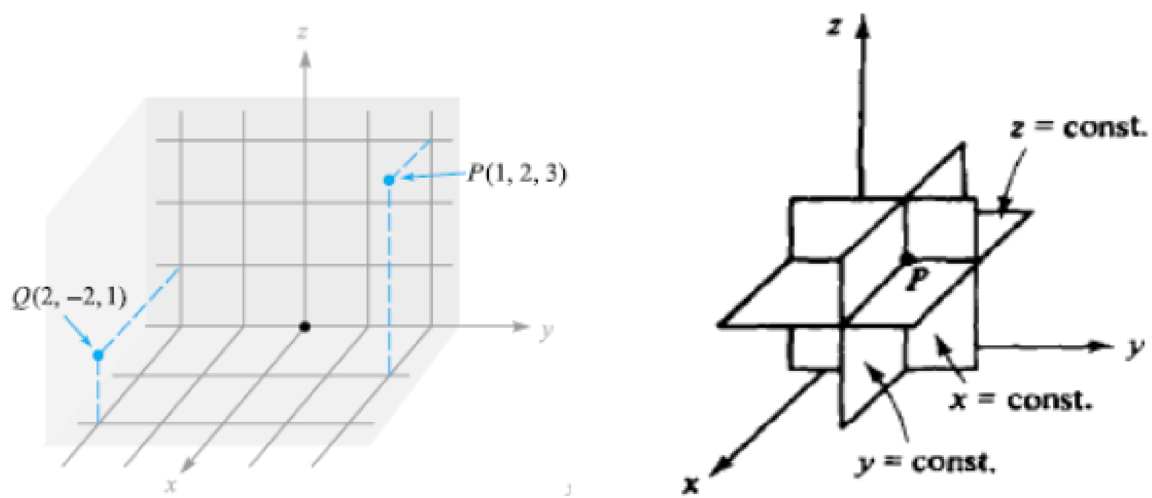
H.W 7: Show that points $P_1(5,2,-4)$, $P_2(1,1,2)$ and $P_3(-3,0,8)$ all lie on a straight line. Determine the shortest distance between the line and point $P_4(3,-1,0)$.

Ans.: 2.426

1.4 Systems of Coordinates

1.4.1 Cartesian (Rectangular) Coordinates (x, y, z)

A point $P(x, y, z)$ in Cartesian coordinates is located by giving its x, y and z coordinates. **Fig. 1.7a** shows the points P and Q whose coordinates are (1, 2, 3) and (2,-2, 1), respectively. Intersection of three mutually perpendicular planes defines a point in Cartesian coordinates, and as shown in **Fig. 1.7b**.



(a) **(b)**

Fig. 1.7 (a) The Location of point P and Q. **(b)** The three mutually perpendicular planes of the Cartesian coordinate system.

A vector $\bar{A}$ in Cartesian coordinates may be represented as:

$\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$, and shown in **Fig. 1.8**

where A_x , A_y and A_z are called the components of $\bar{A}$ in the x, y and z directions respectively; $\bar{a}x$, $\bar{a}y$ *and* $\bar{a}z$ are unit vectors in the x, y and z directions, respectively.

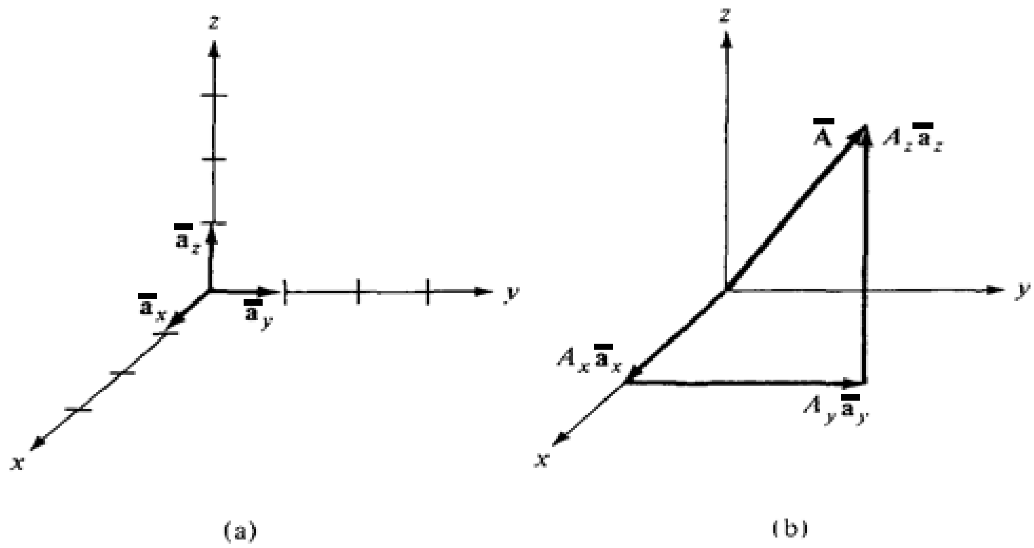


Fig. 1.8 (a) Unit vectors $\bar{a}x$, $\bar{a}y$, and $\bar{a}z$, **(b)** components of $\bar{A}$ along $\bar{a}x$, $\bar{a}y$, and $\bar{a}z$

Any vector can be written as:

$\bar{A} = |\bar{A}|\,\bar{a}_A$, where:

$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ The magnitude of the vector $\bar{A}$

$$\bar{a}_A = \frac{\bar{A}}{|\bar{A}|} = \frac{A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Unit vector along the vector $\bar{A}$.

$|\bar{a}_A| = 1$, $\bar{a}_A$ is a vector of unity magnitude.

If $\overline{A} = A_x\, \bar{a}x + A_y\, \bar{a}y + A_z\, \bar{a}z$ and $\overline{B} = B_x\, \bar{a}x + B_y\, \bar{a}y + B_z\, \bar{a}z$, then:

■ $\overline{A} + \overline{B} = (A_x + B_x)\, \bar{a}x + (A_y + B_y)\, \bar{a}y + (A_z + B_z)\, \bar{a}z$

■ $\overline{A} - \overline{B} = (A_x - B_x)\, \bar{a}x + (A_y - B_y)\, \bar{a}y + (A_z - B_z)\, \bar{a}z$

Position Vector:

The position vector $\bar{r}_p$ (or radius vector) of point P(x,y,z) is as the directed distance from the origin O to P; i. e.,

$$\bar{r}_p = \overline{OP} = x\, \bar{a}x + y\, \bar{a}y + z\, \bar{a}z$$

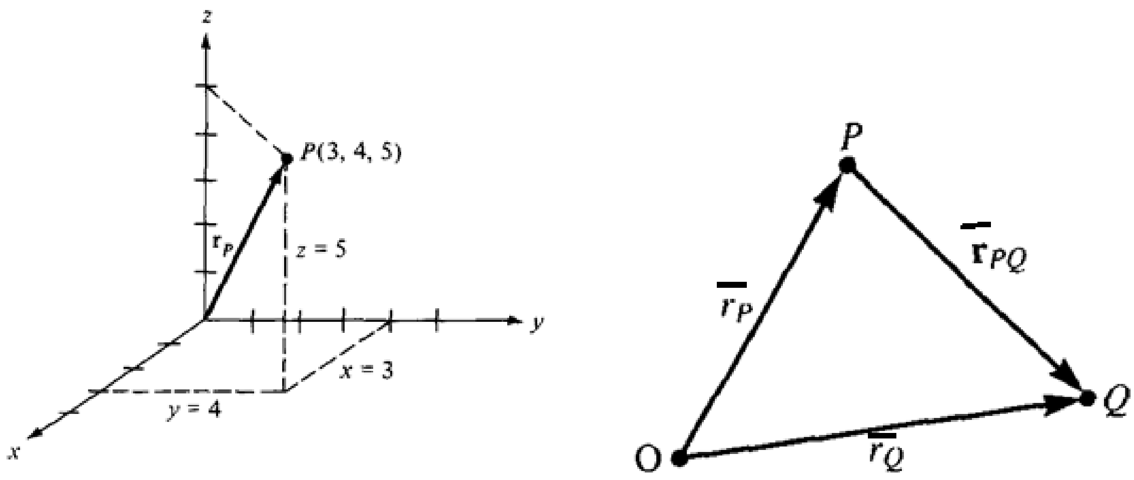
The position vector for point P is useful in defining its position in space. Point P(3,4,5), for example, and its position vector

$\bar{r}_p = \overline{OP} = 3\, \bar{a}x + 4\, \bar{a}y + 5\, \bar{a}z$, Are shown in **Fig. 1.9a.**

Distance Vector:

The distance vector is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), the distance vector (or separation vector) is the displacement from P to Q as shown in **Fig. 1.9b**; that is



(a) (b)

Fig. 1.9 (a) Illustration of position vector $\bar{r}_p = 3\,\bar{a}x + 4\,\bar{a}y + 5\,\bar{a}z$ (b) Distance vector $\bar{r}_{PQ}$.

$$\bar{r}_{PQ} = \bar{r}_Q - \bar{r}_P = (x_Q - x_P)\,\bar{a}x + (y_Q - y_P)\,\bar{a}y + (z_Q - z_P)\,\bar{a}z$$

The distance between the points P and Q is given by:

$$d = |\bar{r}_{PQ}| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$$

Differential Length, Area and Volume in Cartesian Coordinates:

From **Fig. 1.10**, we notice that:

1. Differential length is given by:

$$d\bar{L} = dx\,\bar{a}x + dy\,\bar{a}y + dz\,\bar{a}z, \quad \text{Vector Quantity}$$

$$dL = \sqrt{dx^2 + dy^2 + dz^2}, \quad \text{Scalar Quantity}$$

2. Differential normal Area is given by:

$$d\bar{s} = dy\,dz\,\bar{a}x$$
$$= dx\,dz\ \ \bar{a}y, \qquad \text{Vector Quantity}$$
$$= dx\,dy\,\bar{a}z$$

And illustrated in **Fig. 1.11**

3. Differential Volume is given by:

$$dV = dx\,dy\,dz, \qquad \text{Scalar Quantity}$$

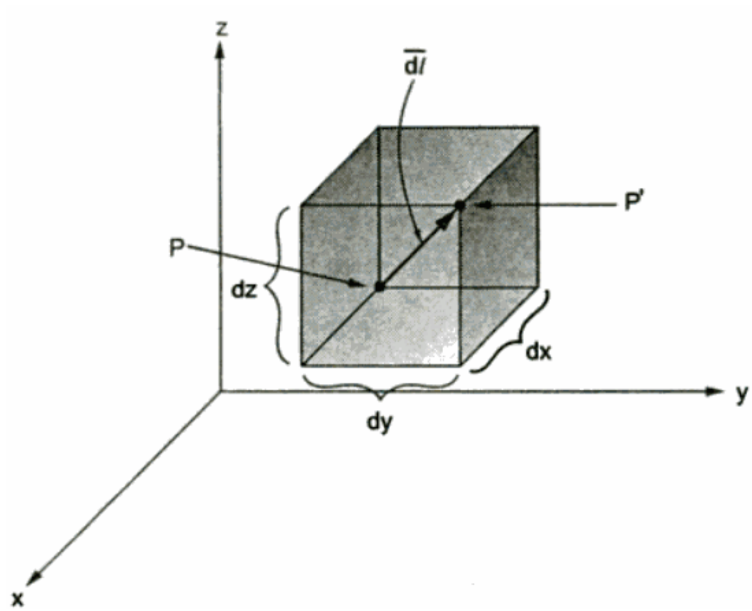


Fig. 1.10 Differential elements in Cartesian coordinates.

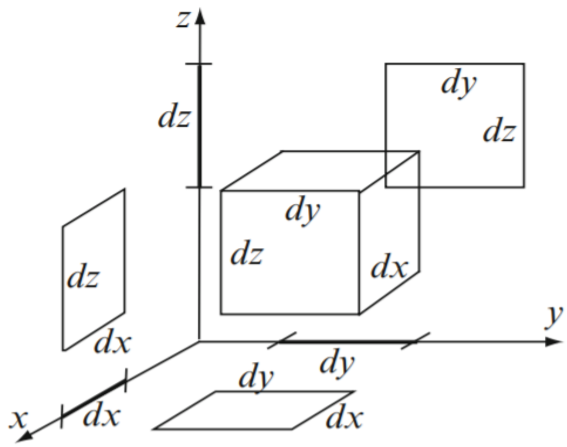


Fig. 1.11 Differential normal areas in Cartesian coordinates:

Example 1: -

Given the points M(2, -1,1) and T(-4, -2,6). Find: (a) the position vector for point M and T; (b) a unit vector from M to T; (c) the distance from M to T.

Solution:

(a) $\bar{r}_M = 2\,\bar{a}x - \bar{a}y + \bar{a}z$ and $\bar{r}_T = -4\,\bar{a}x - 2\,\bar{a}y + 6\,\bar{a}z$

(b) The vector from M to T is given by:

$$\bar{r}_{MT} = \bar{r}_T - \bar{r}_M = (-4-2)\bar{a}x + \big(-2-(-1)\big)\bar{a}y + (6-1)\bar{a}z = -6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z$$

$$\therefore\ \bar{a}_{r_{MT}} = \frac{\bar{r}_{MT}}{|\bar{r}_{MT}|} = \frac{-6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z}{\sqrt{(-6)^2 + (-1)^2 + (5)^2}} = \frac{-6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z}{\sqrt{62}}$$

$$\therefore\ \bar{a}_{r_{MT}} = -0.762\,\bar{a}x - 0.127\,\bar{a}y + 0.635\,\bar{a}z$$

(c) The distance from M to T is given by:

$$d = |\bar{r}_{MT}| = \sqrt{62} = 7.874\ [\text{m}]$$

Example 2: -

Given vectors $\overline{A} = \bar{a}x + 3\,\bar{a}z$ and $\overline{B} = 5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z$, determine:

(a) $|\overline{A} + \overline{B}|$; (b) $5\overline{A} - \overline{B}$; (c) The component of $\overline{A}$ along $\bar{a}y$; (d) A unit vector along $3\overline{A} + \overline{B}$.

Solution:

(a) $\overline{A} + \overline{B} = (\bar{a}x + 3\,\bar{a}z) + (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z) = 6\,\bar{a}x + 2\,\bar{a}y - 3\,\bar{a}z$

$\therefore\ |\overline{A} + \overline{B}| = \sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36 + 4 + 9} = 7$

(b) $5\overline{A} - \overline{B} = 5(\bar{a}x + 3\,\bar{a}z) - (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z)$

$= (5\,\bar{a}x + 15\,\bar{a}z) - (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z)$

$\therefore\ 5\overline{A} - \overline{B} = -2\,\bar{a}y + 21\,\bar{a}z$

(c) The component of $\overline{A}$ along $\bar{a}y$ is $A_y = 0$

(d) Let $\bar{C} = 3\overline{A} + \overline{B} = 3\,(\bar{a}x + 3\,\bar{a}z) + (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z) = 8\,\bar{a}x + 2\,\bar{a}y + 3\,\bar{a}z$

$$\bar{a}_C = \frac{\bar{C}}{|\bar{C}|} = \frac{8\,\bar{a}x + 2\bar{a}y + 3\,\bar{a}z}{\sqrt{64 + 4 + 9}} = 0.9117\,\bar{a}x + 0.2279\,\bar{a}y + 0.3419\,\bar{a}z$$

H.W 1: Given points M(-1,2,1), N(3,-3,0) and P(-2,-3,-4), find

(a) $\bar{r}_{MN}$; (b) $\bar{r}_{MN} + \bar{r}_{MP}$; (c) $|\bar{r}_M|$; (d) $\bar{a}_{r_{MP}}$; (e) $|2\,\bar{r}_P - 3\,\bar{r}_N|$.

Ans.: $4\,\bar{a}x - 5\,\bar{a}y - \bar{a}z$; $3\,\bar{a}x - 10\,\bar{a}y - 6\,\bar{a}z$; 2.45 ; $-0.14\,\bar{a}x - 0.7\,\bar{a}y - 0.7\,\bar{a}z$; 15.56

H.W 2: Express the unit vector directed toward the point P(1,-2,3) from an arbitrary point on the line described by $x = -3\ , y = 1$.

$$\textbf{Ans.}: \frac{4\bar{a}x - 3\bar{a}y + (3-z)\bar{a}z}{\sqrt{25 + (3-z)^2}}$$

H.W 3: An airplane has a ground speed of 350 [Km/hr] in the direction due west. If there is a wind blowing northwest at 40 [Km/hr], calculate the true air speed and heading of the airplane.

Ans.: 379.3 [Km/hr], 4.275° north of west.

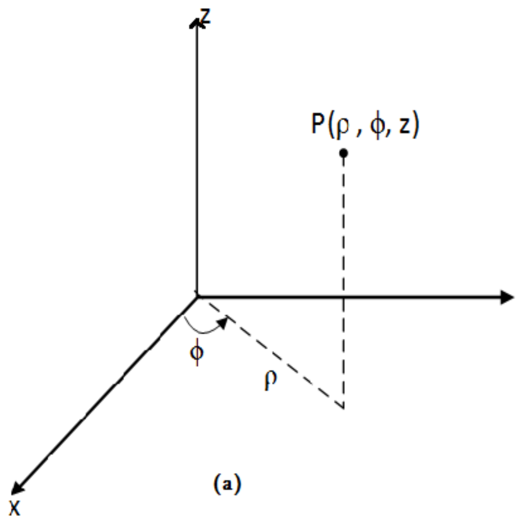
1.4.2 Circular Cylindrical Coordinates:

The circular cylindrical coordinates system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in **Fig. 1.12a** : ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ is the angle measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$$0 \le \rho \le \infty, 0 \le \phi \le 2\pi\ , -\infty \le z \le \infty$$

Intersection of three surfaces defined by $\rho =$ constant, $\phi =$ constant and $z =$ constant is also a point in cylindrical coordinates, and is as shown in **Fig. 1.12b**



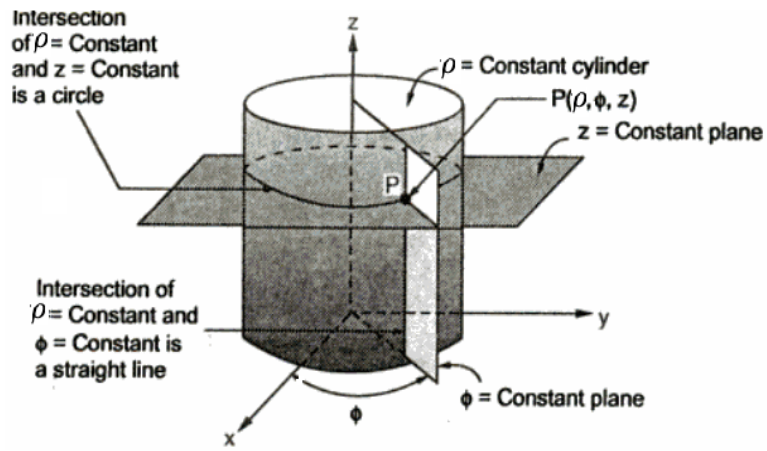


Fig. 1.12 (a) The three cylindrical coordinates; (b) Points P as intersection of three surfaces.

A vector $\bar{A}$ in cylindrical coordinates can be written as

$$\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$$

Where $\bar{a}_\rho$, $\bar{a}_\phi$ and $\bar{a}_z$ are unit vectors in the $\rho -$, $\phi -$ and $z -$directions as illustrated in **Fig. 1.13**.

The magnitude of $\bar{A}$ is:

$$|\bar{A}| = \sqrt{{A_\rho}^2 + {A_\phi}^2 + {A_z}^2}$$

Notice that the unit vectors $\bar{a}_\rho$, $\bar{a}_\phi$ and

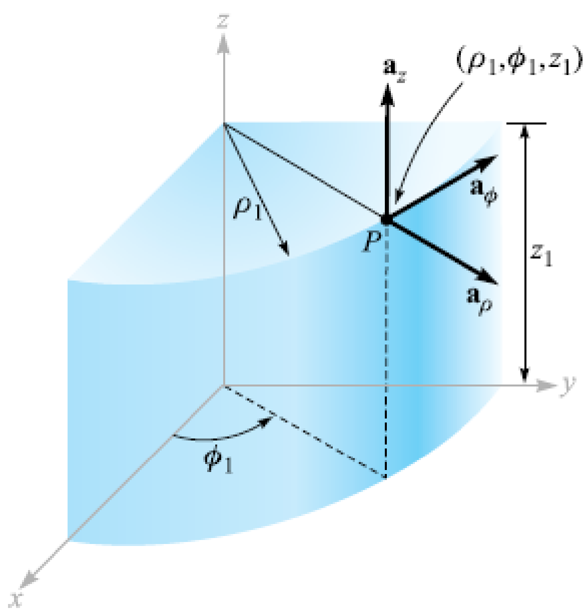


Fig. 1.13 The three unit vectors of the circular cylindrical coordinate system

$\bar{a}_z$ are mutually perpendicular because our coordinates system is orthogonal; $\bar{a}_\rho$ points in the direction of increasing ρ, $\bar{a}_\phi$ points in the direction of increasing ϕ, and $\bar{a}_z$ in the positive z-direction. Thus,

$\bar{a}_\rho.\bar{a}_\rho = \bar{a}_\phi.\bar{a}_\phi = \bar{a}_z.\bar{a}_z = 1$

$\bar{a}_\rho.\bar{a}_\phi = \bar{a}_\phi.\bar{a}_z = \bar{a}_z.\bar{a}_\rho = 0$

$\bar{a}_\rho \times \bar{a}_\rho = \bar{a}_\phi \times \bar{a}_\phi = \bar{a}_z \times \bar{a}_z = 0$

$\bar{a}_\rho \times \bar{a}_\phi = \bar{a}_z$; $\bar{a}_\phi \times \bar{a}_z = \bar{a}_\rho$; $\bar{a}_z \times \bar{a}_\rho = \bar{a}_\phi$, see **Fig. 1.6** with replacing

$(\bar{a}_x , \bar{a}_y , \bar{a}_z)$ with $(\bar{a}_\rho , \bar{a}_\phi , \bar{a}_z)$

If $\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$ and $\bar{B} = B_\rho \bar{a}_\rho + B_\phi \bar{a}_\phi + B_z \bar{a}_z$, then:

$\bar{A}.\bar{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z$

And

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix}$$

<u>Differential Length, Area, and Volume in Cylindrical Coordinates:</u>

From **Fig. 1.14**, we notice that:

(1) Differential length is given by:

$d\bar{L} = d\rho\, \bar{a}\rho + \rho d\phi\, \bar{a}\phi + dz\, \bar{a}z,$ Vector Quantity

$dL = \sqrt{d\rho^2 + (\rho d\phi)^2 + dz^2}\,,$ Scalar Quantity

(2) Differential normal area is given by:

$d\bar{s} = \rho \, d\phi \, dz \, \bar{a}\rho$

$= d\rho \, dz \;\; \bar{a}\phi,$ Vector Quantity

$= \rho d\rho \, d\phi \, \bar{a}z$

And illustrated in **Fig. 1.15**

(3) Differential volume is given by:

$dV \; = \rho d\rho \, d\phi \, dz \,,$ Scalar Quantity

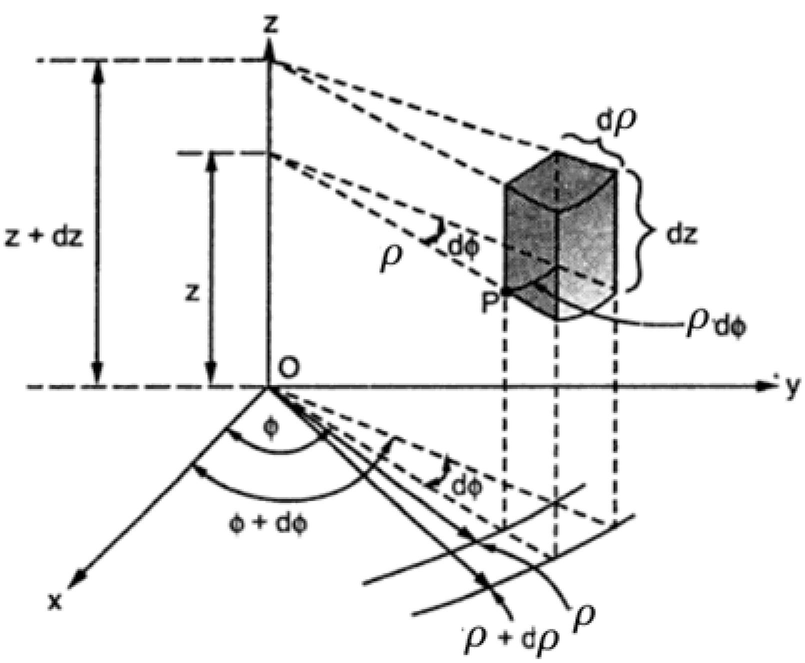


Fig. 1.14 Differential elements in cylindrical coordinates

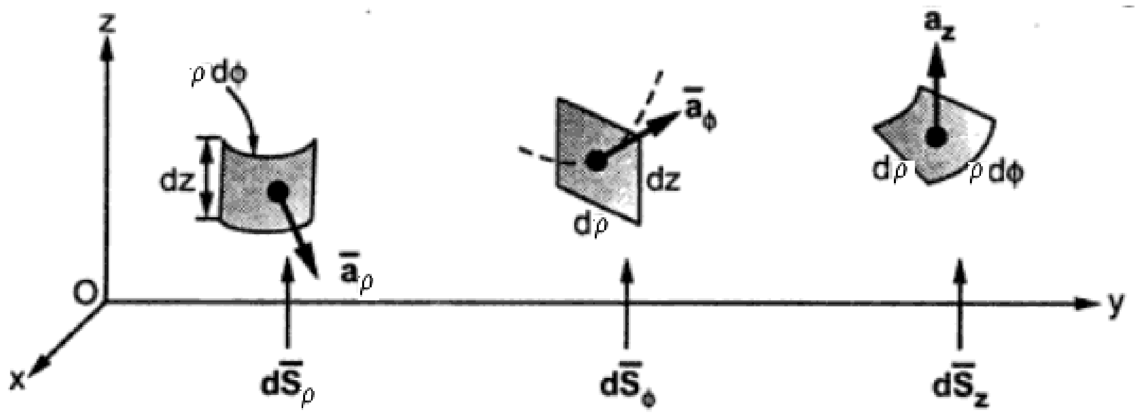


Fig. 1.15 Differential normal areas in cylindrical coordinates:

The relationship between the variables $(x\,, y\,, z)$ of the Cartesian coordinates and those of the cylindrical system $(\rho\,, \phi\,, z)$ are illustrated in Fig. 1.16, and given by:

1- **From Cartesian To Cylindrical**:

$$x = \rho\ \cos\phi$$

$$y = \rho\ \sin\phi$$

$$z = z$$

2- **From Cylindrical To Cartesian**:

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$z = z$$

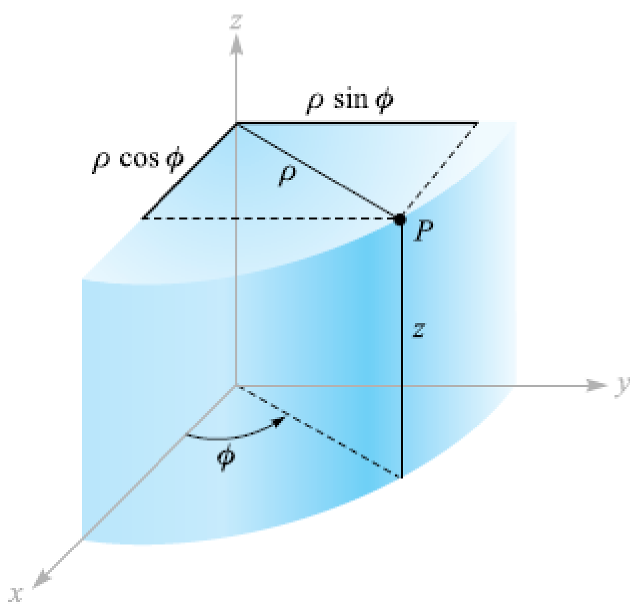


Fig. 1.16 The relationship between $(x\,, y\,, z)$ and $(\rho\,, \phi\,, z)$.

The dot product between $(\bar{a}_x\,, \bar{a}_y\,, \bar{a}_z)$ and $(\bar{a}_\rho\,, \bar{a}_\phi\,, \bar{a}_z)$ are obtained geometrically from **Fig. 1.17**:

$\bar{a}_x.\bar{a}_\rho = \cos\phi$ $\bar{a}_y.\bar{a}_\rho = \cos(90° - \phi) = \sin\phi$

$\bar{a}_x.\bar{a}_\phi = -\cos(90° - \phi) = -\sin\phi$ $\bar{a}_y.\bar{a}_\phi = \cos\phi$

$\bar{a}_x.\bar{a}_z = 0$ $\bar{a}_y.\bar{a}_z = 0$

Thus:

$$\bar{a}_x = \cos\phi\,\bar{a}_\rho - \sin\phi\,\bar{a}_\phi \qquad \bar{a}_\rho = \cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y$$

$$\bar{a}_y = \sin\phi\,\bar{a}_\rho + \cos\phi\,\bar{a}_\phi \qquad \bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y$$

$$\bar{a}_z = \bar{a}_z \qquad \bar{a}_z = \bar{a}_z$$

$$\alpha = 90^\circ - \phi$$

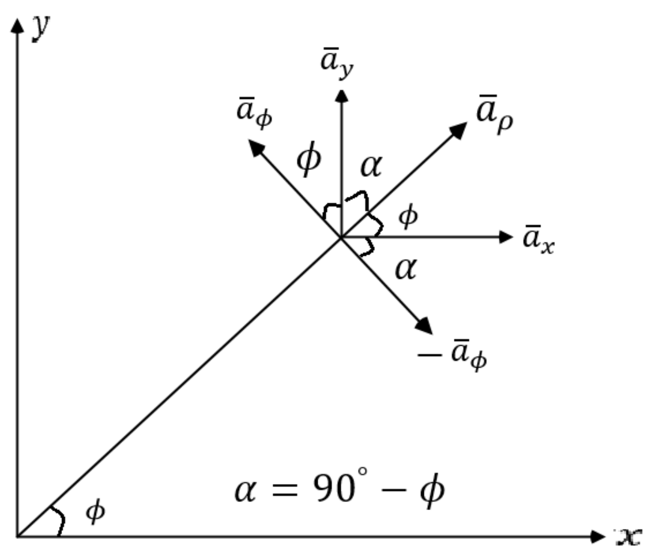


Fig. 1.17 Relationship between unit vectors of Cartesian and cylindrical coordinates.

The vector $\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$ can be transformed into cylindrical coordinates as:

$$A_\rho = \bar{A}.\bar{a}_\rho = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_\rho = A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \bar{A}.\bar{a}_\phi = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_z = \bar{A}.\bar{a}_z = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_z = A_z$$

The vector $\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$ can be transformed into Cartesian coordinates as:

$$A_x = \bar{A}.\bar{a}_x = \left(A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z\right).\bar{a}_x = A_\rho \cos\phi - A_\phi \sin\phi$$

$$A_y = \bar{A}.\bar{a}_y = \left(A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z\right).\bar{a}_y = A_\rho \sin\phi + A_\phi \cos\phi$$

$$A_z = \bar{A}.\bar{a}_z = \left(A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z\right).\bar{a}_z = A_z$$

<u>Example 8</u>: -

(a) Transform the vector $\bar{B} = y\bar{a}x - x\bar{a}y + z\bar{a}z$ into cylindrical coordinates.

(b) Express the vector filed $\bar{S} = \cos\phi\, \bar{a}_\rho + \sin\phi\, \bar{a}_\phi$ in Cartesian coordinates.

(c) Find at P(1, 2, -2) the vector projection of $\bar{B}$ in the direction of $\bar{S}$.

Solution:

(a) $B_\rho = \bar{B}.\bar{a}_\rho = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_\rho = y\ \cos\phi - x\ \sin\phi$

$\because\ x = \rho\ \cos\phi\ \ and\ y = \ \rho\ \sin\phi$

$B_\rho = \rho\ \sin\phi\ \cos\phi - \rho\ \cos\phi\ \sin\phi = 0$

$B_\phi = \bar{B}.\bar{a}_\phi = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_\phi = -y\ \sin\phi - x\cos\phi$

$\therefore B_\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$

$B_z = \bar{B}.\bar{a}_z = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_z = z$

$\therefore\ \bar{B} = -\rho\ \bar{a}_\phi + z\ \bar{a}_z$ in cylindrical coordinates

(b) $S_x = \bar{S}.\bar{a}_x = \left(\cos\phi\, \bar{a}_\rho + \sin\phi\, \bar{a}_\phi\right).\bar{a}_x = \cos^2\phi - \sin^2\phi$

$$\because\ \cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \; S_x = \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$S_y = \bar{S}.\bar{a}_y = \left(\cos\phi\,\bar{a}_\rho + \sin\phi\,\bar{a}_\phi\right).\bar{a}_y = \cos\phi\sin\phi + \sin\phi\cos\phi = 2\cos\phi\sin\phi$$

$$\therefore \; S_y = 2\frac{x}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$$

$$S_z = \bar{S}.\bar{a}_z = \left(\cos\phi\,\bar{a}_\rho + \sin\phi\,\bar{a}_\phi\right).\bar{a}_z = 0$$

$$\bar{S} = \frac{x^2-y^2}{x^2+y^2}\bar{a}_x + \frac{2xy}{x^2+y^2}\bar{a}_y \qquad \text{in Cartesian Coordinates}$$

(c) $\because \bar{B} = y\bar{a}x - x\bar{a}y + z\bar{a}z$

$\therefore \bar{B} = 2\bar{a}x - \bar{a}y - 2\bar{a}z$

$$\because \bar{S} = \frac{x^2-y^2}{x^2+y^2}\bar{a}_x + \frac{2xy}{x^2+y^2}\bar{a}_y$$

$$\therefore \bar{S} = \frac{1-4}{1+4}\bar{a}_x + \frac{2(1)(2)}{1+4}\bar{a}_y = -0.6\bar{a}_x + 0.8\bar{a}_y$$

$$\therefore \bar{B}_s = (\bar{B}.\bar{a}_s)\bar{a}_s = \frac{\bar{B}.\bar{S}}{|\bar{S}|^2}\bar{S}$$

$$\therefore \bar{B}_s = \frac{(2\bar{a}x - \bar{a}y - 2\bar{a}z).\left(-0.6\bar{a}_x + 0.8\bar{a}_y\right)}{(0.6^2+0.8^2)}\left(-0.6\bar{a}_x + 0.8\bar{a}_y\right)$$

$$\therefore \bar{B}_s = \frac{-1.2-0.8}{1}\left(-0.6\bar{a}_x + 0.8\bar{a}_y\right) = 1.2\bar{a}_x + 1.6\bar{a}_y$$

H.W 8: Transform

$$\bar{A} = \frac{-xy\bar{a}_x + x^2\bar{a}_y + y^2\bar{a}_z}{x^2+y^2} \qquad \text{From Cartesian to cylindrical coordinates.}$$

Ans.: $\bar{A} = \cos\phi\bar{a}_\phi + \sin^2\phi\,\bar{a}_z$

H.W 9: Express the field $\bar{E} = \sin\phi\bar{a}_\rho + \cos^2\phi\,\bar{a}_z$ In Cartesian coordinates.

Ans.: $\bar{E} = \dfrac{xy\bar{a}_x + y^2\bar{a}_y + x^2\bar{a}_z}{x^2+y^2}$

H.W 10: Decompose the vector $\bar{A} = 2\bar{a}_x - \bar{a}_y + 5\bar{a}_z$ into vectors parallel and perpendicular to the cylinder $\rho = 1$ at point P(1,30°, 0).

Ans.: $\bar{A}_T = -1.866\bar{a}_\phi + 5\bar{a}_z$ and $\bar{A}_N = 1.232\bar{a}_\rho$

1.4.3 Spherical Coordinates System:

The spherical coordinates system is most appropriate when dealing with problems having of spherical symmetry. A point P can be represented as $P(r,\theta,\phi)$ and illustrated in **Fig. 1.18a**, we notice that r is defined as the distance from the origin to point P or the radius of sphere centered at the origin and passing through P; θ is the angle between the z-axis and the position vector of P; ϕ is measured from the x-axis (ϕ is the same as in the cylindrical coordinates). According to these definitions, the ranges of the variables are:

$$0 \le r \le \infty, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$$

Intersection of three orthogonal surfaces defined by $r = constant$, $\theta = constant$ and $\phi = constant$ is also a point in spherical coordinates, and is shown in Fig. **1.18b.**

A vector $\bar{A}$ in spherical

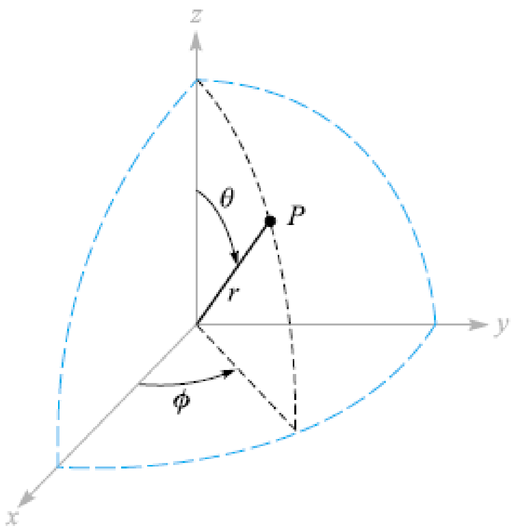


Fig. 1.18 (a) The three spherical coordinates.

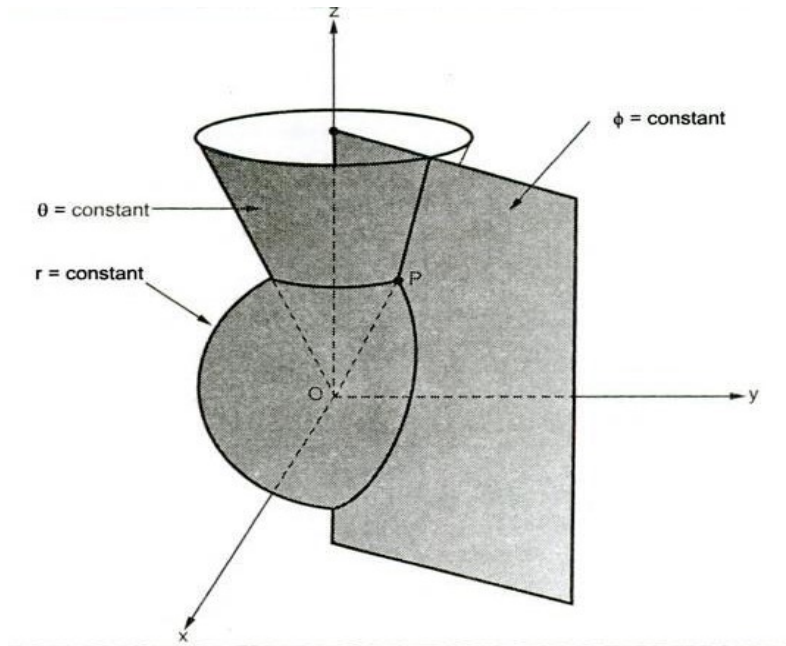


Fig. 1.18 (b) Point P as intersection of three surfaces.

coordinates can be written as:

$$\bar{A} = A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi$$

Where $\bar{a}_r$, $\bar{a}_\theta$, $\bar{a}_\phi$ are unit vectors along the $r-, \theta-,\ and\ \phi-$ directions as illustrated in Fig. 1.19 the magnitude of $\bar{A}$ is:

$$|\bar{A}| = \sqrt{A_r{}^2 + A_\theta{}^2 + A_\phi{}^2}$$

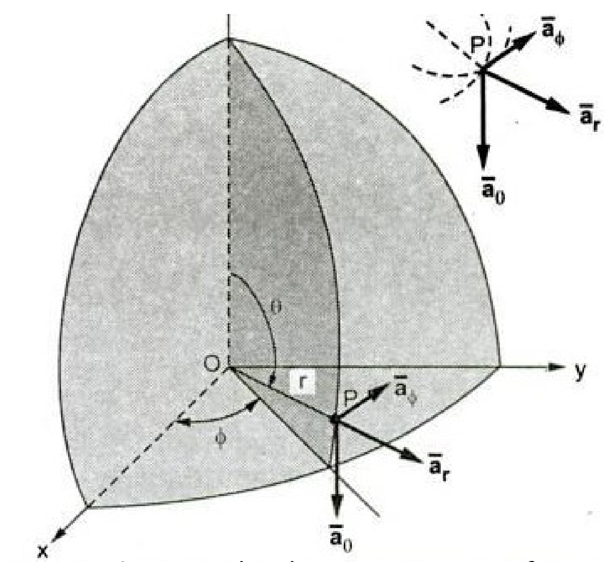


Fig. 1.19 The three unit vectors for spherical coordinates.

The unit vectors $\bar{a}_r$, $\bar{a}_\theta$ and $\bar{a}_\phi$ are mutually orthogonal; $\bar{a}_r$ being directed along the radius or points in the direction of increasing r, $\bar{a}_\theta$ points in the direction of increasing θ, and $\bar{a}_\phi$ in the direction of increasing ϕ. Thus,

$$\bar{a}_r.\bar{a}_r = \bar{a}_\theta.\bar{a}_\theta = \bar{a}_\phi.\bar{a}_\phi = 1$$

$$\bar{a}_r.\bar{a}_\theta = \bar{a}_\theta.\bar{a}_\phi = \bar{a}_\phi.\bar{a}_r = 0$$

$$\bar{a}_r \times \bar{a}_r = \bar{a}_\theta \times \bar{a}_\theta = \bar{a}_\phi \times \bar{a}_\phi = 0$$

$\bar{a}_r \times \bar{a}_\theta = \bar{a}_\phi$; $\bar{a}_\theta \times \bar{a}_\phi = \bar{a}_r$; $\bar{a}_\phi \times \bar{a}_r = \bar{a}_\theta$, see Fig. 1.12 with replacing

$(\bar{a}_x , \bar{a}_y , \bar{a}_z)$ with $(\bar{a}_r , \bar{a}_\theta , \bar{a}_\phi)$.

If $\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$ and $\bar{B} = B_r\bar{a}_r + B_\theta\bar{a}_\theta + B_\phi\bar{a}_\phi$, then:

$$\bar{A}.\bar{B} = A_r\, B_r + A_\theta\, B_\theta + A_\phi\, B_\phi$$

And

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}r & \bar{a}\theta & \bar{a}\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$$

<u>Differential Length, Area, and Volume in Cylindrical Coordinates:</u>

From Fig. 1.20, we notice that:

(1) Differential length is given by:

$d\bar{L} = dr\,\bar{a}r + rd\theta\bar{a}\theta + r\sin\theta\, d\phi\,\bar{a}\phi$, Vector Quantity

$dL = \sqrt{dr^2 + (rd\theta)^2 + (r\sin\theta\, d\phi)^2}$, Scalar Quantity

(2) Differential normal area is given by:

$$d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}r$$

$$= r \sin\theta \, dr \, d\phi \;\; \bar{a}\theta,$$ Vector Quantity

$$= rdr \, d\theta \, \bar{a}\phi$$

And illustrated in Fig. 1.21

(3) Differential volume is given by:

$$dV \; = r^2 \sin\theta \, dr \, d\theta \, d\phi \;\; ,$$

Scalar Quantity

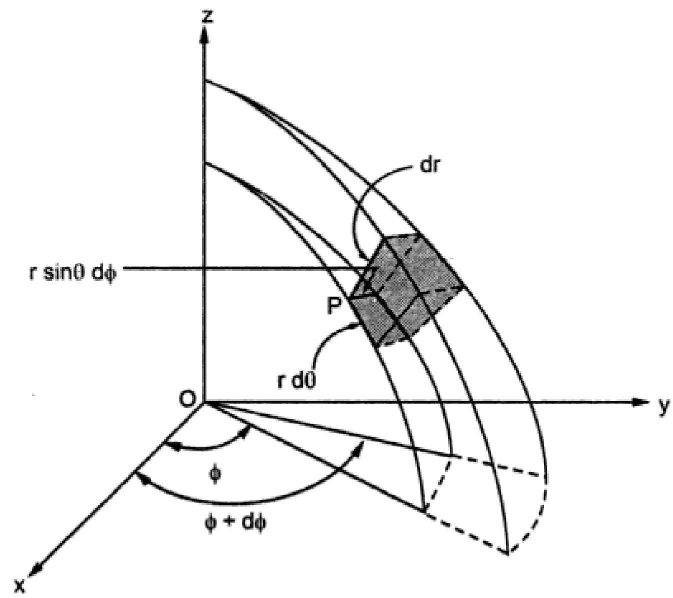


Fig. 1.20 Differential elements in spherical coordinates.

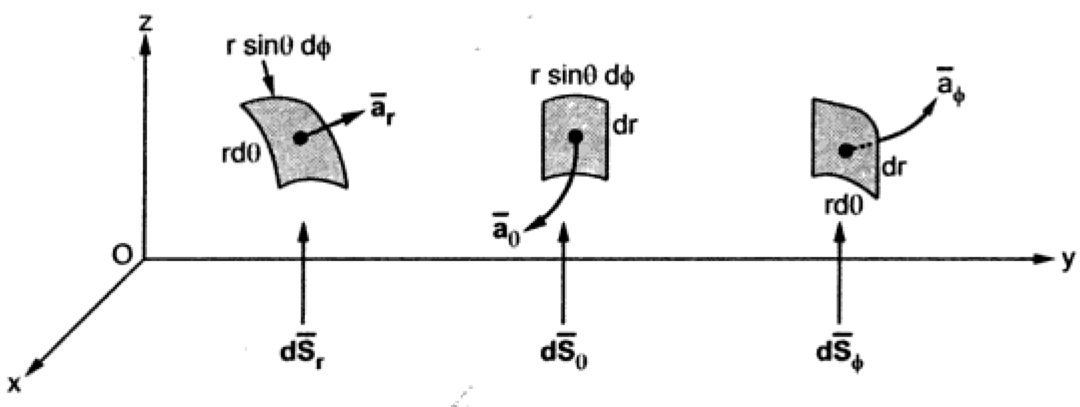


Fig. 1.21 Differential normal areas in spherical coordinates.

The space variables $(x\,, y\,, z)$ of the Cartesian coordinates can be related to variables $(r\,, \theta\,, \phi)$ of a spherical coordinates system. From Fig. 1.22, it is easy to notice that:

1- From Cartesian To Spherical:

$$x = r\ \sin\theta\cos\phi$$

$$y = r\ \sin\theta\sin\phi$$

$$z = r\ \cos\theta$$

2- From Spherical To Cartesian:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

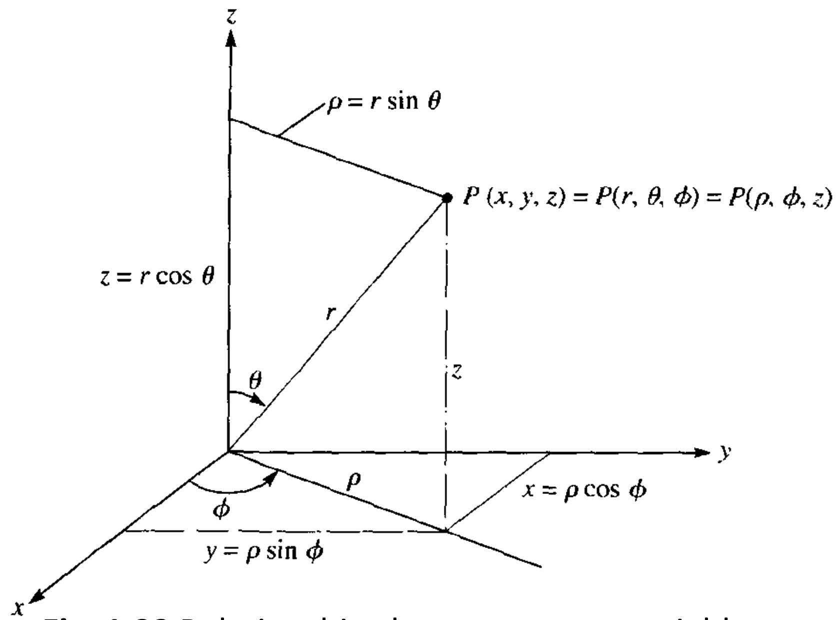


Fig. 1.22 Relationships between space variables $(x\,, y\,, z)$ and $(r\,, \theta\,, \phi)$.

The dot product between $(\bar{a}_x\,, \bar{a}_y\,, \bar{a}_z)$ and $(\bar{a}_r\,, \bar{a}_\theta\,, \bar{a}_\phi)$ are obtained geometrically from **Fig. 1.25**:

$$\bar{a}_x.\bar{a}_r = \bar{a}_x.\left(\cos(90-\theta)\bar{a}_\rho + \cos\theta\,\bar{a}_z\right) = \bar{a}_x.(\sin\theta\,\bar{a}_\rho + \cos\theta\,\bar{a}_z) = \sin\theta\cos\phi$$

$$\bar{a}_x.\bar{a}_\theta = \bar{a}_x.\left(\cos\theta\bar{a}_\rho - \cos(90-\theta)\,\bar{a}_z\right) = \bar{a}_x.(\cos\theta\,\bar{a}_\rho - \sin\theta\,\bar{a}_z) = \cos\theta\cos\phi$$

$$\bar{a}_x.\bar{a}_\phi = -\sin\phi$$

$$\bar{a}_y.\bar{a}_r = \bar{a}_y.\left(\sin\theta\,\bar{a}_\rho + \cos\theta\,\bar{a}_z\right) = \sin\theta\sin\phi$$

$$\bar{a}_y.\bar{a}_\theta = \bar{a}_y.\left(\cos\theta\,\bar{a}_\rho - \sin\theta\,\bar{a}_z\right) = \cos\theta\sin\phi$$

$\bar{a}_y.\bar{a}_\phi = \cos\phi$

$\bar{a}_z.\bar{a}_r = \bar{a}_z.\left(\sin\theta\,\bar{a}_\rho + \cos\theta\,\bar{a}_z\right) = \cos\theta$

$\bar{a}_z.\bar{a}_\theta = \bar{a}_z.\left(\cos\theta\,\bar{a}_\rho - \sin\theta\,\bar{a}_z\right) = -\sin\theta$

$\bar{a}_z.\bar{a}_\phi = 0$

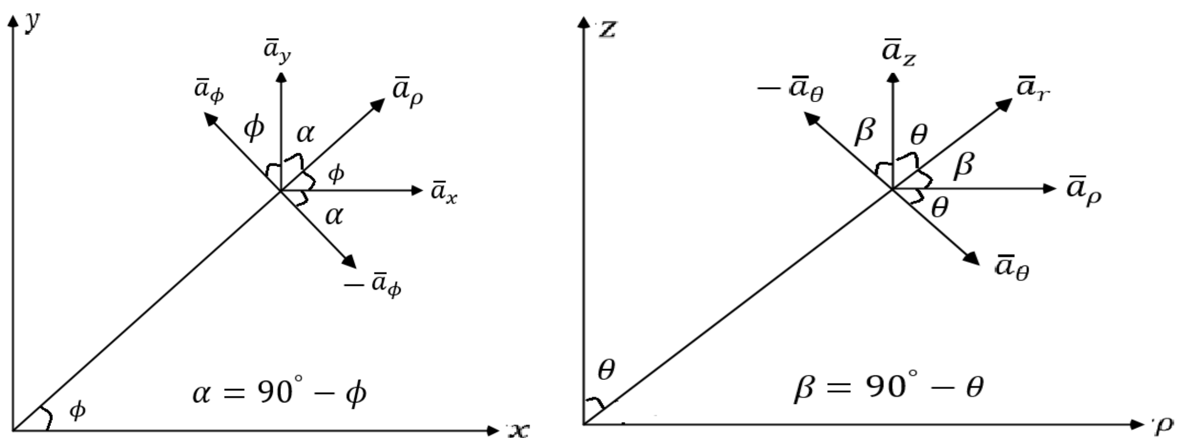


Fig. 1.25 Relationship between the unit vectors of three coordinate systems.

The vector $\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$ can be transformed into spherical coordinates as:

$A_r = \bar{A}.\bar{a}_r = \left(A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z\right).\bar{a}_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$

$A_\theta = \bar{A}.\bar{a}_\theta = \left(A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z\right).\bar{a}_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$

$A_\phi = \bar{A}.\bar{a}_\phi = \left(A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z\right).\bar{a}_\phi = -A_x\sin\phi + A_y\cos\phi$

The vector $\bar{A} = A_\rho\bar{a}_\rho + A_\phi\bar{a}_\phi + A_z\bar{a}_z$ can be transformed into Cartesian coordinates as:

$A_x = \bar{A}.\bar{a}_x = \left(A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi\right).\bar{a}_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$

$A_y = \bar{A}.\bar{a}_y = \left(A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi\right).\bar{a}_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$

$$A_z = \bar{A}.\bar{a}_z = \left(A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi\right).\bar{a}_z = A_r\cos\theta - A_\theta\sin\theta$$

Example 9: -

A vector field is given by:

$$\bar{D} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}\left[(x-y)\bar{a}_x + (x+y)\bar{a}_y\right]$$

Express this field in spherical coordinates.

Solution:

$$r = \sqrt{x^2+y^2+z^2}\,, \qquad \rho = r\sin\theta = \sqrt{x^2+y^2}$$

$$x = r\ \sin\theta\cos\phi\ \ ,\ \ y = r\ \sin\theta\sin\phi$$

$$\therefore\ \bar{D} = \frac{r}{r\ \sin\theta}\left[(r\ \sin\theta\cos\phi - r\ \sin\theta\sin\phi)\bar{a}_x + (r\ \sin\theta\cos\phi + r\ \sin\theta\sin\phi)\bar{a}_y\right]$$

$$\therefore \bar{D} = r\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right]$$

$$D_r = \bar{D}.\bar{a}_r = r\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right].\bar{a}_r$$

$$= r[(\cos\phi - \sin\phi)\sin\theta\cos\phi + (\cos\phi + \sin\phi)\sin\theta\sin\phi]$$

$$= r\sin\theta\,[\cos^2\phi - \sin\phi\cos\phi + \cos\phi\sin\phi + \sin^2\phi] = r\sin\theta$$

$$\therefore D_r = r\sin\theta$$

$$D_\theta = \bar{D}.\bar{a}_\theta = r\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right].\bar{a}_\theta$$

$$= r[(\cos\phi - \sin\phi)\cos\theta\cos\phi + (\cos\phi + \sin\phi)\cos\theta\sin\phi]$$

$$= r\cos\theta\,[\cos^2\phi - \sin\phi\cos\phi + \cos\phi\sin\phi + \sin^2\phi] = r\cos\theta$$

$\therefore D_\theta = r\cos\theta$

$D_\phi = \bar{D}.\bar{a}_\phi = r\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right].\bar{a}_\phi$

$= r[-(\cos\phi - \sin\phi)\sin\phi + (\cos\phi + \sin\phi)\cos\phi]$

$= r[-\cos\phi\sin\phi + \sin^2\phi + \cos^2\phi + \sin\phi\cos\phi] = r$

$\therefore D_\phi = r$

$\therefore \bar{D} = r\sin\theta\,\bar{a}_r + r\cos\theta\,\bar{a}_\theta + r\bar{a}_\phi$

Example 10: -

Given vectors $\bar{A} = 2\bar{a}_x - \bar{a}_y + 5\bar{a}_z$ and $\bar{B} = 4\bar{a}_\theta$, find the angle between $\bar{A}$ and $\bar{B}$ at P(1, 15°, 50°).

Solution:

$B_x = \bar{B}.\bar{a}_x = 4\bar{a}_\theta.\bar{a}_x = 4\ \cos\theta\cos\phi$

$B_y = \bar{B}.\bar{a}_y = 4\bar{a}_\theta.\bar{a}_y = 4\ \cos\theta\sin\phi$

$B_z = \bar{B}.\bar{a}_z = 4\bar{a}_\theta.\bar{a}_z = -4\ \sin\theta$

$\therefore \bar{B} = 4\ \cos\theta\cos\phi\,\bar{a}_x +\ 4\ \cos\theta\sin\phi\,\bar{a}_y - 4\ \sin\theta\ \bar{a}_z$

At P(1, 15°, 50°),

$\bar{B} = 2.4835\bar{a}_x +\ 2.9597\bar{a}_y - 1.0352\ \bar{a}_z$

$\bar{A}.\bar{B} = \left(2\bar{a}_x - \bar{a}_y + 5\bar{a}_z\right).\left(\ 2.4835\bar{a}_x +\ 2.9597\bar{a}_y - 1.0352\ \bar{a}_z\right) = -3.1687$

$|\bar{A}| = \sqrt{2^2 + 1^2 + 5^2} = 5.4772$ and $|\bar{B}| = 4$

$\because \bar{A}.\bar{B} = |\bar{A}||\bar{B}|\cos\theta_{AB}$

$$\therefore \ \theta_{AB} = \cos^{-1}\left[\frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|}\right] = \cos^{-1}\left[\frac{-3.1687}{5.4772 * 4}\right] = \cos^{-1}[-0.1446]$$

$$\therefore \ \theta_{AB} = 98.31^\circ$$

Example 11: -

A spherical region is defined by:

$$1 \leq r \leq \ 3, 15^\circ \leq \theta \leq \ 60^\circ \text{ , and } 10^\circ \leq \phi \leq 80^\circ$$

Find the volume V.

Solution:

$$V = \iiint_v dv = \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\int_{r=1}^{3} r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\left(\frac{r^3}{3}\right)_1^3 \sin\theta \ d\theta \, d\phi$$

$$= \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\frac{26}{3}\sin\theta \ d\theta \, d\phi = \int_{\phi=10^\circ}^{80^\circ}\frac{26}{3}(-\cos\theta)_{15^\circ}^{60^\circ} d\phi = 4.038\int_{\phi=10^\circ}^{80^\circ} d\phi$$

$$= 4.038 \ (\phi)_{10^\circ}^{80^\circ} = \ 4.038 \ (80 - 10) * \frac{\pi}{180} = 4.9333 \ \text{ Unit}^3$$

Example 12: -

Find the area of the surface defined by:

$$\theta = 45^\circ, \qquad 3 \leq r \leq \ 5, \ \text{ and } \ 0.1\pi \leq \phi \leq \pi$$

Solution:

$$S = \iint_s ds = \int_{\phi=0.1\pi}^{\pi} \int_{r=3}^{5} (dr)(r\sin\theta\, d\phi) = \int_{\phi=0.1\pi}^{\pi} \int_{r=3}^{5} r\sin 45^{\circ}\, dr\, d\phi =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{r^2}{2}\right)_3^5 (\phi)_{0.1\pi}^{\pi} = \frac{1}{\sqrt{2}}\left(\frac{25-9}{2}\right)(0.9\pi) = 15.9943\ \text{Unit}^2$$

<u>H.W 11:</u> Find the angle between vector $\bar{A} = \bar{a}_x + 3\bar{a}_y + 2\bar{a}_z$ and the sphere $r = 1$ at the point P(1,20°, 30°).

Ans.: 45°.93

<u>H.W 12:</u> Prove that the field $\bar{A} = \sin\theta\, \bar{a}_\theta$ in Cartesian coordinates is given by:

$$\bar{A} = \frac{xz\bar{a}_x + yz\bar{a}_y - (x^2+y^2)\bar{a}_z}{x^2+y^2+z^2}$$

<u>H.W 13:</u> Obtain the expression for the volume of a sphere of radius a [m] from the differential volume.

Ans.: $V = \frac{4}{3}\pi a^3$

<u>H.W 14:</u> Use the spherical coordinates system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius a [m] (Figure below). What results when $\alpha = 0$, $and\ \beta = \pi$.

Ans.: $2\pi a^2\,(\cos\alpha - \cos\beta)$ and $4\pi a^2$