Chapter One

Vector Analysis

CS309 & PM309

Third Year Class

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1.1 Scalars and Vectors:

A scalar is a quantity that has only magnitude. Quantities such as time, mass, distance, temperature, entropy, electric potential and population are scalars. Symbolically, a scalar is represented by either lower or upper case letters.

A vector is described by two quantities: a magnitude and a direction in space at any point and for any given time. Therefore, vectors may be space and time dependent. Vector quantities include velocity, force, displacement and electric field intensity.

Graphically, a vector is represented by directed line segment in the direction of the vector with its length proportional to its magnitude. Symbolically, a vector is represented by placing a bar over the letter symbol used for a given quantity, such as $\overline{A}\ and\ \overline{B}$, or by a letter in boldface type such as $\boldsymbol{A}$ and $\boldsymbol{B}$.

1.2 Vector Addition and Subtraction:

Two vectors $\overline{A}\ and\ \overline{B}$ can be added (subtracted) together to give another vector $\overline{C}$ ($\overline{D}$); i.e., $\bar{C} = \overline{A} + \bar{B}$; $\overline{D} = \overline{A} - \overline{B} = \overline{A} + (-\overline{B})$.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head to tail rule as portrayed in **Fig. 1.1** and **1.2**, respectively.

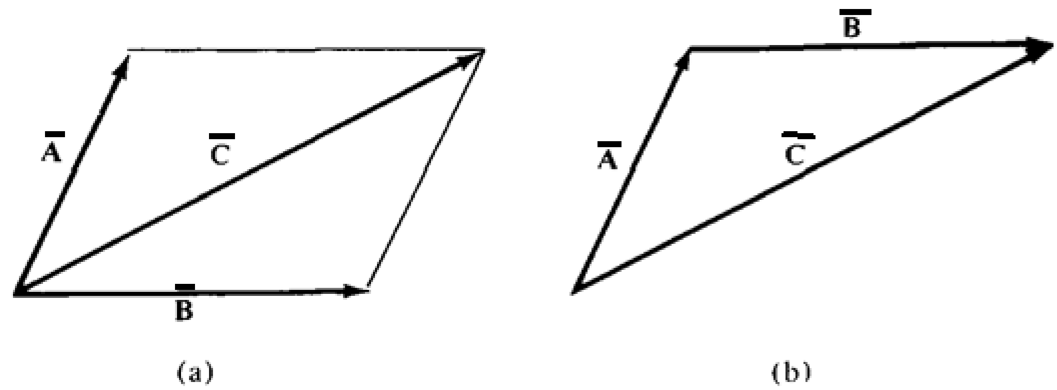


Fig. 1.1 Vector addition $\bar{C} = \bar{A} + \bar{B}$: (a) parallelogram rule, (b) head to tail rule

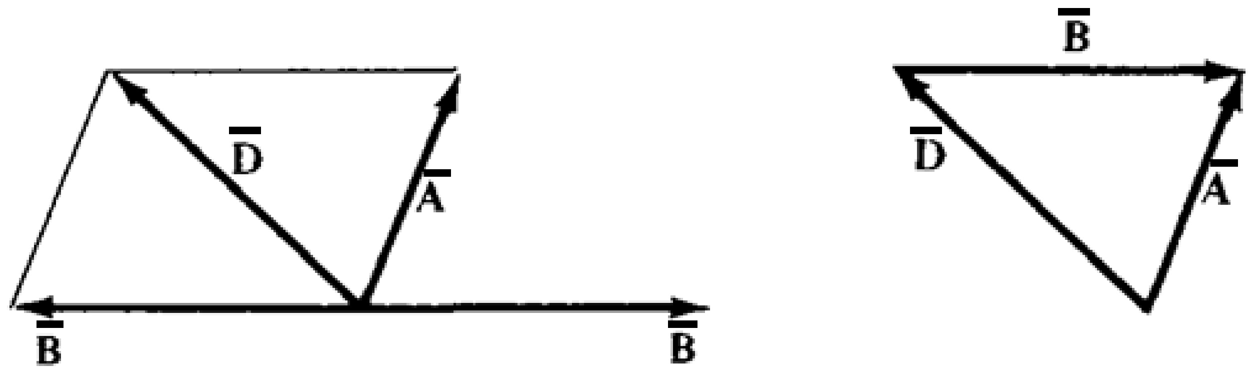


Fig. 1.2 Vector subtraction $\bar{D} = (\bar{A}) - \bar{B}$: (a) parallelogram rule, (b) head-to-tail rule.

The three basic laws of algebra obeyed by any given vectors $\bar{A}, \bar{B}$ *and* $\bar{C}$ are summarized as follows:

Law	Addition	Multiplication
Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$k\,\bar{A} = \bar{A}\,k$
Associative	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$k\,(L\bar{A}) = (kL)\,\bar{A}$
Distributive	$k\,(\bar{A} + \bar{B}) = k\bar{A} + k\bar{B}$	

Where k and L are scalars.

1.3 Products of Vectors

The multiplication of two vectors is called a product. two types of products based on the result obtained from the product. The first type is the scalar product. This is a product of two vectors which results in a scalar. The second is a vector product of two vectors, which results in a vector.

1.3.1 The Dot Product:

The dot product of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A} \,.\, \bar{B}$, is defined geometrically as the product of the magnitude of $\bar{A}$ and $\bar{B}$ and the cosine of the smaller angle between them.

Thus:

$$\bar{A} \,.\, \bar{B} = |\bar{A}||\bar{B}| \cos\theta_{AB}$$

If $\bar{A} = A_x \,\bar{a}x + A_y \,\bar{a}y + A_z \,\bar{a}z$ and $\bar{B} = B_x \,\bar{a}x + B_y \,\bar{a}y + B_z \,\bar{a}z$, then:

$$\bar{A} \,.\bar{B} = A_x B_x + A_y B_y + A_z B_z$$

Notes:

1- $\bar{A} \,.\, \bar{B} = \bar{B} \,.\, \bar{A}$ (Commutative Law)
2- $\bar{A} \,.(\bar{B} + \bar{C}) = \bar{A}.\bar{B} + \bar{A}.\bar{C}$ (Distributive Law)
3- $\bar{A} \,.\, \bar{A} = |\bar{A}|^2$
4- $\bar{a}x.\bar{a}y = \bar{a}y.\bar{a}z = \bar{a}x.\bar{a}z = 0$ and $\bar{a}x.\bar{a}x = \bar{a}y.\bar{a}y = \bar{a}z.\bar{a}z = 1$

A direct application of dot product is its use in determining the projection (or Component) of a vector in a given direction. The projection can be scalar or vector. Given a vector $\bar{A}$, we define the scalar projection A_B of $\bar{A}$ along $\bar{B}$ as [see **Fig. 1.3a**]

$$A_B = |\bar{A}| \cos\theta_{AB} = |\bar{A}||\bar{a}_B| \cos\theta_{AB}$$

Or $A_B = \bar{A} \,.\bar{a}_B$

The vector projection $\bar{A}_B$ of $\bar{A}$ along $\bar{B}$ is simply the scalar projection A_B multiplied by a unit vector along $\bar{B}$; is:

$$\bar{A}_B = A_B \,\bar{a}_B = (\bar{A} \,.\bar{a}_B) \,\bar{a}_B$$

Both the scalar and vector projections of $\bar{A}$ are illustrated in **Fig. 1.3**.

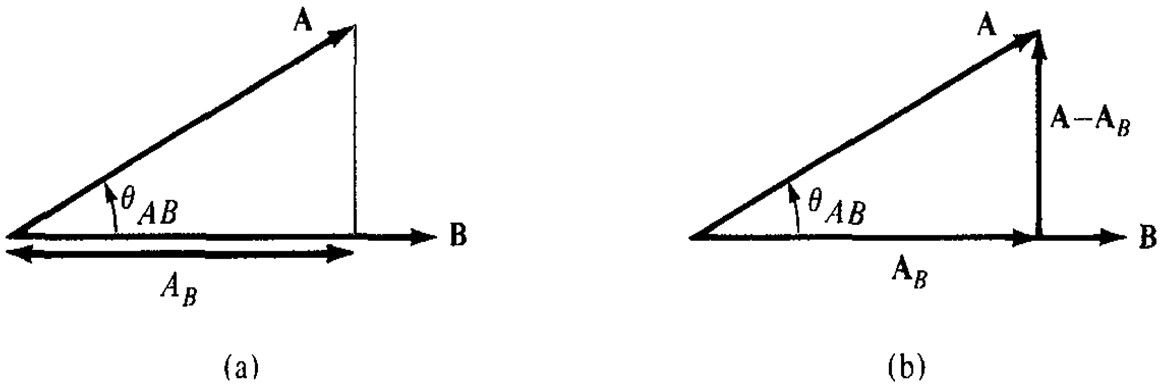


Fig. 1.3 Components of $\bar{A}$ along $\bar{B}$: **(a)** scalar component A_B ; **(b)** vector component $\bar{A}_B$.

<u>Example 1</u>: -

Given vectors $\bar{A} = 3\bar{a}x + 4\bar{a}y + \bar{a}z$ and $\bar{B} = 2\bar{a}y - 5\bar{a}z$. Find: (a) $\bar{A} \,.\, \bar{B}$; (b) θ_{AB} ; (c) The scalar component of $\bar{A}$ along $\bar{B}$; (d) The vector projection of $\bar{A}$ along $\bar{B}$.

Solution:

(a) $\bar{A} \,.\, \bar{B} = (3\bar{a}x + 4\bar{a}y + \bar{a}z).(2\bar{a}x - 5\bar{a}z) = 3(0) + 4(2) + 1(-5) = 3$

(b) $|\bar{A}| = \sqrt{9+16+1} = \sqrt{26}$ and $|\bar{B}| = \sqrt{0+4+25} = \sqrt{29}$

$$\bar{A} \,.\, \bar{B} = |\bar{A}||\bar{B}| \cos\theta_{AB} \implies \cos\theta_{AB} = \frac{\bar{A} \,.\, \bar{B}}{|\bar{A}||\bar{B}|} = \frac{3}{\sqrt{26}\sqrt{29}} = 0.1092$$

$$\therefore \ \theta_{AB} = \cos^{-1}(0.1092) = 83.73°$$

(c) $A_B = \bar{A} \,.\, \bar{a}_B = \dfrac{\bar{A} \,.\, \bar{B}}{|\bar{B}|} = \dfrac{3}{\sqrt{29}} = 0.557$

(d) $\bar{A}_B = (\bar{A} \,.\, \bar{a}_B)\, \bar{a}_B = 0.557\, \bar{a}_B = 0.557 \dfrac{\bar{B}}{|\bar{B}|} = \dfrac{0.557\,(2\bar{a}x - 5\bar{a}z)}{\sqrt{29}}$

$$\bar{A}_B = 0.207\bar{a}x - 0.517\bar{a}z$$

H.W 1:

Decompose the vector $\bar{A} = -2\bar{a}x + 3\bar{a}y + 5\bar{a}z$ on to vectors parallel and perpendicular to the vector $\bar{B} = \bar{a}x - 2\bar{a}y - 2\bar{a}z$.

Ans.: $-2\bar{a}x + 4\bar{a}y + 4\bar{a}z$; $-\bar{a}y + \bar{a}z$

1.3.2 The Cross Product:

The cross product of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A} \times \bar{B}$, is a vector quantity whose magnitude is the area of the parallelepiped formed by $\bar{A}$ and $\bar{B}$ (see **Fig. 1.4**) and is in the direction of advanced of right-handed screw as $\bar{A}$ is turned in to $\bar{B}$.

Thus: $\bar{A} \times \bar{B} = |\bar{A}||\bar{B}| \sin\theta_{AB}\ \bar{a}n$

Where $\bar{a}n$ is a unit vector normal to the plane containing $\bar{A}$ and $\bar{B}$. The direction of $\bar{a}n$ is taken as the direction of the right thumb when the fingers of the right hand rotate from $\bar{A}$ to $\bar{B}$ as shown in **Fig. 1.5a**. Alternatively, the direction of $\bar{a}n$ is taken as that of the advance of a right-handed screw as $\bar{A}$ is turned into $\bar{B}$ as shown in **Fig. 1.5b**. If $\bar{A} = A_x\,\bar{a}x + A_y\ \bar{a}y + A_z\,\bar{a}z$ and $\bar{B} = B_x\,\bar{a}x + B_y\ \bar{a}y + B_z\,\bar{a}z$, then:

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}x & \bar{a}y & \bar{a}z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(A_yB_z - B_yA_z\right)\bar{a}x - (A_xB_z\text{-}A_zB_x)\ \bar{a}y + \left(A_xB_y - A_yB_x\right)\bar{a}z$$

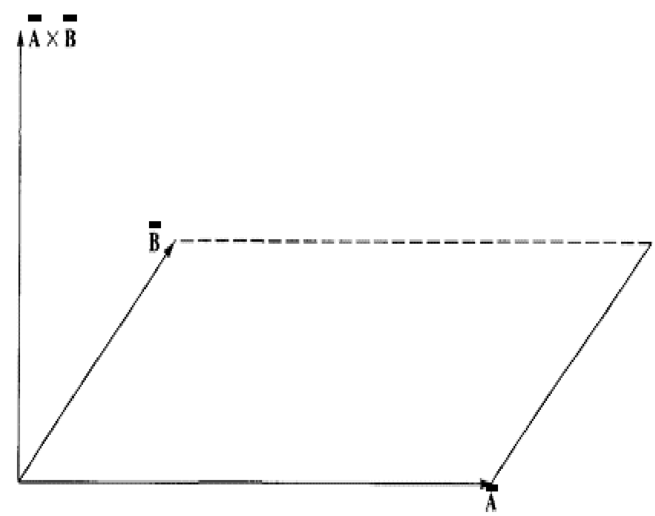


Fig. 1.4 The cross product of $\bar{A}$ and $\bar{B}$ is a vector with magnitude equal to the area of the parallelogram and direction as indicated.

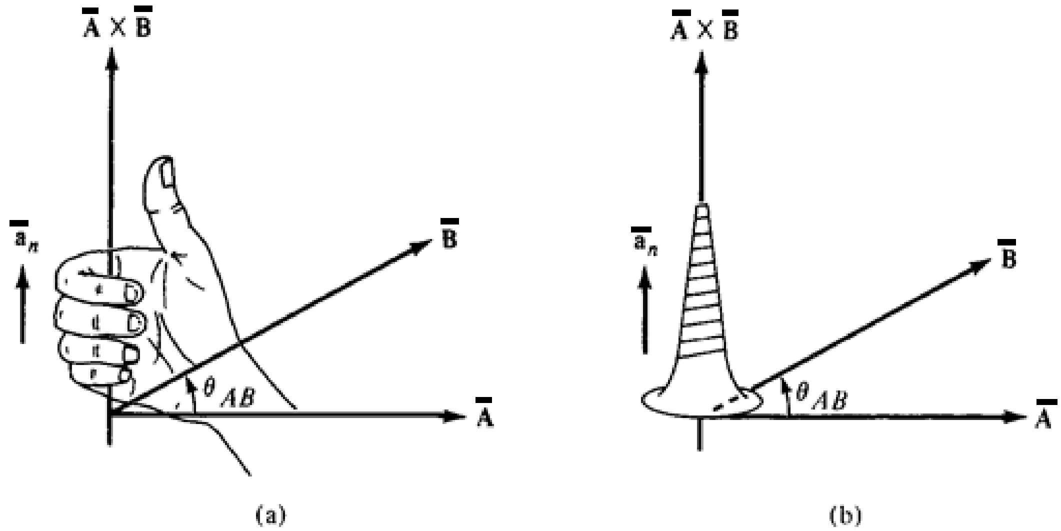


Fig. 1.5: Direction of $\bar{A} \times \bar{B}$ and $\bar{a}n$ using: **(a)** right-hand rule, **(b)** right-handed screw rule.

Notes:

1- $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$ (it is not commutative)

$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ (it is anti-commutative)

2- $\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C}$ (It is not associative)

3- $\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$ (It is distributive)

4- $\bar{A} \times \bar{A} = 0$

5- $\bar{a}x \times \bar{a}y = \bar{a}z$; $\bar{a}y \times \bar{a}z = \bar{a}x$; $\bar{a}x \times \bar{a}z = \bar{a}y$

6- $\bar{a}x \times \bar{a}x = \bar{a}y \times \bar{a}y = \bar{a}z \times \bar{a}z = 0$

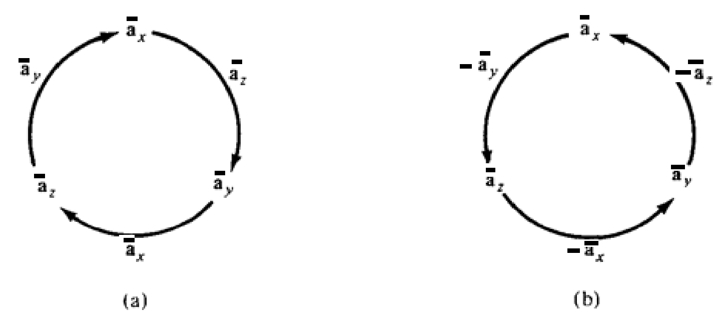


Fig. 1.6 Cross product using cyclic permutation: **(a)** moving clockwise leads to positive results: **(b)** moving counterclockwise leads to negative results.

Example 2: -

Points $P_1(1,2,3)$, $P_2(-5,2,0)$ and $P_3(2,7,-3)$ form a triangle in space. Calculate (a) The area of the triangle; (b) The unit vector perpendicular to the plane containing the triangle.

Solution:

$\bar{r}_{p1} = \bar{a}x + 2\bar{a}y + 3\bar{a}z$; $\bar{r}_{p2} = -5\bar{a}x + 2\bar{a}y$ and $\bar{r}_{p3} = 2\bar{a}x + 7\bar{a}y - 3\bar{a}z$

(a) $\bar{r}_{p1p2} = \bar{r}_{p2} - \bar{r}_{p1} = -6\bar{a}x - 3\bar{a}z$ and $\bar{r}_{p1p3} = \bar{r}_{p3} - \bar{r}_{p1} = \bar{a}x + 5\bar{a}y - 6\bar{a}z$

$$\bar{r}_{p1p2} \times \bar{r}_{p1p3} = \begin{vmatrix} \bar{a}x & \bar{a}y & \bar{a}z \\ -6 & 0 & -3 \\ 1 & 5 & -6 \end{vmatrix} = (0+15)\bar{a}x - (36+3)\,\bar{a}y + (-30-0)\bar{a}z$$

$\bar{r}_{p1p2} \times \bar{r}_{p1p3} = 15\bar{a}x - 39\bar{a}y - 30\bar{a}z$

Area of the triangle $= \frac{1}{2}\left|\bar{r}_{p1p2} \times \bar{r}_{p1p3}\right| = \frac{1}{2}\sqrt{15^2 + 39^2 + 30^2} = 25.72$

(b) $\bar{a}_n = \mp \dfrac{\bar{r}_{p1p2} \times \bar{r}_{p1p3}}{\left|\bar{r}_{p1p2} \times \bar{r}_{p1p3}\right|} = \mp \dfrac{15\bar{a}x - 39\bar{a}y - 30\bar{a}z}{51.44}$

$\therefore\ \bar{a}_n = \mp(0.291\bar{a}x - 0.758\bar{a}y - 0.583\bar{a}z)$

Example 3: -

The vertices of triangle are located at $P_1(4,1,-3)$, $P_2(-2,5,4)$ and $P_3(0,1,6)$. Find the three angles of the triangle.

Solution:

$\bar{r}_{p1} = 4\bar{a}x + \bar{a}y - 3\bar{a}z$; $\bar{r}_{p2} = -2\bar{a}x + 5\bar{a}y + 4\bar{a}z$ and $\bar{r}_{p3} = \bar{a}y + 6\bar{a}z$

Let $\bar{A} = \bar{r}_{p1p2} = \bar{r}_{p2} - \bar{r}_{p1} = -6\bar{a}x + 4\bar{a}y + 7\bar{a}$

$\bar{B} = \bar{r}_{p2p3} = \bar{r}_{p3} - \bar{r}_{p2} = 2\bar{a}x - 4\bar{a}y + 2\bar{a}z$

$\bar{C} = \bar{r}_{p3p1} = \bar{r}_{p1} - \bar{r}_{p3} = 4\bar{a}x - 9\bar{a}z$

Note that $\bar{A} + \bar{B} + \bar{C} = 0$

$$\bar{A}.\bar{B} = |\bar{A}||\bar{B}|\cos\alpha_1 \qquad \Rightarrow \quad \cos\alpha_1 = \frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|} = \frac{-12-16-14}{\sqrt{101}\sqrt{24}}$$

$$\therefore\ \alpha_1 = \cos^{-1}\frac{-14}{\sqrt{101}\sqrt{24}} = 106.52° \ \Rightarrow\ \theta_1 = 180 - \alpha_1 = 73.48°$$

$$\bar{B}.\bar{C} = |\bar{B}||\bar{C}|\cos\alpha_2 \qquad \Rightarrow \quad \cos\alpha_2 = \frac{\bar{B}.\bar{C}}{|\bar{B}||\bar{C}|} = \frac{8+0-18}{\sqrt{24}\sqrt{97}}$$

$$\therefore\ \alpha_2 = \cos^{-1}\frac{-10}{\sqrt{24}\sqrt{97}} = 101.96° \ \Rightarrow\ \theta_2 = 180 - \alpha_2 = 78.04°$$

$$\bar{C}.\bar{A} = |\bar{C}||\bar{A}|\cos\alpha_3 \qquad \Rightarrow \quad \cos\alpha_3 = \frac{\bar{C}.\bar{A}}{|\bar{C}||\bar{A}|} = \frac{-24+0-63}{\sqrt{97}\sqrt{101}}$$

$$\therefore\ \alpha_3 = \cos^{-1}\frac{-87}{\sqrt{97}\sqrt{101}} = 151.52° \ \Rightarrow\ \theta_3 = 180 - \alpha_3 = 28.48°$$

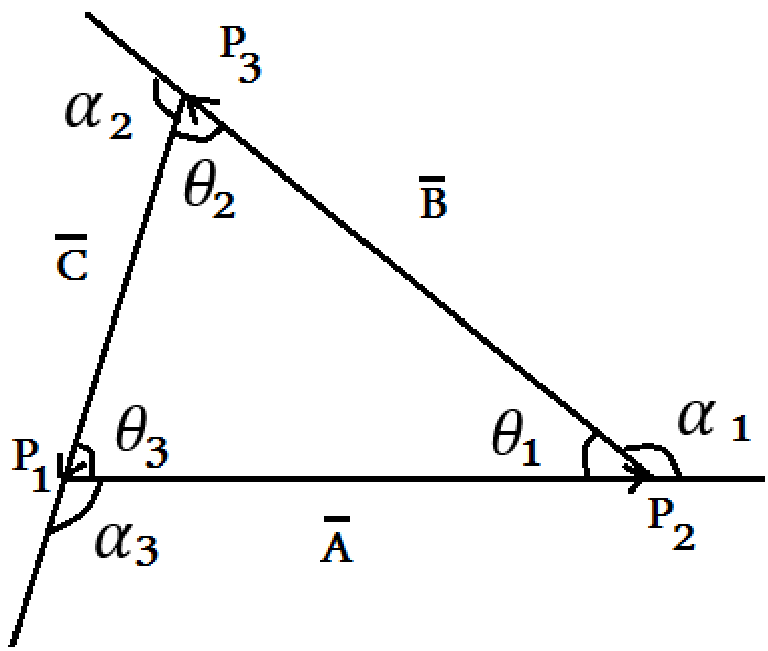


Fig. 1.7 for Example 3.

Note that $\theta_1 + \theta_2 + \theta_3 = 180°$

H.W 2: Show that vectors $\bar{A} = -5\bar{a}x - 3\bar{a}y - 3\bar{a}z$, $\bar{B} = \bar{a}x + 3\bar{a}y + 4\bar{a}z$ and $\bar{C} = 4\bar{a}x - \bar{a}z$ form the sides of a triangle. Is this a right-angle triangle? Calculate the area of the triangle.

Ans.: Yes; 10.5

H.W 3: Show that points $P_1(5,2,-4)$, $P_2(1,1,2)$ and $P_3(-3,0,8)$ all lie on a straight line. Determine the shortest distance between the line and point $P_4(3,-1,0)$.

Ans.: 2.426

1.4 Scalar and Vector Fields:

A field is a function that specifies a particular quantity everywhere in a region. If the quantity is scalar (or vector), the field is said to be a scalar (or vector) field. Examples of scalar fields are temperature distribution in a building, sound intensity in a theater, and electric potential in a region. The gravitational force on a body in space, the velocity of raindrops in the atmosphere, and the electric field intensity are examples of vector fields.

Example 4: -

A vector field $\bar{S}$ is expressed in Cartesian coordinates as:

$$\bar{S} = 125\,\frac{(x-1)\bar{a}x + (y-2)\bar{a}y + (z+1)\bar{a}z}{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

(a) Evaluate $\bar{S}$ at P(2,4,3). (b) Determine a unit vector that gives the direction of $\bar{S}$ at P. (c) Specify the surface f(x,y,z) on which $|\bar{S}| = 1$.

Solution:

(a) at P(2,4,3)

$$\Rightarrow \bar{S} = 125\,\frac{\bar{a}x + 2\bar{a}y + 4\bar{a}z}{1^2 + 2^2 + 4^2}$$

$$\therefore\ \bar{S} = 5.95\bar{a}x + 11.9\bar{a}y + 23.8\bar{a}z$$

(b) at P(2,4,3)

$$\Rightarrow \bar{a}_S = \frac{\bar{S}}{|\bar{S}|} = \frac{5.95\bar{a}x + 11.9\bar{a}y + 23.8\bar{a}z}{27.277}$$

$$\therefore\ \bar{a}_S = 0.218\bar{a}x + 0.436\bar{a}y + 0.873\bar{a}z$$

(c) $\because \bar{S} = 125 \dfrac{(x-1)\bar{a}x + (y-2)\bar{a}y + (z+1)\bar{a}z}{(x-1)^2 + (y-2)^2 + (z+1)^2}$

$$\therefore\ |\bar{S}| = \frac{125}{(x-1)^2 + (y-2)^2 + (z+1)^2}\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 1$$

$$\therefore |\bar{S}| = \frac{125}{\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}} = 1$$

$$\therefore\ \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$$

H.W 4: Two vector field are: $\bar{F} = -10\bar{a}x + 20x(y-1)\bar{a}y$ and $\bar{G} = 2x^2y\bar{a}x - 4\bar{a}y + z\bar{a}z$. For the point P(2,3,-4), find: (a) $|\bar{F}|$; (b) $|\bar{G}|$; (c) a unit vector in the direction of $\bar{F} - \bar{G}$; (d) a unit vector in the direction of $\bar{F} + \bar{G}$.

Ans.: 80.6 ; 24.7 ; $-0.37\bar{a}x + 0.92\bar{a}y + 0.04\bar{a}z$; $0.18\bar{a}x + 0.98\bar{a}y - 0.05\bar{a}z$

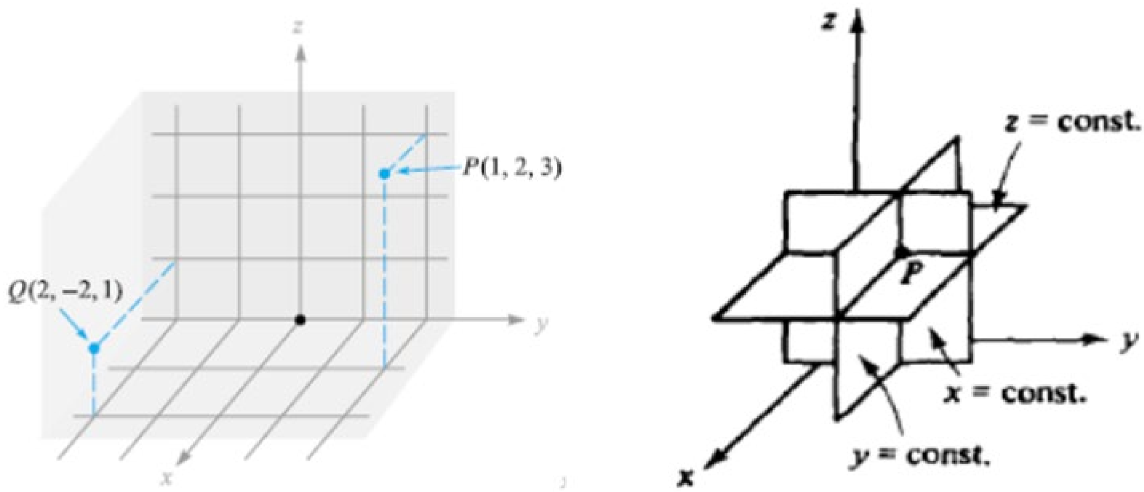
1.5 Systems of Coordinates

In this section, three orthogonal systems will be discussed which include: Cartesian, cylindrical, and the spherical system of coordinates.

1.5.1 Cartesian (Rectangular) Coordinates (x, y, z)

A point $P(x, y, z)$ in Cartesian coordinates is located by giving its x, y and z coordinates. **Fig. 1.8a** shows the points P and Q whose coordinates are (1, 2, 3) and (2,-2, 1), respectively. Intersection of three mutually perpendicular planes defines a point in Cartesian coordinates, and as shown in **Fig. 1.8b**.

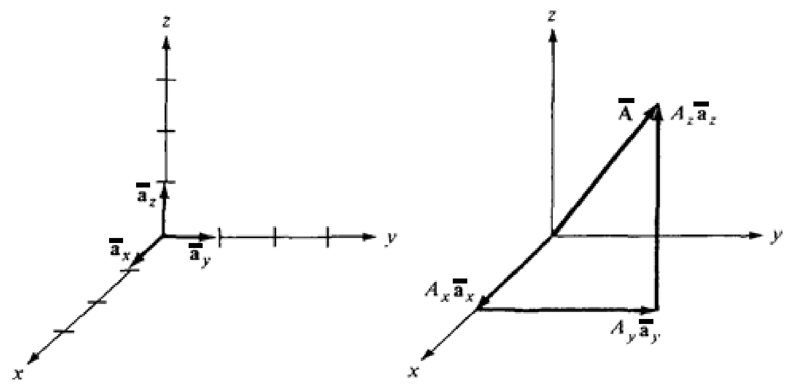
A vector $\bar{A}$ in Cartesian coordinates may be represented as: $\bar{A} = A_x\ \bar{a}x + A_y\ \bar{a}y + A_z\ \bar{a}z$, and shown in **Fig. 1.9** where A_x , A_y and A_z are called the components of $\bar{A}$ in the x, y and z directions respectively; $\bar{a}x$, $\bar{a}y$ and $\bar{a}z$ are unit vectors in the x, y and z directions, respectively.



(a) **(b)**

Fig. 1.8 (a) The Location of point P and Q. **(b)** The three mutually perpendicular planes of the Cartesian coordinate system.

Fig. 1.9 (a) Unit vectors $\bar{a}x$, $\bar{a}y$, and $\bar{a}z$, **(b)** components of $\bar{A}$ along $\bar{a}x$, $\bar{a}y$, and $\bar{a}z$



(a) (b)

Any vector can be written as:

$\bar{A} = |\bar{A}|\, \bar{a}_A$, where:

$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ The magnitude of the vector $\bar{A}$

$$\bar{a}_A = \frac{\bar{A}}{|\bar{A}|} = \frac{A_x\, \bar{a}x + A_y\, \bar{a}y + A_z\, \bar{a}z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Unit vector along the vector $\bar{A}$.

$|\bar{a}_A| = 1,$ $\bar{a}_A$ is a vector of unity magnitude.

If $\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$ and $\bar{B} = B_x\,\bar{a}x + B_y\,\bar{a}y + B_z\,\bar{a}z$, then:

■ $\bar{A} + \bar{B} = (A_x + B_x)\,\bar{a}x + (A_y + B_y)\,\bar{a}y + (A_z + B_z)\,\bar{a}z$

■ $\bar{A} - \bar{B} = (A_x - B_x)\,\bar{a}x + (A_y - B_y)\,\bar{a}y + (A_z - B_z)\,\bar{a}z$

***Position Vector*:**

The position vector $\bar{r}_p$ (or radius vector) of point P(x,y,z) is as the directed distance from the origin O to P; i. e.,

$\bar{r}_p = \overline{OP} = x\,\bar{a}x + y\,\bar{a}y + z\,\bar{a}z$

The position vector for point P is useful in defining its position in space. Point P(3,4,5), for example, and its position vector

$\bar{r}_p = \overline{OP} = 3\,\bar{a}x + 4\,\bar{a}y + 5\,\bar{a}z$, are shown in **Fig. 1.10a.**

***Distance Vector*:**

The distance vector is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), the distance vector (or separation vector) is the displacement from P to Q as shown in **Fig. 1.10b**; that is

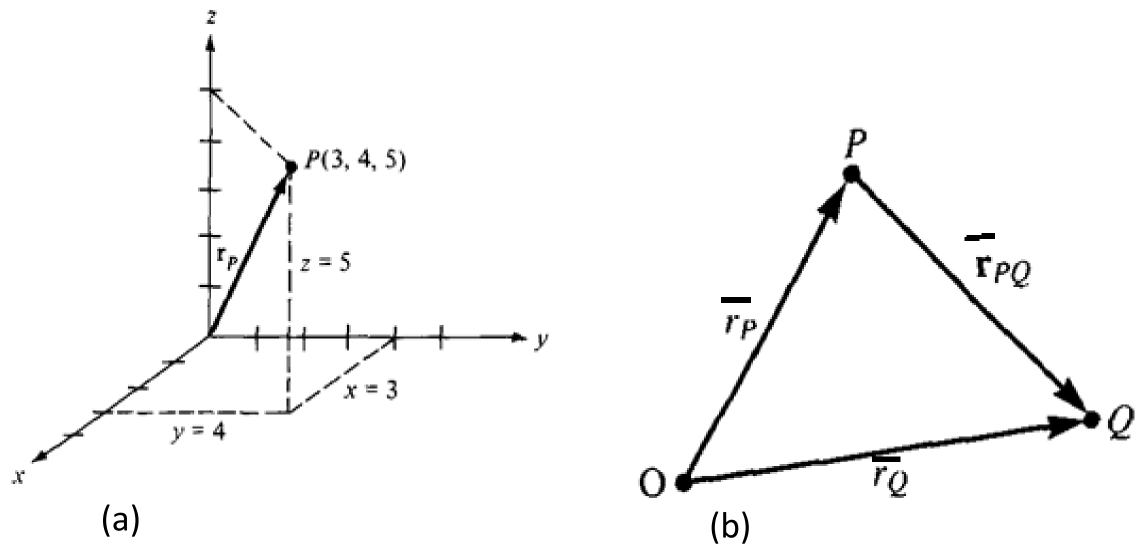


Fig. 1.10 (a) Illustration of position vector $\bar{r}_p = 3\ \bar{a}x + 4\ \bar{a}y + 5\ \bar{a}z$ (b) Distance vector $\bar{r}_{PQ}$.

$$\bar{r}_{PQ} = \bar{r}_Q - \bar{r}_P = (x_Q - x_P)\ \bar{a}x + (y_Q - y_P)\ \bar{a}y + (z_Q - z_P)\ \bar{a}z$$

The distance between the points P and Q is given by:

$$d = |\bar{r}_{PQ}| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$$

<u>Differential Length, Area and Volume in Cartesian Coordinates:</u>

From **Fig. 1.11**, we notice that:

1. Differential length is given by:

$d\bar{L} = dx\ \bar{a}x + dy\ \bar{a}y + dz\ \bar{a}z,$ Vector Quantity

$dL = \sqrt{dx^2 + dy^2 + dz^2}\,,$ Scalar Quantity

2. Differential normal area is given by:

$d\bar{s} = dy\ dz\ \bar{a}x$

$= dx\ dz\ \bar{a}y,$ Vector Quantity

$= dx\ dy\ \bar{a}z$

3. Differential volume is given by:

$dV = dx\, dy\, dz,$	Scalar Quantity

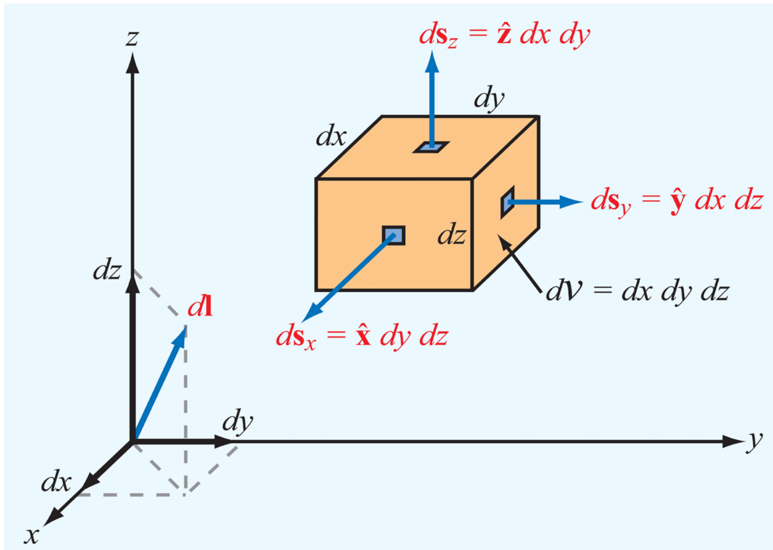


Fig. 1.11 Differential length, area, and volume in Cartesian coordinates.

__Example 5__: **-** Given the points M(2, -1,1) and T(-4, -2,6). Find: (a) the position vector for point M and T; (b) a unit vector from M to T; (c) the distance from M to T.

Solution:

(a) $\bar{r}_M = 2\,\bar{a}x - \bar{a}y + \bar{a}z$ and $\bar{r}_T = -4\,\bar{a}x - 2\,\bar{a}y + 6\,\bar{a}z$

(b) The vector from M to T is given by:

$$\bar{r}_{MT} = \bar{r}_T - \bar{r}_M = (-4-2)\bar{a}x + \big(-2-(-1)\big)\bar{a}y + (6-1)\bar{a}z = -6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z$$

$$\therefore \bar{a}_{r_{MT}} = \frac{\bar{r}_{MT}}{|\bar{r}_{MT}|} = \frac{-6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z}{\sqrt{(-6)^2 + (-1)^2 + (5)^2}} = \frac{-6\,\bar{a}x - \bar{a}y + 5\,\bar{a}z}{\sqrt{62}}$$

$$\therefore \bar{a}_{r_{MT}} = -0.762\,\bar{a}x - 0.127\,\bar{a}y + 0.635\,\bar{a}z$$

(c) The distance from M to T is given by:

$d = |\bar{r}_{MT}| = \sqrt{62} = 7.874$ [m]

<u>Example 6</u>: - Given vectors $\bar{A} = \bar{a}x + 3\,\bar{a}z$ and $\bar{B} = 5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z$, determine:

(a) $|\bar{A} + \bar{B}|$; (b) $5\bar{A} - \bar{B}$; (c) The component of $\bar{A}$ along $\bar{a}y$; (d) A unit vector along $3\bar{A} + \bar{B}$.

Solution:

(a) $\bar{A} + \bar{B} = (\bar{a}x + 3\,\bar{a}z) + (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z) = 6\,\bar{a}x + 2\,\bar{a}y - 3\,\bar{a}z$

$\therefore\ |\bar{A} + \bar{B}| = \sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36 + 4 + 9} = 7$

(b) $5\bar{A} - \bar{B} = 5(\bar{a}x + 3\,\bar{a}z) - (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z)$

$= (5\,\bar{a}x + 15\,\bar{a}z) - (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z)$

$\therefore\ 5\bar{A} - \bar{B} = -2\,\bar{a}y + 21\,\bar{a}z$

(c) The component of $\bar{A}$ along $\bar{a}y$ is $A_y = 0$

(d) Let $\bar{C} = 3\bar{A} + \bar{B} = 3\,(\bar{a}x + 3\,\bar{a}z) + (5\,\bar{a}x + 2\,\bar{a}y - 6\,\bar{a}z) = 8\,\bar{a}x + 2\,\bar{a}y + 3\,\bar{a}z$

$$\bar{a}_C = \frac{\bar{C}}{|\bar{C}|} = \frac{8\,\bar{a}x + 2\bar{a}y + 3\,\bar{a}z}{\sqrt{64 + 4 + 9}} = 0.9117\,\bar{a}x + 0.2279\,\bar{a}y + 0.3419\,\bar{a}z$$

<u>H.W 5:</u> Given points M(-1,2,1), N(3,-3,0) and P(-2,-3,-4), find:

(a) $\bar{r}_{MN}$; (b) $\bar{r}_{MN} + \bar{r}_{MP}$; (c) $|\bar{r}_M|$; (d) $\bar{a}_{r_{MP}}$; (e) $|2\,\bar{r}_P - 3\,\bar{r}_N|$.

Ans.: $4\,\bar{a}x - 5\,\bar{a}y - \bar{a}z$; $3\,\bar{a}x - 10\,\bar{a}y - 6\,\bar{a}z$; 2.45 ; $-0.14\,\bar{a}x - 0.7\,\bar{a}y - 0.7\,\bar{a}z$; 15.56

<u>H.W 6:</u> Express the unit vector directed toward the point P(1,-2,3) from an arbitrary point on the line described by $x = -3$, $y = 1$.

Ans.: $\dfrac{4\bar{a}x - 3\bar{a}y + (3 - z)\bar{a}z}{\sqrt{25 + (3 - z)^2}}$

1.5.2 Circular Cylindrical Coordinates:

The circular cylindrical coordinates system is very convenient whenever we are dealing with problems having cylindrical symmetry. A point P in cylindrical coordinates is represented as (r, ϕ, z) and is as shown in **Fig. 1.12**. r is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ is the angle measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$$0 \leq r \leq \infty, 0 \leq \phi \leq 2\pi, -\infty \leq z \leq \infty$$

Intersection of three surfaces defined by $r =$ constant, $\phi =$ constant and $z =$ constant is also a point in cylindrical coordinates, and is as shown in **Fig. 1.12.**

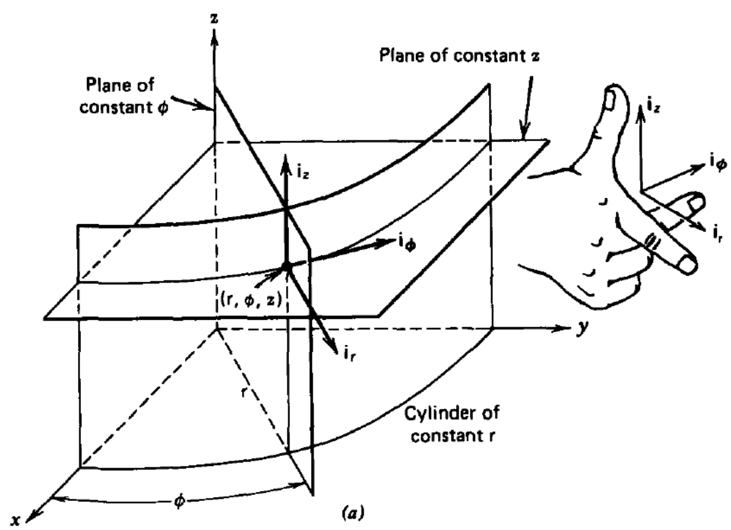


Fig. 1.12 Cylindrical coordinate system.

A vector $\bar{A}$ in cylindrical coordinates can be written as $\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$

where $\bar{a}_r$, $\bar{a}_\phi$ and $\bar{a}_z$ are unit vectors in the $r-$, $\phi -$ and $z -$directions.

The magnitude of $\bar{A}$ is:

$$|\bar{A}| = \sqrt{{A_r}^2 + {A_\phi}^2 + {A_z}^2}$$

Notice that the unit vectors $\bar{a}_r$, $\bar{a}_\phi$ and $\bar{a}_z$ are mutually perpendicular because our coordinates system is orthogonal; $\bar{a}_r$ points in the direction of increasing r, $\bar{a}_\phi$ points in the direction of increasing ϕ, and $\bar{a}_z$ in the positive z-direction. Thus,

$\bar{a}_r.\bar{a}_r = \bar{a}_\phi.\bar{a}_\phi = \bar{a}_z.\bar{a}_z = 1$

$\bar{a}_r.\bar{a}_\phi = \bar{a}_\phi.\bar{a}_z = \bar{a}_z.\bar{a}_r = 0$

$\bar{a}_r \times \bar{a}_r = \bar{a}_\phi \times \bar{a}_\phi = \bar{a}_z \times \bar{a}_z = 0$

$\bar{a}_r \times \bar{a}_\phi = \bar{a}_z$; $\bar{a}_\phi \times \bar{a}_z = \bar{a}_r$; $\bar{a}_z \times \bar{a}_r = \bar{a}_\phi$, see **Fig. 1.6** with replacing $(\bar{a}_x , \bar{a}_y , \bar{a}_z)$ with $(\bar{a}_r , \bar{a}_\phi , \bar{a}_z)$

If $\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$ and $\bar{B} = B_r\bar{a}_r + B_\phi\bar{a}_\phi + B_z\bar{a}_z$, then:

$\bar{A}.\bar{B} = A_r\, B_r + A_\phi\, B_\phi + A_z\, B_z$

And

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

Differential Length, Area, and Volume in Cylindrical Coordinates:

From **Fig. 1.13**, we notice that:

(1) Differential length is given by:

$d\bar{L} = dr\, \bar{a}r + rd\phi\ \bar{a}\phi + dz\, \bar{a}z,$ Vector Quantity

$dL = \sqrt{dr^2 + (rd\phi)^2 + dz^2}\,,$ Scalar Quantity

(2) Differential normal area is given by:

$d\bar{s} = r\, d\phi\, dz\, \bar{a}r$

$= dr\, dz\ \bar{a}\phi,$

$= rdr\, d\phi\, \bar{a}z$ Vector Quantity

(3) Differential volume is given by:

$dV = rdr\, d\phi\, dz$, Scalar Quantity

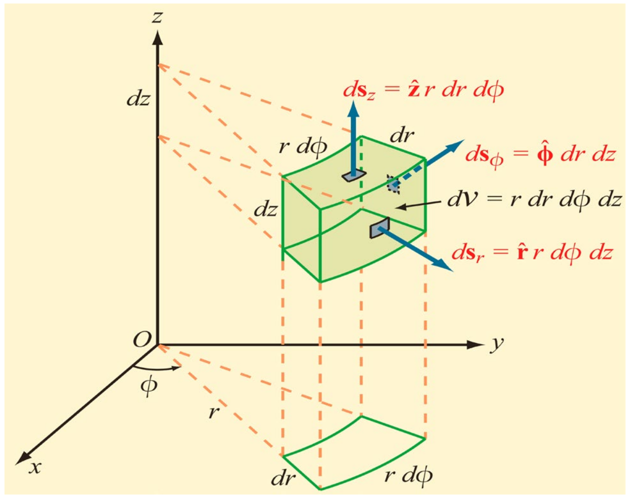


Fig. 1.13 Differential quantities in the cylindrical system.

The relationship between the variables $(x\,, y\,, z)$ of the Cartesian coordinates and those of the cylindrical system $(r\,, \phi\,, z)$ are illustrated in Fig. 1.14, and given by:

1- **From Cartesian To Cylindrical**:

$x = r\ \cos\phi$

$y = r\ \sin\phi$

$z = z$

2- **From Cylindrical To Cartesian**:

$r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

$z = z$

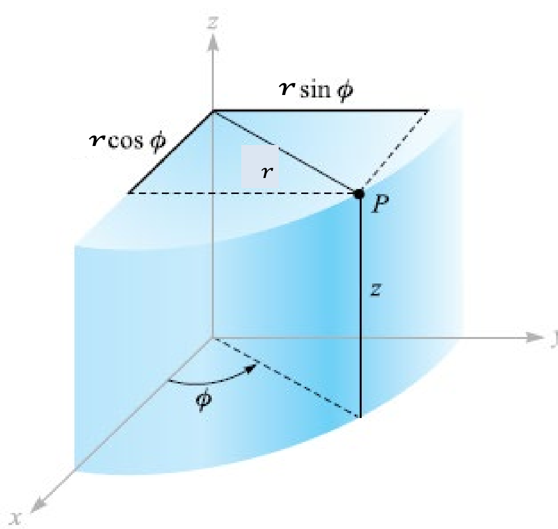


Fig. 1.14 The relationship between $(x\,, y\,, z)$ and $(r\,, \phi\,, z)$.

The dot product between $(\bar{a}_x\,, \bar{a}_y\,, \bar{a}_z)$ and $(\bar{a}_r\,, \bar{a}_\phi\,, \bar{a}_z)$ are obtained geometrically from **Fig. 1.15**:

$\bar{a}_x.\bar{a}_r = \cos\phi$

$\bar{a}_x.\bar{a}_\phi = -\cos(90° - \phi) = -\sin\phi$

$\bar{a}_x.\bar{a}_z = 0$

$\bar{a}_y.\bar{a}_r = \cos(90° - \phi) = \sin\phi$

$\bar{a}_y.\bar{a}_\phi = \cos\phi$

$\bar{a}_y.\bar{a}_z = 0$

Thus:

$\bar{a}_x = \cos\phi\,\bar{a}_r - \sin\phi\,\bar{a}_\phi$

$\bar{a}_y = \sin\phi\,\bar{a}_r + \cos\phi\,\bar{a}_\phi$

$\bar{a}_z = \bar{a}_z$

$\bar{a}_r = \cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y$

$\bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y$

$\bar{a}_z = \bar{a}_z$

$\alpha = 90^\circ - \phi$

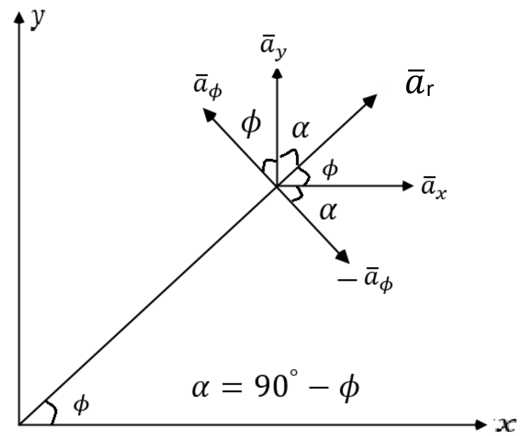


Fig. 1.15 Relationship between unit vectors of Cartesian and cylindrical coordinates.

The vector $\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$ can be transformed into cylindrical coordinates as:

$$A_r = \bar{A}.\bar{a}_r = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_r = A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \bar{A}.\bar{a}_\phi = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_z = \bar{A}.\bar{a}_z = (A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z).\bar{a}_z = A_z$$

The vector $\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$ can be transformed into Cartesian coordinates as:

$$A_x = \bar{A}.\bar{a}_x = (A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z).\bar{a}_x = A_r\cos\phi - A_\phi\sin\phi$$

$$A_y = \bar{A}.\bar{a}_y = (A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z).\bar{a}_y = A_r\sin\phi + A_\phi\cos\phi$$

$$A_z = \bar{A}.\bar{a}_z = (A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z).\bar{a}_z = A_z$$

Example 7: -

(a) Transform the vector $\bar{B} = y\bar{a}x - x\bar{a}y + z\bar{a}z$ into cylindrical coordinates.
(b) Express the vector filed $\bar{S} = \cos\phi\,\bar{a}_r + \sin\phi\,\bar{a}_\phi$ in Cartesian coordinates.
(c) Find at P(1, 2, -2) the vector projection of $\bar{B}$ in the direction of $\bar{S}$.

Solution:

(a) $B_r = \bar{B}.\bar{a}_r = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_r = y\ \cos\phi - x\ \sin\phi$

$\because\ x = r\ \cos\phi\ \ and\ y = \ r\ \sin\phi$

$B_r = r\ \sin\phi\ \cos\phi - r\ \cos\phi\ \sin\phi = 0$

$B_\phi = \bar{B}.\bar{a}_\phi = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_\phi = -y\ \sin\phi - x\cos\phi$

$\therefore B_\phi = -r\sin^2\phi - r\cos^2\phi = -r$

$B_z = \bar{B}.\bar{a}_z = (y\bar{a}x - x\bar{a}y + z\bar{a}z).\bar{a}_z = z$

$\therefore\ \bar{B} = -r\ \bar{a}_\phi + z\ \bar{a}_z$ in cylindrical coordinates

(b) $S_x = \bar{S}.\bar{a}_x = (\cos\phi\,\bar{a}_r + \sin\phi\,\bar{a}_\phi).\bar{a}_x = \cos^2\phi - \sin^2\phi$

$$\because\ \cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}\ , \quad \sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore\ S_x = \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$S_y = \bar{S}.\bar{a}_y = (\cos\phi\,\bar{a}_r + \sin\phi\,\bar{a}_\phi).\bar{a}_y = \cos\phi\sin\phi + \sin\phi\cos\phi = 2\cos\phi\sin\phi$

$$\therefore\ S_y = 2\frac{x}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$$

$S_z = \bar{S}.\bar{a}_z = (\cos\phi\,\bar{a}_r + \sin\phi\,\bar{a}_\phi).\bar{a}_z = 0$

$$\bar{S} = \frac{x^2-y^2}{x^2+y^2}\bar{a}_x + \frac{2xy}{x^2+y^2}\bar{a}_y \qquad \text{in Cartesian Coordinates}$$

(c) $\because \bar{B} = y\bar{a}x - x\bar{a}y + z\bar{a}z$

$\therefore \bar{B} = 2\bar{a}x - \bar{a}y - 2\bar{a}z$

$$\because \bar{S} = \frac{x^2 - y^2}{x^2 + y^2}\bar{a}_x + \frac{2xy}{x^2 + y^2}\bar{a}_y$$

$$\therefore \bar{S} = \frac{1-4}{1+4}\bar{a}_x + \frac{2(1)(2)}{1+4}\bar{a}_y = -0.6\bar{a}_x + 0.8\bar{a}_y$$

$$\therefore \bar{B}_s = (\bar{B}.\bar{a}_s)\bar{a}_s = \frac{\bar{B}.\bar{S}}{|\bar{S}|^2}\bar{S}$$

$$\therefore \bar{B}_s = \frac{(2\bar{a}x - \bar{a}y - 2\bar{a}z).(-0.6\bar{a}_x + 0.8\bar{a}_y)}{(0.6^2 + 0.8^2)}(-0.6\bar{a}_x + 0.8\bar{a}_y)$$

$$\therefore \bar{B}_s = \frac{-1.2 - 0.8}{1}(-0.6\bar{a}_x + 0.8\bar{a}_y) = 1.2\bar{a}_x + 1.6\bar{a}_y$$

H.W 7: Transform

$\bar{A} = \dfrac{-xy\bar{a}_x + x^2\bar{a}_y + y^2\bar{a}_z}{x^2 + y^2}$ from Cartesian to cylindrical coordinates.

Ans.: $\bar{A} = \cos\phi\bar{a}_\phi + \sin^2\phi\,\bar{a}_z$

H.W 8: Express the field $\bar{E} = \sin\phi\bar{a}_r + \cos^2\phi\,\bar{a}_z$ In Cartesian coordinates.

Ans.: $\bar{E} = \dfrac{xy\bar{a}_x + y^2\bar{a}_y + x^2\bar{a}_z}{x^2 + y^2}$

H.W 9: Decompose the vector $\bar{A} = 2\bar{a}_x - \bar{a}_y + 5\bar{a}_z$ into vectors parallel and perpendicular to the cylinder $r = 1$ at point P(1,30°, 0).

Ans.: $\bar{A}_T = -1.866\bar{a}_\phi + 5\bar{a}_z$ and $\bar{A}_N = 1.232\bar{a}_r$

1.5.3 Spherical Coordinates System:

The spherical coordinates system is most appropriate when dealing with problems having of spherical symmetry. A point P can be represented as $P(r,\theta,\phi)$ and illustrated in **Fig. 1.16a**, R is defined as the distance from the origin to point P or the radius of sphere centered at the origin and passing through P; θ is the angle between the z-axis and the position vector of P; ϕ is measured from the x-axis (ϕ is the same as in the cylindrical coordinates). According to these definitions, the ranges of the variables are:

$$0 \le R \le \infty, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$$

Intersection of three orthogonal surfaces defined by $R = constant$, $\theta = constant$ and $\phi = constant$ is also a point in spherical coordinates, and is shown in Fig. **1.16b.**

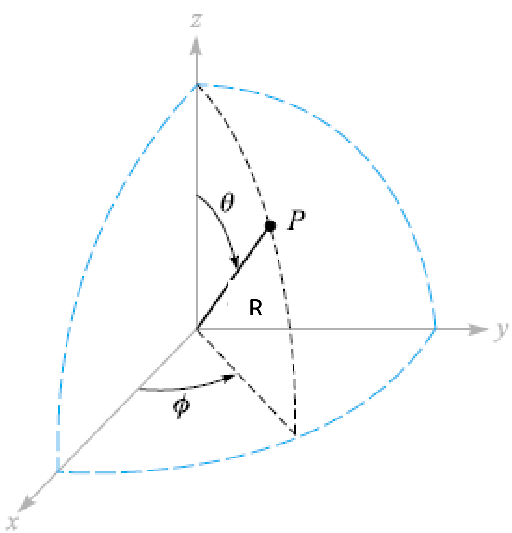


Fig. 1.16 (a) The three spherical coordinates.

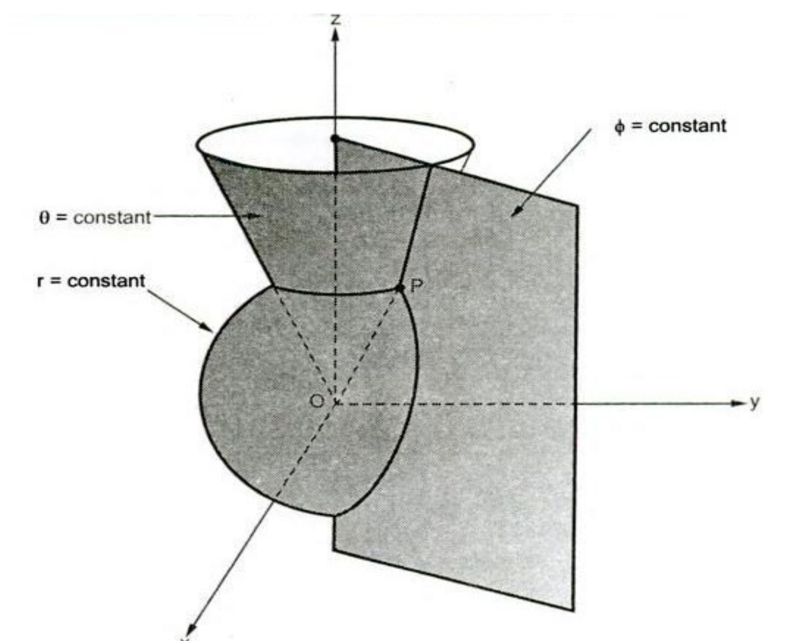


Fig. 1.16 (b) Point P as intersection of three surfaces.

A vector $\bar{A}$ in spherical coordinates can be written as:

$\bar{A} = A_R\bar{a}_R + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$

where $\bar{a}_R$, $\bar{a}_\theta$, $\bar{a}_\phi$ are unit vectors along the $R-, \theta-,$ $and\ \phi -$ directions as illustrated in Fig. 1.17 the magnitude of $\bar{A}$ is:

$$|\bar{A}| = \sqrt{{A_R}^2 + {A_\theta}^2 + {A_\phi}^2}$$

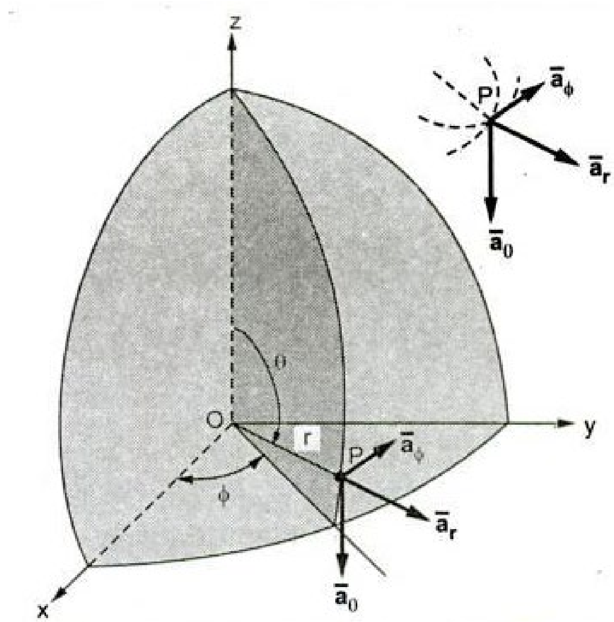


Fig. 1.17 The three unit vectors for spherical coordinates.

The unit vectors $\bar{a}_R$, $\bar{a}_\theta$ and $\bar{a}_\phi$ are mutually orthogonal; $\bar{a}_r$ being directed along the radius or points in the direction of increasing r, $\bar{a}_\theta$ points in the direction of increasing θ, and $\bar{a}_\phi$ in the direction of increasing ϕ. Thus,

$\bar{a}_R.\bar{a}_R = \bar{a}_\theta.\bar{a}_\theta = \bar{a}_\phi.\bar{a}_\phi = 1$

$\bar{a}_R.\bar{a}_\theta = \bar{a}_\theta.\bar{a}_\phi = \bar{a}_\phi.\bar{a}_R = 0$

$\bar{a}_R \times \bar{a}_R = \bar{a}_\theta \times \bar{a}_\theta = \bar{a}_\phi \times \bar{a}_\phi = 0$

$\bar{a}_R \times \bar{a}_\theta = \bar{a}_\phi$; $\bar{a}_\theta \times \bar{a}_\phi = \bar{a}_R$; $\bar{a}_\phi \times \bar{a}_R = \bar{a}_\theta$, see Fig. 1.12 with replacing

$(\bar{a}_x , \bar{a}_y , \bar{a}_z)$ with $(\bar{a}_R , \bar{a}_\theta , \bar{a}_\phi)$.

If $\bar{A} = A_R\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$ and $\bar{B} = B_R\bar{a}_R + B_\theta\bar{a}_\theta + B_\phi\bar{a}_\phi$, then:

$\bar{A}.\bar{B} = A_R\, B_R + A_\theta\, B_\theta + A_\phi\, B_\phi$

and

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_R & \bar{a}_\theta & \bar{a}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$$

Differential Length, Area, and Volume in Cylindrical Coordinates:

From Fig. 1.18, we notice that:

(1) Differential length is given by:

$$d\bar{L} = dR\ \bar{a}R + Rd\theta\bar{a}\theta + R\sin\theta\ d\phi\ \bar{a}\phi, \qquad \text{Vector Quantity}$$

$$dL = \sqrt{dR^2 + (Rd\theta)^2 + (R\sin\theta\ d\phi)^2}, \qquad \text{Scalar Quantity}$$

(2) Differential normal area is given by (Fig.1.19):

$$d\bar{s} = R^2\sin\theta\ d\theta\ d\phi\ \bar{a}r$$

$$= R\sin\theta\ dR\ d\phi\ \ \bar{a}\theta, \qquad \text{Vector Quantity}$$

$$= RdR\ d\theta\ \bar{a}\phi$$

(3) Differential volume is given by:

$$dV = R^2\sin\theta\ dr\ d\theta\ d\phi\ , \qquad \text{Scalar Quantity}$$

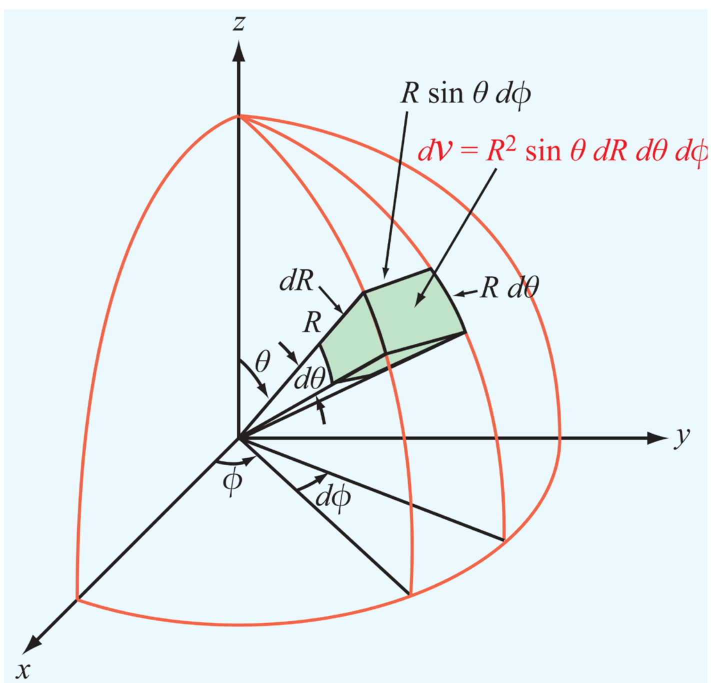


Fig. 1.18 Differential elements in spherical coordinates.

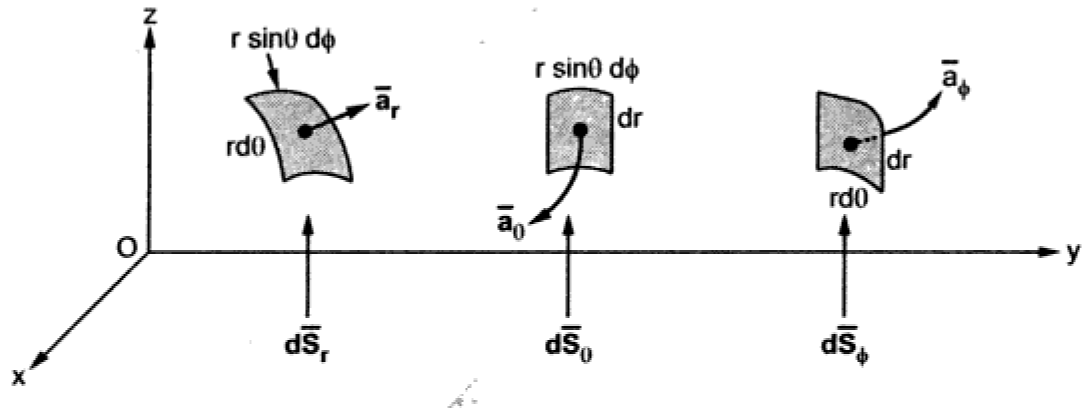


Fig. 1.19 Differential normal areas in spherical coordinates.

The space variables $(x\,, y\,, z)$ of the Cartesian coordinates can be related to variables $(R\,, \theta\,, \phi)$ of a spherical coordinates system. From Fig. 1.20, it is easy to notice that:

<u>1- From Cartesian To Spherical</u>:

$$x = R\ \sin\theta \cos\phi$$

$$y = R\ \sin\theta \sin\phi$$

$$z = R\ \cos\theta$$

<u>2- From Spherical To Cartesian</u>

$$R = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

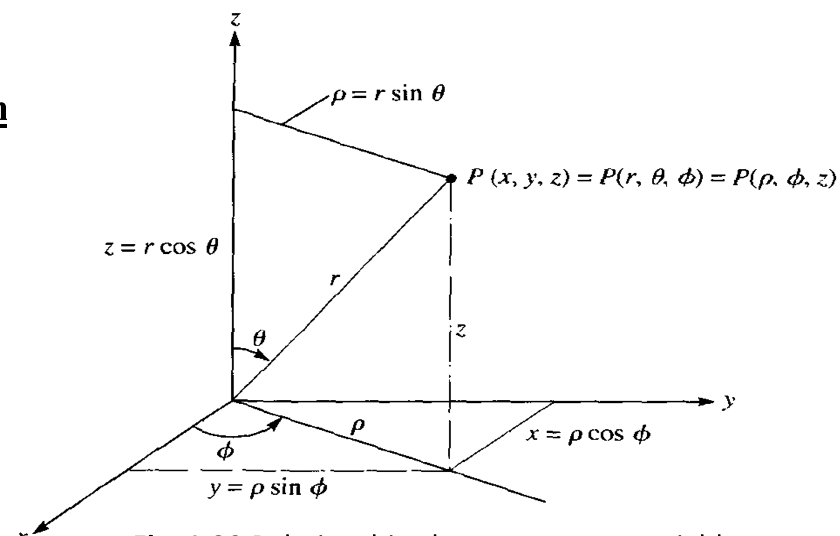


Fig. 1.20 Relationships between space variables $(x\,, y\,, z)$ and $(r\,, \theta\,, \phi)$.

The dot product between $(\bar{a}_x\,, \bar{a}_y\,, \bar{a}_z)$ and $(\bar{a}_R\,, \bar{a}_\theta\,, \bar{a}_\phi)$ are obtained geometrically from **Fig. 1.21**:

$$\bar{a}_x.\bar{a}_R = \bar{a}_x.(\cos(90-\theta)\bar{a}_r + \cos\theta\,\bar{a}_z) = \bar{a}_x.(\sin\theta\,\bar{a}_r + \cos\theta\,\bar{a}_z) = \sin\theta\cos\phi$$

$\bar{a}_x.\bar{a}_\theta = \bar{a}_x.(\cos\theta\bar{a}_r - \cos(90-\theta)\,\bar{a}_z) = \bar{a}_x.(\cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_z) = \cos\theta\cos\phi$

$\bar{a}_x.\bar{a}_\phi = -\sin\phi$

$\bar{a}_y.\bar{a}_R = \bar{a}_y.(\sin\theta\,\bar{a}_r + \cos\theta\,\bar{a}_z) = \sin\theta\sin\phi$

$\bar{a}_y.\bar{a}_\theta = \bar{a}_y.(\cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_z) = \cos\theta\sin\phi$

$\bar{a}_y.\bar{a}_\phi = \cos\phi$

$\bar{a}_z.\bar{a}_R = \bar{a}_z.(\sin\theta\,\bar{a}_r + \cos\theta\,\bar{a}_z) = \cos\theta$

$\bar{a}_z.\bar{a}_\theta = \bar{a}_z.(\cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_z) = -\sin\theta$

$\bar{a}_z.\bar{a}_\phi = 0$

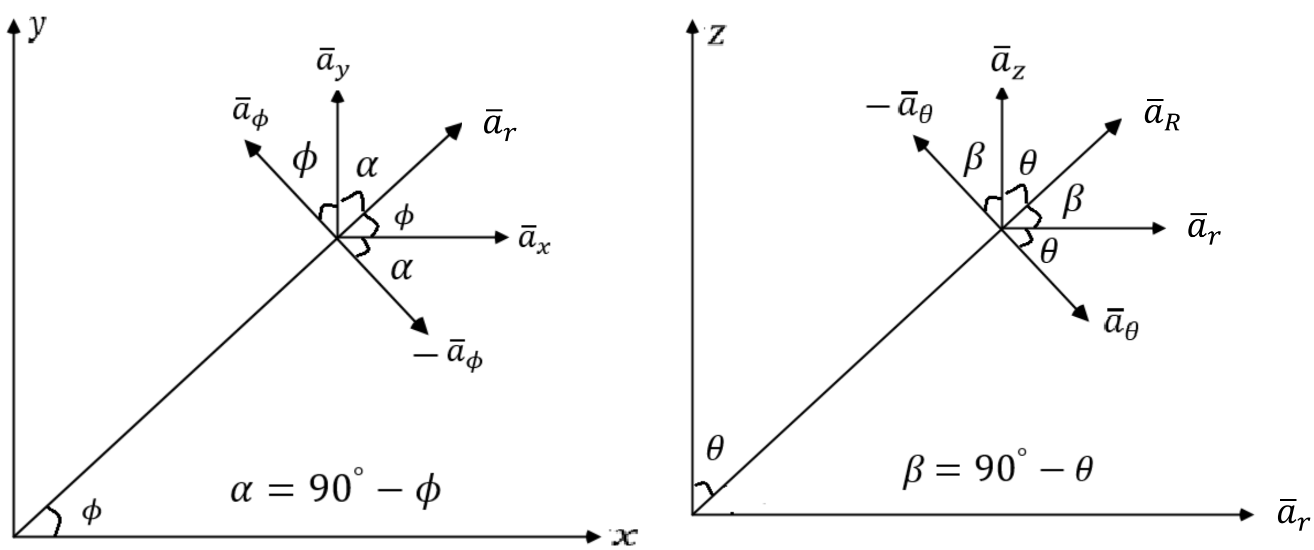


Fig. 1.21 Relationship between the unit vectors of three coordinate systems.

The vector $\bar{A} = A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z$ can be transformed into spherical coordinates as:

$$A_R = \bar{A}.\bar{a}_R = \left(A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z\right).\bar{a}_R$$
$$= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$$

$$A_\theta = \bar{A}.\bar{a}_\theta = \left(A_x\,\bar{a}x + A_y\,\bar{a}y + A_z\,\bar{a}z\right).\bar{a}_\theta$$
$$= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$A_\phi = \bar{A}.\bar{a}_\phi = (A_x\ \bar{a}x + A_y\ \ \bar{a}y + A_z\ \bar{a}z).\bar{a}_\phi = -A_x\ \sin\phi + A_y \cos\phi$

The vector $\bar{A} = A_R\bar{a}_R + A_\phi\bar{a}_\phi + A_z\bar{a}_z$ can be transformed into Cartesian coordinates as:

$$A_x = \bar{A}.\bar{a}_x = (A_R\bar{a}_R + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi).\bar{a}_x$$
$$= A_R \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_y = \bar{A}.\bar{a}_y = (A_R\bar{a}_R + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi).\bar{a}_y$$
$$= A_R \sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_z = \bar{A}.\bar{a}_z = (A_R\bar{a}_R + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi).\bar{a}_z = A_R\cos\theta - A_\theta\sin\theta$$

Example 8: -

A vector field is given by:

$$\bar{D} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}\left[(x-y)\bar{a}_x + (x+y)\bar{a}_y\right]$$

Express this field in spherical coordinates.

Solution:

$R = \sqrt{x^2+y^2+z^2}\,,\qquad r = R\sin\theta = \sqrt{x^2+y^2}$

$x = R\ \sin\theta\cos\phi\ \ ,\ \ y = R\ \sin\theta\sin\phi$

$$\therefore\ \bar{D} = \frac{R}{R\ \sin\theta}\big[(R\ \sin\theta\cos\phi - R\ \sin\theta\sin\phi)\bar{a}_x$$
$$+ (R\ \sin\theta\cos\phi + R\ \sin\theta\sin\phi)\bar{a}_y\big]$$

$$\therefore \bar{D} = R\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right]$$

$$D_r = \bar{D}.\bar{a}_R = R\left[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\right].\bar{a}_R$$

$$= R[(\cos\phi - \sin\phi)\sin\theta\cos\phi + (\cos\phi + \sin\phi)\sin\theta\sin\phi]$$

$$= R\sin\theta\,[\cos^2\phi - \sin\phi\cos\phi + \cos\phi\sin\phi + \sin^2\phi] = R\sin\theta$$

$$\therefore D_r = R\sin\theta$$

$D_\theta = \bar{D}.\bar{a}_\theta = R\big[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\big].\bar{a}_\theta$

$= R[(\cos\phi - \sin\phi)\cos\theta\cos\phi + (\cos\phi + \sin\phi)\cos\theta\sin\phi]$

$= R\cos\theta\,[\cos^2\phi - \sin\phi\cos\phi + \cos\phi\sin\phi + \sin^2\phi] = R\cos\theta$

$\therefore D_\theta = R\cos\theta$

$D_\phi = \bar{D}.\bar{a}_\phi = R\big[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y\big].\bar{a}_\phi$

$= R[-(\cos\phi - \sin\phi)\sin\phi + (\cos\phi + \sin\phi)\cos\phi]$

$= R[-\cos\phi\sin\phi + \sin^2\phi + \cos^2\phi + \sin\phi\cos\phi] = R$

$\therefore D_\phi = R$

$\therefore \bar{D} = R\sin\theta\,\bar{a}_R + R\cos\theta\,\bar{a}_\theta + R\bar{a}_\phi$

Example 9: -

Given vectors $\bar{A} = 2\bar{a}_x - \bar{a}_y + 5\bar{a}_z$ and $\bar{B} = 4\bar{a}_\theta$, find the angle between $\bar{A}$ and $\bar{B}$ at P(1, 15°, 50°).

Solution:

$B_x = \bar{B}.\bar{a}_x = 4\bar{a}_\theta.\bar{a}_x = 4\ \cos\theta\cos\phi$

$B_y = \bar{B}.\bar{a}_y = 4\bar{a}_\theta.\bar{a}_y = 4\ \cos\theta\sin\phi$

$B_z = \bar{B}.\bar{a}_z = 4\bar{a}_\theta.\bar{a}_z = -4\ \sin\theta$

$\therefore \bar{B} = 4\ \cos\theta\cos\phi\,\bar{a}_x + 4\ \cos\theta\sin\phi\,\bar{a}_y - 4\ \sin\theta\ \bar{a}_z$

At P(1, 15°, 50°),

$\bar{B} = 2.4835\bar{a}_x + 2.9597\bar{a}_y - 1.0352\,\bar{a}_z$

$\bar{A}.\bar{B} = (2\bar{a}_x - \bar{a}_y + 5\bar{a}_z).(2.4835\bar{a}_x + 2.9597\bar{a}_y - 1.0352\,\bar{a}_z) = -3.1687$

$|\bar{A}| = \sqrt{2^2 + 1^2 + 5^2} = 5.4772$ and $|\bar{B}| = 4$

$\because \bar{A}.\bar{B} = |\bar{A}||\bar{B}|\cos\theta_{AB}$

$$\therefore\ \theta_{AB} = \cos^{-1}\left[\frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|}\right] = \cos^{-1}\left[\frac{-3.1687}{5.4772 * 4}\right] = \cos^{-1}[-0.1446]$$

$$\therefore\ \theta_{AB} = 98.31^\circ$$

Example 10: -

A spherical region is defined by: $1 \le R \le\ 3, 15^\circ \le \theta \le\ 60^\circ$, and $10^\circ \le \phi \le 80^\circ$

Find the volume V.

Solution:

$$V = \iiint_v dv = \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\int_{r=1}^{3} R^2 \sin\theta\, dR\, d\theta\, d\phi = \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\left(\frac{r^3}{3}\right)_1^3 \sin\theta\ d\theta\, d\phi$$

$$= \int_{\phi=10^\circ}^{80^\circ}\int_{\theta=15^\circ}^{60^\circ}\frac{26}{3}\sin\theta\ d\theta\, d\phi = \int_{\phi=10^\circ}^{80^\circ}\frac{26}{3}(-\cos\theta)_{15^\circ}^{60^\circ}\, d\phi = 4.038\int_{\phi=10^\circ}^{80^\circ} d\phi$$

$$= 4.038\ (\phi)|_{10}^{80} =\ 4.038\ (80-10) * \frac{\pi}{180} = 4.9333\ \ \text{Unit}^3$$

Example 11: -

Find the area of the surface defined by:

$$\theta = 45^\circ, \qquad 3 \le R \le\ 5,\ \text{ and }\ 0.1\pi \le \phi \le \pi$$

Solution:

$$S = \iint_s ds = \int_{\phi=0.1\pi}^{\pi}\int_{r=3}^{5}(dR)(R\sin\theta\ d\phi) = \int_{\phi=0.1\pi}^{\pi}\int_{r=3}^{5} R\sin 45^\circ\ dr\ d\phi =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{R^2}{2}\right)_3^5 (\phi)_{0.1\pi}^{\pi} = \frac{1}{\sqrt{2}}\left(\frac{25-9}{2}\right)(0.9\pi) = 15.9943\ \text{Unit}^2$$

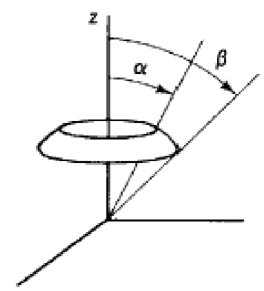
H.W 10: Find the angle between vector $\bar{A} = \bar{a}_x + 3\bar{a}_y + 2\bar{a}_z$ and the sphere R= 1 at the point P(1,20°, 30°).
Ans.: 45°.93

H.W 11: Prove that the field $\bar{A} = \sin\theta\, \bar{a}_\theta$ in Cartesian coordinates is given by:

$$\bar{A} = \frac{xz\bar{a}_x + yz\bar{a}_y - (x^2 + y^2)\bar{a}_z}{x^2 + y^2 + z^2}$$

H.W 12: Obtain the expression for the volume of a sphere of radius a [m] from the differential volume.
Ans.: $V = \frac{4}{3}\pi a^3$

H.W 13: Use the spherical coordinates system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius a [m] (Figure below). What results when $\alpha = 0,\ and\ \beta = \pi$.
Ans.: $2\pi a^2\ (\cos\alpha - \cos\beta)$ and $4\pi a^2$



1.6 Integration of Vector Functions:

1.6.1 Line Integral:

Consider a vector field $\bar{A}$ as shown in **Fig. 1.22** and an arbitrary path C. The line integral of the vector $\bar{A}$ over the path C is written as:

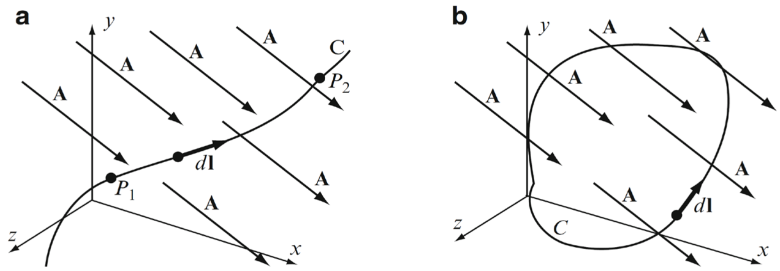
$$Q = \int_c \bar{A}.\overline{dl} = \int_c |\bar{A}||\overline{dl}| \cos\theta_{\overline{A},\overline{dl}}$$

$$\int_{p_1}^{p_2} \bar{A}.\bar{d}l = \int_{p_1}^{p_2} |A||\bar{d}l| \cos\theta_{\overline{A},\overline{dl}}$$

closed contour integral or a *loop integral*

$$\int_c \bar{A}.\bar{d}l = \int_c |\bar{A}||\overline{dl}| \cos\theta_{\overline{A},\overline{dl}}$$

$$\oint \bar{A}\cdot\bar{d}l = \oint |\bar{A}||\bar{d}l| \cos\theta_{\overline{A},\overline{dl}}$$



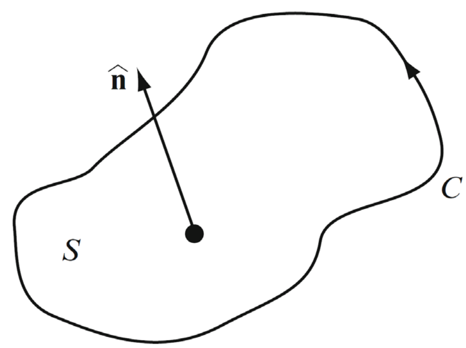
Fig, 1.22 The line integral. (a) Open contour integration. (b) Closed

1.6.2 Surface Integral

The surface integral of a vector is the flux (flow) of this vector through the surface. The surface integral is also written as:

$$Q = \int_s \bar{A} \cdot \bar{d}s$$

where $\bar{d}s = ds\bar{a}_n$ and where $\bar{a}_n$is the unit vector normal to surface S.



Surface Integral

If this surface is a closed surface, integration becomes as a *closed surface integration*:

$$Q = \oint_s \bar{A} \cdot \bar{d}s$$

Closed surface integration gives the total or net flux through a closed surface.

1.6.3 Volume Integral

The volume integral of a vector field is a vector and is written as

$$\bar{P} = \int_v \bar{p}\, dv$$

In Cartesian coordinates,

$$\bar{P} = \int_v p_x\, dv\, \overline{ax} + \int_v p_y\, dv\, \overline{ay} + \int_v p_Z\, dv\, \overline{az}$$

This type of vector integral is often called a regular or ordinary vector integral because it is essentially a scalar integral with the unit vectors added.

1.6 Del Operator and Gradient:

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\bar{a}_X + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

This vector differential operator, otherwise known as the *gradient operator*, when it operates on a scalar function. The operator is useful in defining:

1. The **gradient** of a scalar V, written. as ∇V

2. The **divergence** of a vector A, written as $\nabla \cdot \bar{A}$

3. The **curl** of a vector A, written as $\nabla \times \bar{A}$

4. The Laplacian of a scalar V, written as $\nabla^2 V$

The gradient of a scalar function gives both the magnitude and direction of the maximum spatial rate of change of the scalar function.

In Cartesian coordinates, the gradient of a scalar function is written as

$$\text{grad } U = \nabla U = \left(\frac{\partial}{\partial_X}\ \overline{ax} + \frac{\partial}{\partial y}\overline{ay} + \frac{\partial}{\partial_z}\ \overline{az}\right)U$$

and is read as *grad* U or *del* U.

The gradient has the following general properties:

- It operates on a scalar function and results in a vector function.
- The gradient is normal to a constant value surface.
- The gradient always points in the direction of maximum change in the scalar function.

for cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$$

and for spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial R}\bar{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

Example 12: Find the gradient of the following scalar fields:

(a) $V = e^{-z}\sin 2x \cosh y$

(b) $U = r^2 z \cos 2\phi$

(c) $W = 10R\sin^2\theta\cos\phi$

Solution:

(a) $\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$

$$= 2e^{-z}\cos 2x \cosh y\bar{a}_X + e^{-z}\sin 2x \sinh y\bar{a}_y - e^{-z}\sin 2x\cosh y\bar{a}_z$$

(b) $\nabla U = \frac{\partial U}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\bar{a}_\phi + \frac{\partial U}{\partial z}\bar{a}_z$

$$= 2rz\cos 2\phi\bar{a}_r - 2rz\sin 2\phi\bar{a}_\phi + r^2 cos2\phi\bar{a}_z$$

(c) $\nabla W = \frac{\partial W}{\partial R}\bar{a}_R + \frac{1}{R}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\phi$

$$= 10\sin^2\theta\cos\phi\bar{a}_R + 10\sin 2\theta\cos\phi\bar{a}_\theta - 10\sin\theta\sin\phi\bar{a}_\phi$$

1.7 Divergence and Divergence Theorem:

The divergence of $\bar{A}$ at a given point *P* is the outward flux per unit volume as the volume shrinks about *P*.

$$div\bar{A} = \nabla\cdot\bar{A} = \lim_{\Delta v\to 0}\frac{\oint_S \bar{A}\cdot\overline{dS}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface *S* in which *P* is located.

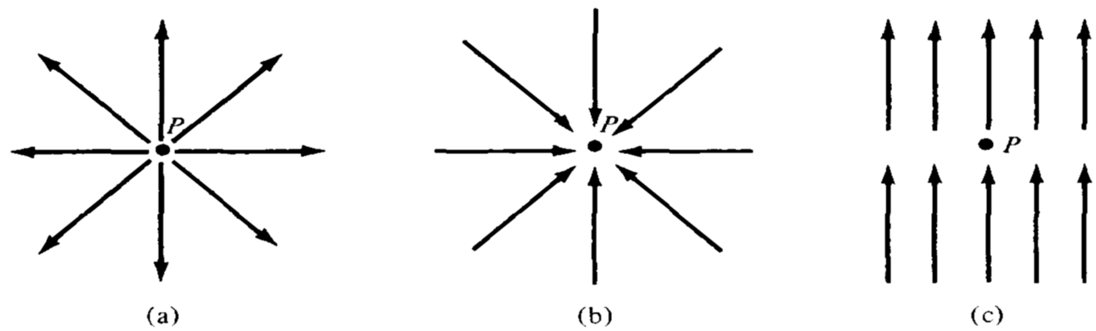


Fig. 1.23. Illustration of the divergence of a vector field at P; (a) positive divergence, (b) negative divergence, (c) zero divergence.

The divergence in different coordinate systems can be written as:

$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	Cartesian
$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	Cylindrical
$\nabla \cdot \bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$	Spherical

Note the following properties of the divergence of a vector field:

1. It produces a scalar field.

2. The divergence of a scalar V, $divV$, makes no sense.

3. $\nabla \cdot (\bar{A} + \bar{B}) = \nabla \cdot \bar{A} + \nabla \cdot \bar{B}$

4. $\nabla \cdot (V\bar{A}) = V\nabla \cdot \bar{A} + \bar{A}.\nabla V$

The divergence theorem follows from the definition of the divergence, stating that ***the volume integral of*** $\bar{\nabla} \cdot \bar{A}$ ***over a volume is equal to the closed surface integral of*** $\bar{\mathbf{A}}$ ***over the surface bounding the volume***. The divergence theorem is expressed as

$$\int_v \bar{\nabla} \cdot \bar{A}\, dv = \oint_s \bar{A} \cdot \overline{ds}$$

where S is the bounding surface of the volume $\mathcal{V}$, and ds is the differential area vector on S, which is always directed out of the enclosed volume.

Its most important use is the conversion of volume integrals of the divergence of a vector field into closed surface integrals.

Example 13 :- Find the divergence of the position vector to an arbitrary point.
Solution :
We will find the solution in Cartesian as well as in spherical coordinates.
$a\rangle$ *Cartesian coordinates*. The expression for the position vector to an arbitrary point (x, y, z) is

$$\overline{OP} = x\overline{a_x} + y\overline{a_y} + z\overline{a_z}$$

Then

$$\nabla \cdot (\overline{OP}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$b\rangle$ *Spherical coordinates*. Here the position vector is simply

$$\overline{OP} = r\overline{a_r}$$

Its divergence in spherical coordinates (r, θ, ϕ) can be obtained by

$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Hence, $\nabla \cdot (\overrightarrow{OP}) = 3$, as expected.

Example 14:- Given $\bar{A} = x^2\overline{a_x} + xy\,\overline{a_y} + yz\,\overline{a_z}$, verify the divergence theorem over a cube one unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin.

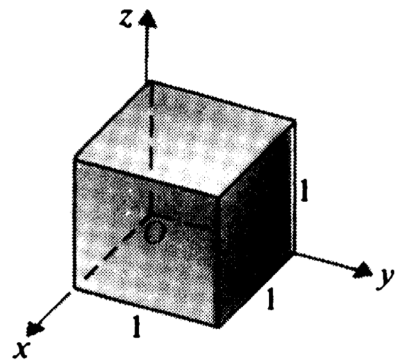
Solution: We first evaluate the surface integral over the six faces.

1. Front face: $x = 1$, $\overline{ds} = dydz\,\overline{a_x}$;

$$\int_{front\ face} \bar{A} \cdot \overline{ds} = \int_0^1 \int_0^1 dydz = 1.$$

2. Back face: $x = 0, \overline{ds} = -\, dydz\, \overline{ax}$;

$$\int_{back\ face} \bar{A} \cdot \overline{ds} = 0$$



3. Left face: $y = 0, \overline{ds} = -aydx\, \overline{dz}$;

$$\int_{left\ face} \bar{A} \cdot \overline{ds} = 0.$$

4. Right face: $y = 1, \overline{ds} = dxdz\, \overline{a_y}$;

$$\int_{right\ face} \bar{A} \cdot \overline{ds} = \int_0^1 \int_0^1 x\, dxdz = \tfrac{1}{2}.$$

5. Top face: $z = 1, \overline{ds} = dxdy\, \overline{a_z}$;

$$\int_{top\ face} \bar{A} \cdot \overline{ds} = \int_0^1 \int_0^1 y\, dxdy = \tfrac{1}{2}.$$

6. Bottom face: $z = 0, \overline{ds} = -dxdy\overline{az}$;

$$\int_{bottom\ face} \bar{A} \cdot \overline{ds} = 0$$

Adding the above six values, we have

$$\oint_S A \cdot ds = 1 + 0 + 0 + \frac{1}{2} + \frac{1}{2} + 0 = 2$$

Now the divergence of $\bar{A}$ is

$$\nabla \,.\, A = \tfrac{\partial}{\partial x}(x^2) + \tfrac{\partial}{\partial y}(xy) + \tfrac{\partial}{\partial z}(yz) = 3x + y.$$

Hence,

$$\int_V \nabla \cdot \bar{A}\, dv = \int_0^1 \int_0^1 \int_0^1 (3x + y) dxdydz = 2$$

1.8 Curl :

The curl of $\bar{A}$ is *the circulation of the vector $\bar{A}$ per unit area, as this area tends to zero and is in the direction normal to the area when the area is oriented such that the circulation is maximum.* The curl of a vector field is, therefore, a vector field, defined at any point in space.

More accurately, we define the curl using the following relation:

$$\text{curl } \bar{A} \equiv \nabla \text{x} \bar{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} [\bar{a}_n \oint_c \bar{A} \cdot \bar{d}\ell]_{\max}$$

The common notation for the curl of a vector A is $\nabla \times \bar{A}$ (read: del cross A), and it can be written in Cartesian coordinates as:

$$\nabla \times \bar{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\overline{ax} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\overline{ay} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\overline{az}$$

The properties of the curl are:
(1) The curl of a vector field is a vector field.
(2) The magnitude of the curl gives the maximum circulation of the vector per unit area at a point.
(3) The direction of the curl is along the normal to the area of maximum circulation at a point.
(4) The curl has the general properties of the vector product: it is distributive but not associative

$$\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B} \text{ and } \nabla \times (\bar{A} \times \bar{B}) \neq (\nabla \times \bar{A}) \times \bar{B}$$

(5) The divergence of the curl of any vector function is identically zero:

$$\nabla \cdot (\nabla \times \bar{A}) \equiv 0$$

(6) The curl of the gradient of a scalar function is also identically zero for any scalar:

$$\nabla \times (\nabla V) \equiv 0$$

For cylindrical coordinate,

$$\nabla \text{ x } \bar{A} = \frac{1}{r} \begin{vmatrix} \bar{a}_r & \bar{a}_\phi r & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix},$$

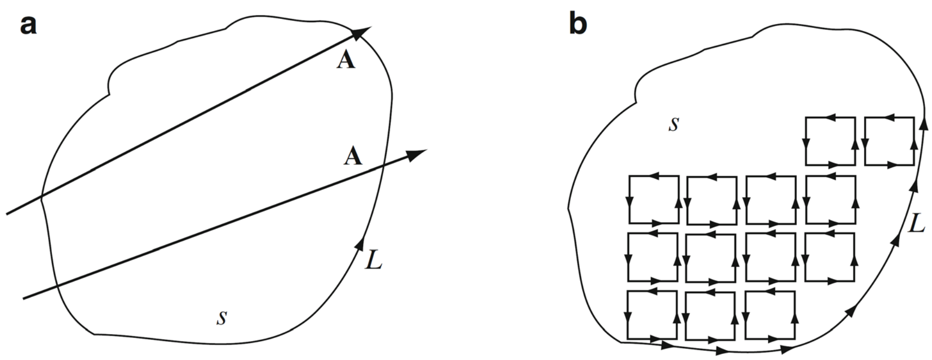
For spherical coordinate,

$$\nabla \times \bar{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \bar{a}_R & \bar{a}_\theta R & \bar{a}_\phi R \sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_R & R A_\theta & R \sin\theta A_\emptyset \end{vmatrix}$$

1.9 <u>Stokes's theorem:</u>

The Stokes's theorem follows from the definition of the curl, stating that the surface integral of $\nabla \times \bar{A}$ over an open surface is equal to the closed line integral of $\bar{A}$ around the loop bounding the surface. The Stokes's theorem is expressed as

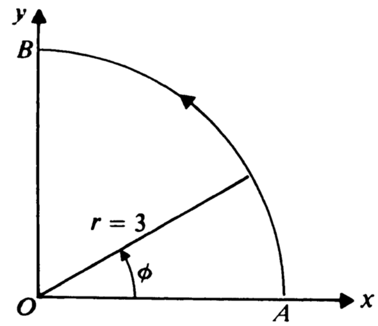
$$\int_s (\nabla \times \bar{A}) \cdot \overline{ds} = \oint_L \bar{A} . \overline{dl}$$



Stokes' theorem. (a) Vector field A and an open surface s. (b) The only components of the contour integrals on the small loops that do not cancel are along the outer contour L

<u>Example 15</u>:- Given $\bar{F} = \bar{a}_x xy - \bar{a}_y 2x$, verify Stokes's theorem over a quarter-circular disk with a radius 3 in the first quadrant.

Solution Let us first find the surface integral of $\nabla \text{x} \bar{F}$



$$\nabla \mathrm{x} \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -a_z(2+x) \ .$$

$$\int_s \ (\nabla \mathrm{x} \bar{F}) \cdot \bar{d}s = \int_0^3 \int_0^{\sqrt{9-y^2}} (\nabla \mathrm{x} \bar{F}) \ . \ (\bar{a}_z \ \mathrm{dx \ dy})$$

$$= \int_0^3 [\int_0^{\sqrt{9-y^2}} -(2+x)dx]dy$$

$$= -\int_0^3 [\, 2\sqrt{9-y^2} + \frac{1}{2}(9-y^2)]dy$$

$$= -[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3} + \frac{9}{2}y - \frac{y^3}{6}]_0^3$$

$$= -9(1+\frac{\pi}{2}) \ .$$

For the line integral around ABOA

From A to B: $\int_A^B \bar{F} \cdot \bar{d}\ell = \int_0^{\pi/2} -3(9\sin^2\phi \cos\phi + 6\cos^2\phi)d\phi$

$$= -9(\sin^3\phi + \phi + \ \sin\phi\cos\phi)|_0^{\pi/2} = -9(1+\frac{\pi}{2}),$$

From B to O: $x = 0$, and $\bar{F} \cdot \bar{d}\ell = \bar{F} \cdot (-\bar{a}_y dy) = 2xdy = 0.$

From O to A: $y = 0$, and $\bar{F} \cdot \bar{d}\ell = \bar{F} \cdot (\bar{a}_x dx) = xydx = 0.$ Hence,

$$\oint_{ABOA} \bar{F} \cdot \bar{d}\ell = \int_A^B \bar{F} \cdot \bar{d}\ell = \ -9(1+\frac{\pi}{2})$$

Stokes′s theorem is verified.

The Helmholtz Theorem

The Helmholtz theorem states: "***A vector field is uniquely defined by specifying its divergence and its curl***." The Helmholtz theorem is normally written as

$$\bar{B} = -\nabla U + \nabla \times \bar{A}$$

where U is a scalar field and A is a vector field. That is, any vector field can be decomposed into two terms; one is the gradient of a scalar function and the other is the curl of a vector function.

Divergenceless field is called *solenoidal* and *a curl-free field is* called *irrotational*. We may classify vector fields in accordance with their being solenoidal and/or irrotational. A vector field $\bar{A}$ is: -

1). Solenoidal and irrotational if

$$\nabla \cdot \bar{A} = 0 \text{ and } \nabla \text{x} \bar{A} = 0.$$

Ex: A static electric field in a charge-free region.

2). Solenoidal but not irrotational if

$$\nabla \cdot \bar{A} = 0 \text{ and } \nabla \text{x} \bar{A} \neq 0.$$

Ex: A steady magnetic field in a current-carrying conductor.

3). Irrotational but not solenoidal if

$$\nabla \text{x} \bar{A} = 0 \text{ and } \nabla \cdot \bar{A} \neq 0.$$

Ex: A static electric field in a charged region.

4). Neither solenoidal nor irrotational if

$$\nabla \cdot \bar{A} \neq 0 \text{ and } \nabla \text{x} \bar{A} \neq 0.$$

Ex: An electric field in a charged medium with a time-varying magnetic field.

Example16: Given a vector function $\bar{F} = \bar{a}_x(3y - c_1 z) + \bar{a}_y(c_2 x - 2z)$ $-\bar{a}_z(c_3 y + z)$. Determine the constants c_1, c_2, and c_3 if $\bar{F}$ is irrotational.

Solution

For $\bar{F}$ to be irrotational, $\nabla \text{x} \bar{F} = 0$; that is,

$$\nabla x \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y - c_1 z & -2z c_2 x & -(c_3 y + z) \end{vmatrix}$$

$$= a_X(-c_3 + 2) - a_y c_1 + a_z(c_2 - 3) = 0.$$

Each component of $\nabla \text{x} \bar{F}$ must vanish. Hence $c_1 = 0$, $c_2 = 3$, and $c_3 = 2$.

<u>**H.W 14**</u>. Determine the scalar potential function V whose negative gradient equals $\bar{F}$.