Chapter 12: Shear Strength of Soil

The *shear strength* of a soil mass is the internal resistance per unit area that the soil mass can offer to resist failure and sliding along any plane inside it. One must understand the nature of shearing resistance in order to analyze soil stability problems, such as bearing capacity, slope stability, and lateral pressure on earth-retaining structures.

12.1 Normal and Shear Stresses on a Plane

Figure (12.1a) shows a two-dimensional soil element that is being subjected to normal and shear stresses ($\sigma_y > \sigma_x$). To determine the normal stress and the shear stress on a plane *EF* that makes an angle θ with the plane *AB*, we need to consider the free body diagram of *EFB* shown in Figure (12.1b). Let σ_n and τ_n be the normal stress and the shear stress, respectively, on the plane *EF*.

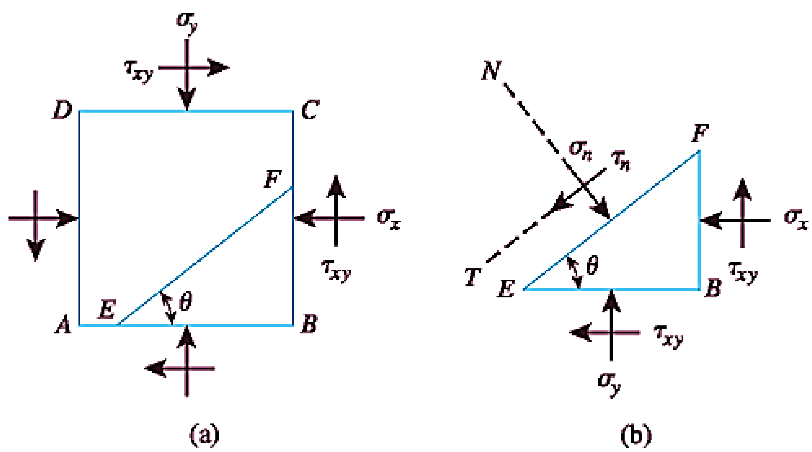

Figure 12.1 (a) A soil element with normal and shear stresses acting on it; (b) free body diagram of *EFB* as shown in (a)

Summing the components of forces that act on the element in the direction of *N* and *T*, we have

$$\sigma_n = \frac{\sigma_y+\sigma_x}{2} + \frac{\sigma_y-\sigma_x}{2}cos2\theta + \tau_{xy}sin2\theta \qquad (12.1)$$

and

$$\tau_n = \frac{\sigma_y-\sigma_x}{2}sin2\theta - \tau_{xy}cos2\theta \qquad (12.2)$$

From eq. (12.2), we can see that we can choose the value of θ in such a way that τ_n will be equal to zero. Substituting $\tau_n = 0$, we get

$$tan2\theta = \frac{2\tau_{xy}}{\sigma_y - \sigma_x} \qquad (12.3)$$

For given values of $\tau_{xy}, \sigma_x, and\ \sigma_y$, Eq. (12.3) will give two values of θ that are 90^o apart. This means that there are two planes that are at right angles to each other on which the shear stress is zero. Such planes are called *principal planes*. The normal stresses that act on the principal planes are referred to as *principal stresses.* The values of principal stresses can be found by substituting Eq. (12.3) into Eq. (12.1), which yields

Major principal stress:

$$\sigma_n = \sigma_1 = \frac{\sigma_y + \sigma_x}{2} + \sqrt{[\frac{(\sigma_y - \sigma_x)}{2}]^2 + \tau_{xy}^2} \qquad (12.4)$$

Minor principal stress:

$$\sigma_n = \sigma_3 = \frac{\sigma_y + \sigma_x}{2} - \sqrt{[\frac{(\sigma_y - \sigma_x)}{2}]^2 + \tau_{xy}^2} \qquad (12.5)$$

The normal stress and shear stress that act on any plane can also be determined by plotting a Mohr's circle, as shown in Figure (12.2). The following sign conventions are used in Mohr's circles: compressive normal stresses are taken as positive, and shear stresses are considered positive if they act on opposite faces of the element in such a way that they tend to produce a counterclockwise rotation.

For plane *AD* of the soil element shown in Figure (12.1a), normal stress equals $+\sigma_x$ and shear stress equals $+\tau_{xy}$. For plane *AB*, normal stress equals $+\sigma_y$ and shear stress equals $-\tau_{xy}$.

The points *R* and *M* in Figure (12.2) represent the stress conditions on planes *AD* and *AB*, respectively. *O* is the point of intersection of the normal stress axis with the line *RM*. The circle *MNQRS* drawn with *O* as the center and *OR* as the radius is the Mohr's circle for the stress conditions considered. The radius of the Mohr's circle is equal to

$$\sqrt{[\frac{(\sigma_y - \sigma_x)}{2}]^2 + \tau_{xy}^2}$$

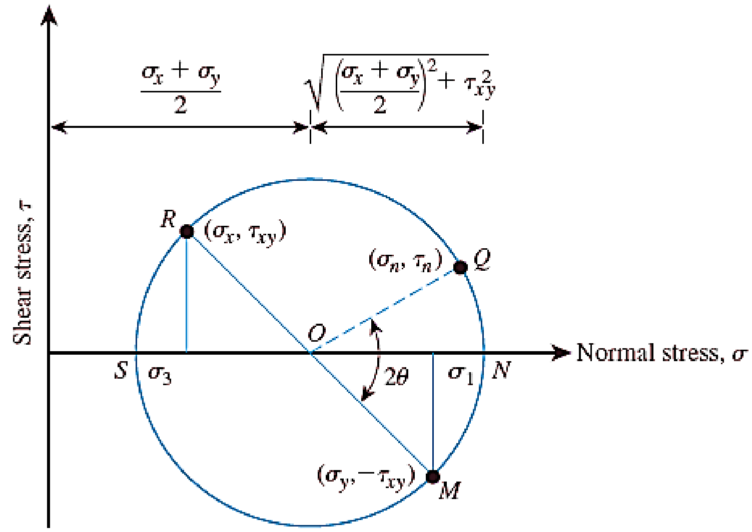

Figure 12.2 Principles of the Mohr's circle

The stress on plane *EF* can be determined by moving an angle 2θ (which is twice the angle that the plane *EF* makes in a counterclockwise direction with plane *AB* in Figure 12.1a) in a counterclockwise direction from point *M* along the circumference of the Mohr's circle to reach point *Q*. The abscissa and ordinate of point *Q*, respectively, give the normal stress σ_n and the shear stress τ_n on plane *EF*.

Because the ordinates (that is, the shear stresses) of points *N* and *S* are zero, they represent the stresses on the principal planes. The abscissa of point *N* is equal to σ_1 [Eq. (12.4)], and the abscissa for point *S* is σ_3 [Eq. (12.5)].

As a special case, if the planes *AB* and *AD* were major and minor principal planes, the normal stress and the shear stress on plane *EF* could be found by substituting $\tau_{xy} = 0$. Equations (12.1) and (12.2) show that $\sigma_y = \sigma_1$ *and* $\sigma_x = \sigma_3$ (Figure 12.3a). Thus,

$$\sigma_n = \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2} cos2\theta \quad (12.6)$$

$$\tau_n = \frac{\sigma_1-\sigma_3}{2} sin2\theta \quad (12.7)$$

The Mohr's circle for such stress conditions is shown in Figure (12.3b). The abscissa and the ordinate of point *Q* give the normal stress and the shear stress, respectively, on the plane *EF*.

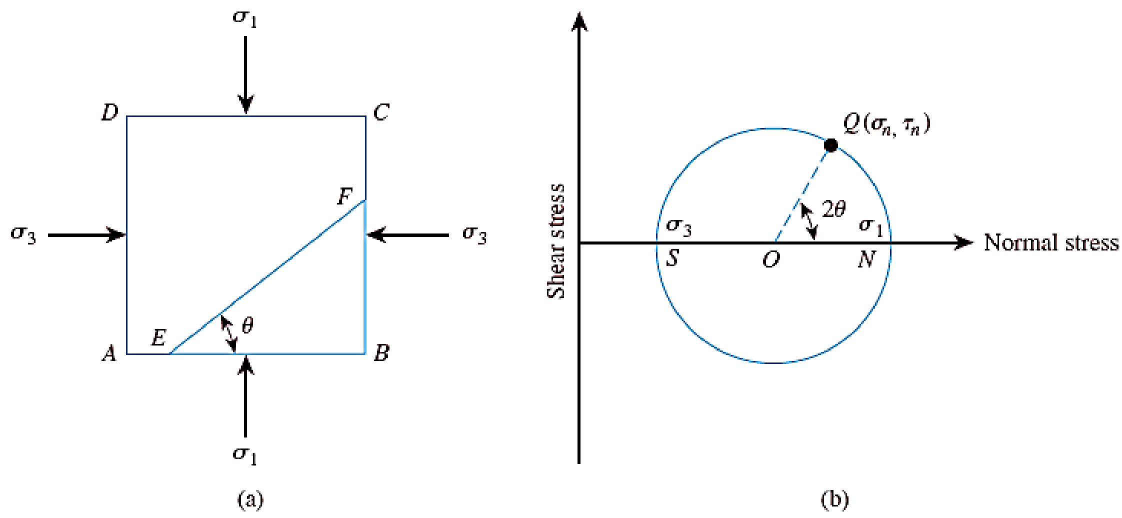

Figure 12.3 (a) Soil element with *AB* and *AD* as major and minor principal planes; (b) Mohr's circle for soil element shown in (a)

Example 12.1

A soil element is shown in Figure (12.4). the magnitudes of stresses are $\sigma_x = 96\ kN/m^2, \tau = 38\ kN/m^2, \sigma_y = 120\ kN/m^2$ and $\theta = 20^o$. Determine

a. Magnitudes of the principal stresses

b. Normal and shear stresses on plane *AB*. Use Eqs. (12.1), (12.2), (12.4), and (12.5).

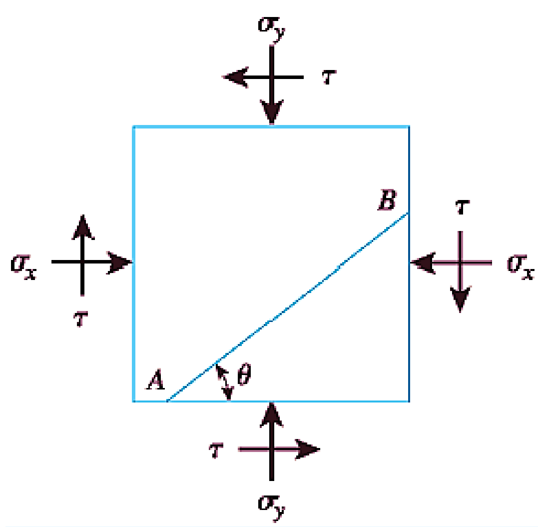

Figure 12.4 Soil element with stresses acting on it

Solution

Part a

From Eqs. (12.4) and (12.5)

$$\left.\begin{matrix}\sigma_1\\\sigma_3\end{matrix}\right\} = \frac{\sigma_y+\sigma_x}{2} \pm \sqrt{[\frac{\sigma_y-\sigma_x}{2}]^2 + \tau_{xy}^2}$$

$$= \frac{120+96}{2} \pm \sqrt{[\frac{120-96}{2}]^2 + (-38)^2}$$

$$\sigma_1 = 147.85\ kN/m^2$$

$$\sigma_3 = 68.15\ kN/m^2$$

Part b

From Eq. (12.1)

$$\sigma_n = \frac{\sigma_y+\sigma_x}{2} + \frac{\sigma_y-\sigma_x}{2} cos2\theta + \tau sin2\theta$$

$$= \frac{120+96}{2} + \frac{120-96}{2}\cos(2\times 20) + (-38)\sin(2\times 20) = 92.76\ kN/m^2$$

From eq. (12.2)

$$\tau_n = \frac{\sigma_y-\sigma_x}{2} sin2\theta - \tau cos2\theta$$

$$= \frac{120-96}{2}\sin(2\times 20) - (-38)\cos(2\times 20) = 36.82\ kN/m^2$$

12.2 The Pole Method of Finding Stresses Along a Plane

Another important technique of finding stresses along a plane from a Mohr's circle is the *pole method*, or the method of *origin of planes*. This is demonstrated in Figure (12.5). Figure (12.5a) is the same stress element that is shown in Figure (12.1a); Figure 10.5b is the Mohr's circle for the stress conditions indicated. According to the pole method, we draw a line from a known point on the Mohr's circle parallel to the plane on which the state of stress acts. The point of intersection of this line with the Mohr's circle is called the *pole*. This is a unique point for the state of stress under consideration. For example, the point *M* on the Mohr's circle in Figure (12.5b) represents the stresses on the plane AB. The line *MP* is drawn parallel to *AB*. So point *P* is the pole (origin of planes) in this case. If we need to find the stresses on a plane *EF*, we draw a line from the pole parallel to *EF*. The point of intersection of this line with the Mohr's circle is *Q*. The coordinates of *Q* give the stresses on the plane *EF*. (*Note*: From geometry, angle *QOM* is twice the angle *QPM*.)

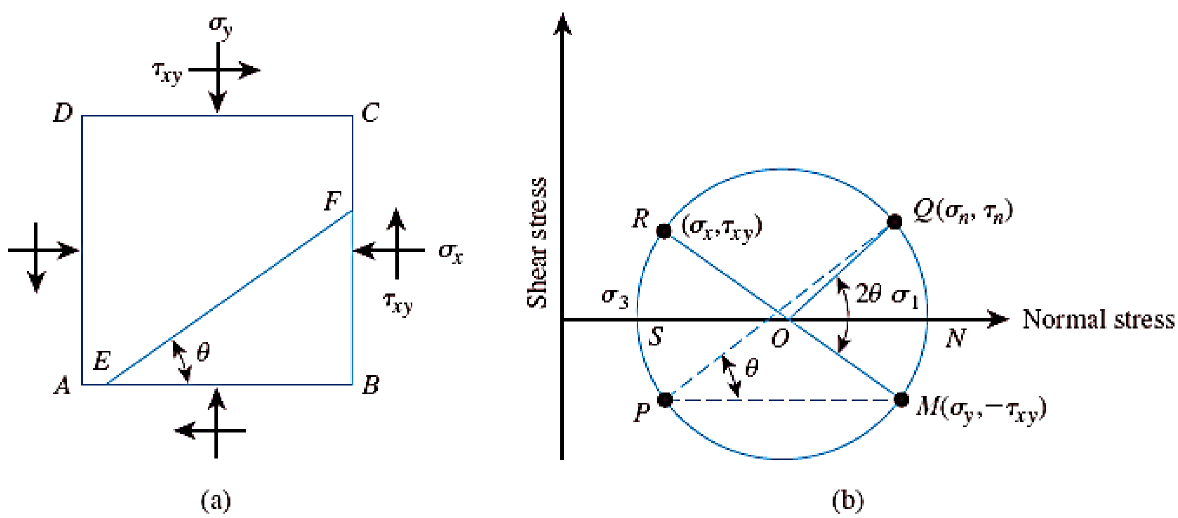

Figure 12.5 (a) Soil element with normal and shear stresses acting on it; (b) use of pole method to find the stresses along a plane

Example 12.2

For the stressed soil element shown in Figure (12.6a), determine

a. Major principal stress
b. Minor principal stress
c. Normal and shear stresses on the plane *AE*

Use the pole method.

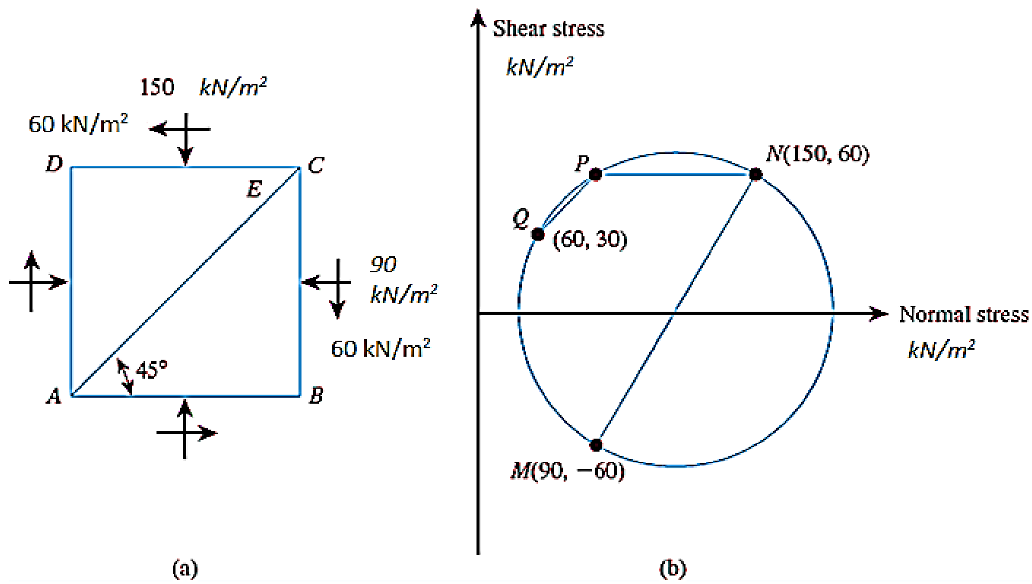

Figure12.6 (a) soi element with stresses acting on it; (b) Mohr's circle

Solution

On plane *AD*:

Normal stress = 90 kN/m^2
Shear stress = -60 kN/m^2

On plane AB:

Normal stress = 150 kN/m^2
Shear stress = 60 kN/m^2

The Mohr's circle is plotted in Figure (12.6b). from the plot,

Part a

Major principal stress = 187.1 kN/m^2

Part b

Minor principal stress = 52.9 kN/m^2

Part c

NP is the line drawn parallel to the plane *CD*. *P* is the pole. *PQ* is drawn parallel to *AE* (see Figure 12.6a). The coordinates of point *Q* give the stresses on the plane *AE*. Thus,

Normal stress = 60 kN/m^2

Shear stress = 30 kN/m^2

12.3 Mohr–Coulomb Failure Criterion

Mohr (1900) presented a theory for rupture in materials that contended that a material fails because of a critical combination of normal stress and shearing stress and not from either maximum normal or shear stress alone. Thus, the functional relationship between normal stress and shear stress on a failure plane can be expressed in the following form:

$$\tau_f = f(\sigma) \tag{12.8}$$

The failure envelope defined by Eq.(12.8) is a curved line. For most soil mechanics problems, it is sufficient to approximate the shear stress on the failure plane as a linear function of the normal stress (Coulomb, 1776). This linear function can be written as

$$\tau_f = c + \sigma tan\emptyset \tag{12.9}$$

where c = cohesion
$\emptyset$ = angle of internal friction
σ = normal stress on the failure plane
τ_f = shear strength

The preceding equation is called the *Mohr–Coulomb failure criterion.*

In saturated soil, the total normal stress at a point is the sum of the effective stress (σ') and pore water pressure (u), or

$$\sigma = \sigma' + u$$

The effective stress σ' is carried by the soil solids. The Mohr–Coulomb failure criterion, expressed in terms of effective stress, will be of the form

$$\tau_f = c' + \sigma' tan\emptyset' \tag{12.10}$$

where c' = cohesion and $\emptyset'$ = friction angle, based on effective stress.

Thus, Eqs. (12.9) and (12.10) are expressions of shear strength based on total stress and effective stress. The value of c' for sand and inorganic silt is 0. For normally consolidated clays, c' can be approximated at 0. Overconsolidated clays have values of c' that are greater than 0. The angle of friction, $\emptyset'$, is sometimes referred to as the *drained angle of friction*. Typical values of $\emptyset'$ for some granular soils are given in Table 12.1.

The significance of Eq. (12.10) can be explained by referring to Fig.(12.7), which shows an elemental soil mass. Let the effective normal stress and the shear stress on the plane *ab* be σ' and τ, respectively. Figure (12.7b) shows the plot of the failure envelope defined by Eq. (12.10). If the magnitudes of σ' and τ on plane ab are such that they plot as point *A* in Figure (12.7b), shear failure will not occur along the plane. If the effective normal stress and the shear stress on plane *ab* plot as point *B* (which falls on the failure envelope), shear failure will occur along that plane. A state of stress on a plane represented by point *C* cannot exist, because it plots above the failure envelope, and shear failure in a soil would have occurred already.

Table 12.1 Typical Values of Drained Angle of Friction for Sands and Silts

Soil type	ϕ' (deg)
Sand: Rounded grains	
Loose	27–30
Medium	30–35
Dense	35–38
Sand: Angular grains	
Loose	30–35
Medium	35–40
Dense	40–45
Gravel with some sand	34–48
Silts	26–35

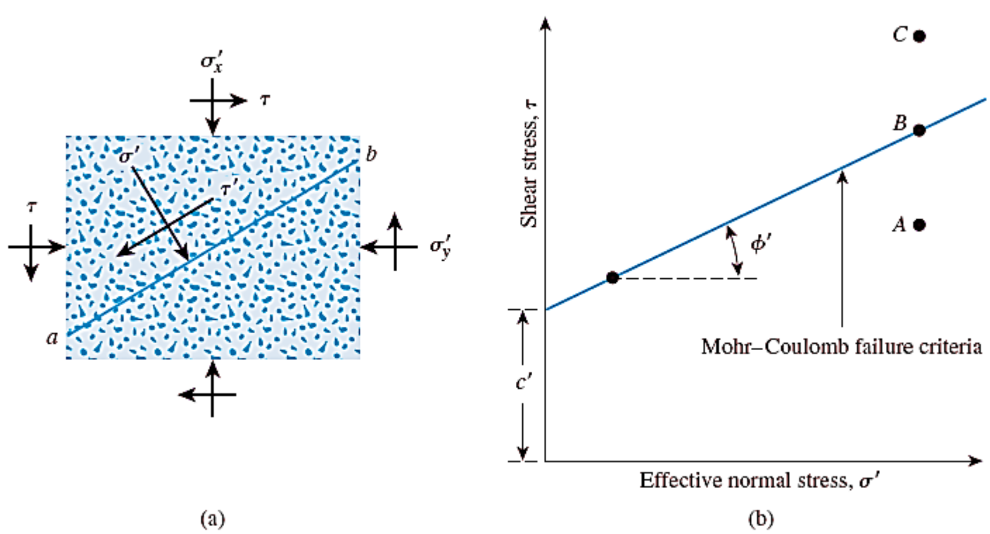

Figure 12.7 Mohr–Coulomb failure criterion

12.4 Inclination of the Plane of Failure Caused by Shear

As stated by the Mohr–Coulomb failure criterion, failure from shear will occur when the shear stress on a plane reaches a value given by Eq. (12.10). To determine the inclination of the failure plane with the major principal plane, refer to Figure (12.8), where σ'_1 and σ'_3 are, respectively, the major and minor effective principal stresses. The failure plane *EF* makes an angle θ with the major principal plane. To determine the angle θ and the relationship between σ'_1 *and* σ'_3, refer to Figure (12.9), which is a plot of the Mohr's circle for the state of stress shown in Figure (12.8). In Figure (12.9),*fgh* is the failure envelope defined by the relationship $\tau_f = c' + \sigma' tan\emptyset'$. The radial line *ab* defines the major principal plane (*CD* in Figure 12.8), and the radial line *ad* defines the failure plane (*EF* in Figure 12.8). It can be shown that the angle $bad = 2\theta = 90 + \emptyset'$, or

$$\theta = 45 + \frac{\emptyset'}{2} \tag{12.11}$$

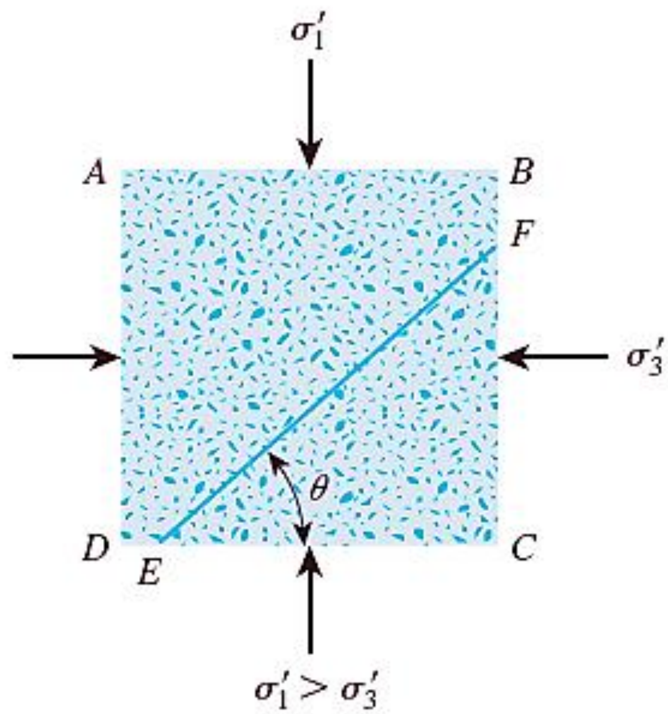

Figure 12.8 Inclination of failure plane in soil with major principal plane

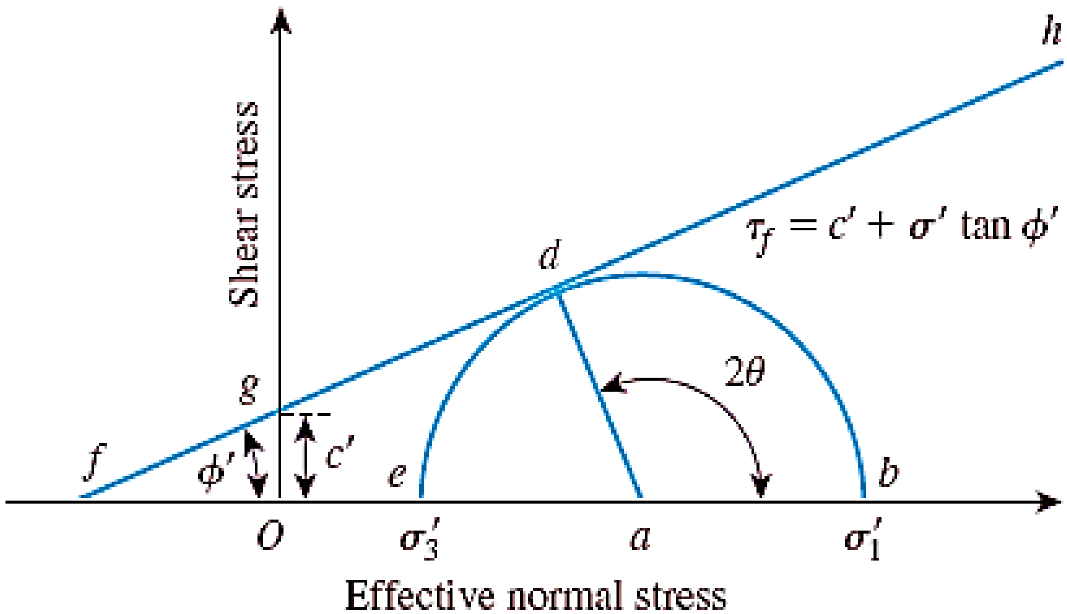

Figure 12.9 Mohr's circle and failure envelope

12.5 Laboratory Test for Determination of Shear Strength Parameters

There are several laboratory methods now available to determine the shear strength parameters (i.e., c , $\emptyset$, c', $\emptyset'$) of various soil specimens in the laboratory. They are as follows:

- Direct shear test
- Triaxial test
- Direct simple shear test
- Plane strain triaxial test
- Torsional ring shear test

The direct shear test and the triaxial test are the two commonly used techniques for determining the shear strength parameters. These two tests will be described in detail in the sections that follow.

12.6 Direct Shear Test (Shear box Test)

The direct shear test is the oldest and simplest form of shear test arrangement. A diagram of the direct shear test apparatus is shown in Figure (12.10). The test equipment consists of a metal shear box in which the soil specimen is placed. The soil specimens may be square or circular in plan. The size of the specimens generally used is about 51 mm × 51 mm or 102 mm × 102 mm (2 in. × 2 in. or 4 in. × 4 in.) across and about 25 mm (1 in.) high. The box is split horizontally into halves. Normal force on the specimen is applied from the top of the shear box. The normal stress on the specimens can be as great as 1050 kN/m^2 (150 $lb/in.^2$). Shear force is applied by moving one-half of the box relative to the other to cause failure in the soil specimen.

Depending on the equipment, the shear test can be either stress controlled or strain controlled. In stress-controlled tests, the shear force is applied in equal increments until the specimen fails. The failure occurs along the plane of split of the shear box. After the application of each incremental load, the shear displacement of the top half of the box is measured by a horizontal dial gauge. The change in the height of the specimen (and thus the volume change of the specimen) during the test can be obtained from the readings of a dial gauge that measures the vertical movement of the upper loading plate.

In strain-controlled tests, a constant rate of shear displacement is applied to one-half of the box by a motor that acts through gears. The constant rate of shear displacement is measured by a horizontal dial gauge. The resisting shear force of the soil corresponding to any shear displacement can be measured by a horizontal proving ring or load cell. The volume change of the specimen during the test is obtained in a manner similar to that in the stress-controlled tests. Figure 12.11 shows a photograph taken from the top of the direct shear test equipment with the dial gages and proving ring in place.

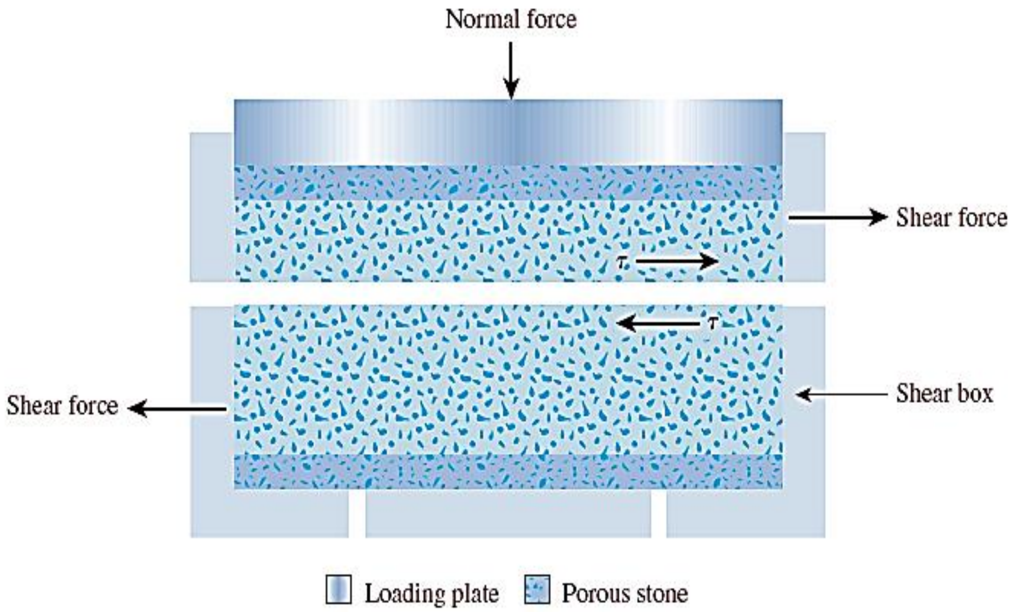

Figure 12.10 Diagram of direct shear test arrangement

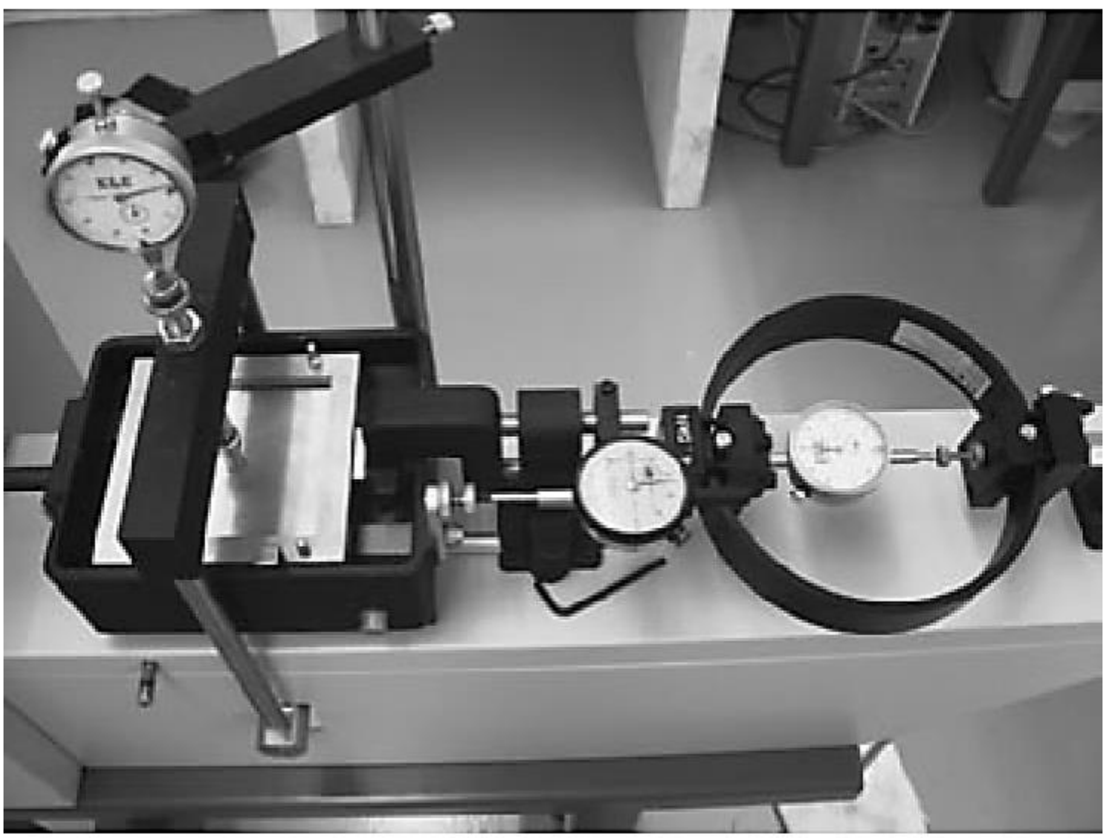

Figure 12.11 A photograph showing the dial gauges and proving ring in place

Figure (12.12) shows a typical plot of shear stress and change in the height of the specimen against shear displacement for dry loose and dense sands. These observations were obtained from a strain-controlled test. The following generalizations can be developed from Figure (12.12) regarding the variation of resisting shear stress with shear displacement:

1. In loose sand, the resisting shear stress increases with shear displacement until a failure shear stress of τ_f is reached. After that, the shear resistance remains approximately constant for any further increase in the shear displacement.
2. In dense sand, the resisting shear stress increases with shear displacement until it reaches a failure stress of τ_f. This τ_f is called the *peak shear strength*. After failure stress is attained, the resisting shear stress gradually decreases as shear displacement increases until it finally reaches a constant value called the *ultimate shear strength*.

Since the height of the specimen changes during the application of the shear force (as shown in Figure 12.12), it is obvious that the void ratio of the sand changes (at least in the vicinity of the split of the shear box). Figure 12.13 shows the nature of variation of the void ratio for loose and dense sands with shear displacement. At large shear displacement, the void ratios of loose and dense sands become practically the same, and this is termed the *critical void ratio.*

It is important to note that, in dry sand,

$$\sigma = \sigma'$$

and

$$c' = 0$$

Direct shear tests are repeated on similar specimens at various normal stresses. The normal stresses and the corresponding values of τ_f obtained from a number of tests are plotted on a graph from which the shear strength parameters are determined. Figure (12.14) shows such a plot for tests on a dry sand. The equation for the average line obtained from experimental results is

$$\tau_f = \sigma' tan\emptyset' \qquad (12.12)$$

So, the friction angle can be determined as follows:

$$\emptyset' = tan^{-1}(\frac{\tau_f}{\sigma'}) \qquad (12.13)$$

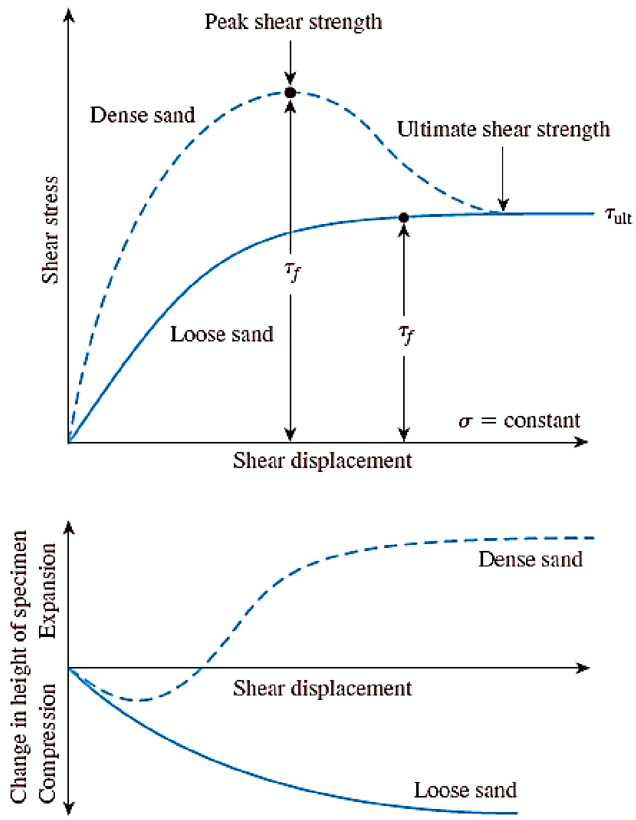

Figure 12.12 Plot of shear stress and change in height of specimen against shear displacement for loose and dense dry sand (direct shear test)

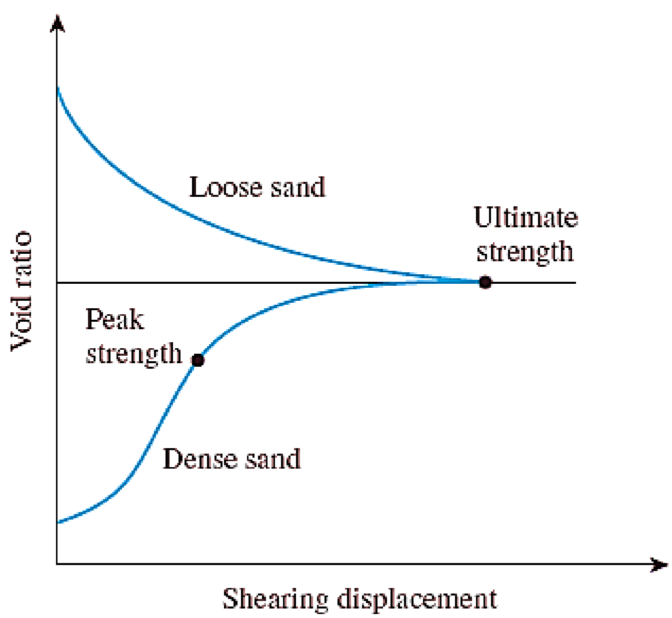

Figure 12.13 Nature of variation of void ratio with shearing displacement

It is important to note that in situ cemented sands may show a c' intercept.
If the variation of the ultimate shear strength (τ_{ult}) with normal stress is known, it can be plotted as shown in Figure (12.14). The average plot can be expressed as

$$\tau_{ult} = \sigma' \tan\emptyset'_{ult} \qquad (12.14a)$$

$$\tau_r = \sigma' \tan\emptyset'_r \qquad (12.14b)$$

or

$$\emptyset'_{ult} = \tan^{-1}\left(\frac{\tau_{ult}}{\sigma'}\right) \qquad (12.15a)$$

$$\emptyset'_r = \tan^{-1}\left(\frac{\tau_r}{\sigma'}\right)$$

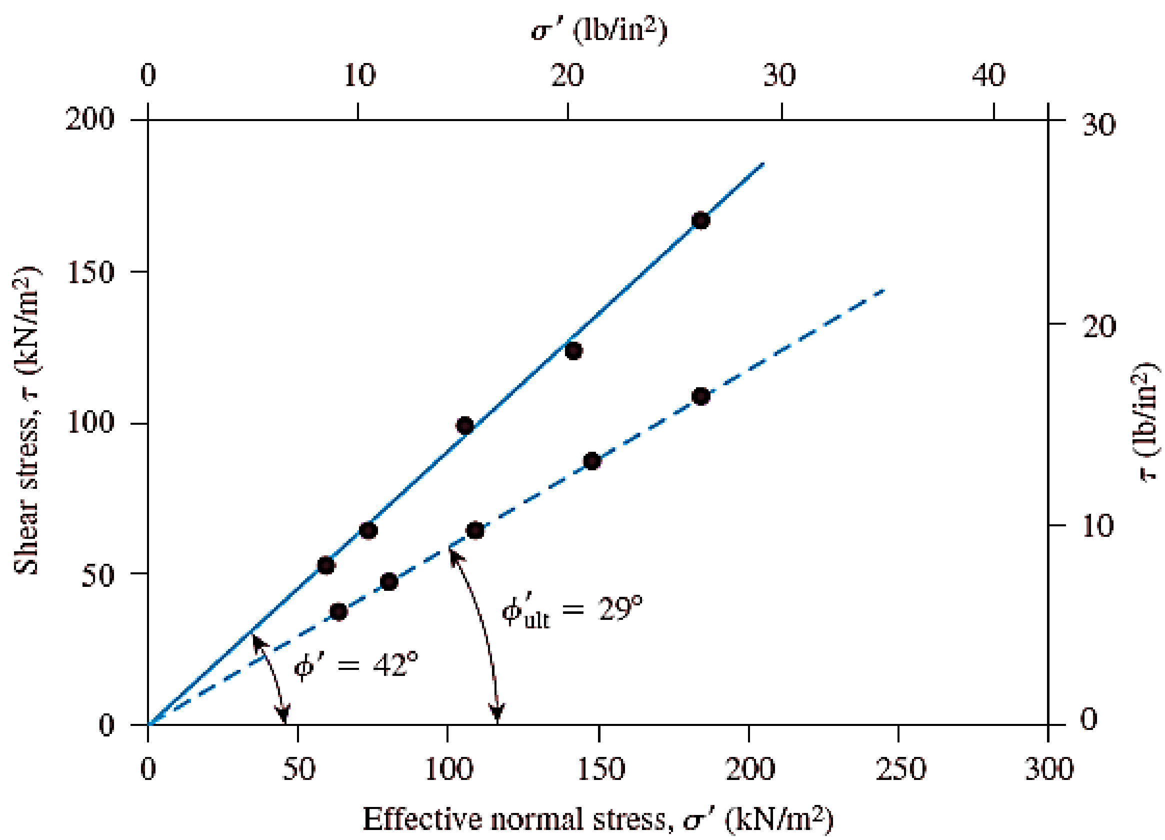

Figure 12.14 Determination of shear strength parameters for a dry sand using the results of direct shear tests

12.7 Advantages and Disadvantages of Shear Box Test

Advantages

1. Simple, quick, and inexpensive test.
2. Used to determine shear parameters (c and $\emptyset$) for both cohesive and cohesionless soils.

Disadvantages

1. The shear area is changing during the test causing unequal distribution of shear stress.
2. The direction and location of failure plane is at the box split and parallel to the horizontal force, practically this condition may not be obtained.
3. The soil is confined.
4. Drainage conditions can not be controlled and *p.w.p.* can not be measured.

Example 12.3

Following are the results of four drained direct shear tests on an overconsolidated clay:

- Diameter of specimen = 50 mm
- Height of specimen = 25 mm

Test no.	Normal force, N (N)	Shear force at failure, S_{peak} (N)	Residual shear force, $S_{residual}$ (N)
1	150	157.5	44.2
2	250	199.9	56.6
3	350	257.6	102.9
4	550	363.4	144.5

Determine the relationships for peak shear strength (τ_f) and residual shear strength (τ_r).

Solution

Area of specimen $(A) = (\frac{\pi}{4})(\frac{50}{1000})^2 = 0.0019634\ m^2$. Now the following table can be prepared.

Test no.	Normal force, N (N)	Normal stress, σ' (kN/m²)	Peak shear force, S_{peak} (N)	$\tau_f = \frac{S_{peak}}{A}$ (kN/m²)	Residual shear force, $S_{residual}$ (N)	$\tau_r = \frac{S_{residual}}{A}$ (kN/m²)
1	150	76.4	157.5	80.2	44.2	22.5
2	250	127.3	199.9	101.8	56.6	28.8
3	350	178.3	257.6	131.2	102.9	52.4
4	550	280.1	363.4	185.1	144.5	73.6

The variations of τ_f and τ_r with σ' are plotted in Figure (12.15). From the plots, we find that

Peak strength: $\tau_f\left(\frac{kN}{m^2}\right) = 40 + \sigma' tan27$

Residual strength: $\tau_r\left(\frac{kN}{m^2}\right) = \sigma' tan14.6$

(*Note*: For all *overconsolidated clays*, the residual shear strength can be expressed as

$$\tau_r = \sigma' tan\emptyset'_r$$

Where $\emptyset'_r$ = effective residual friction angle

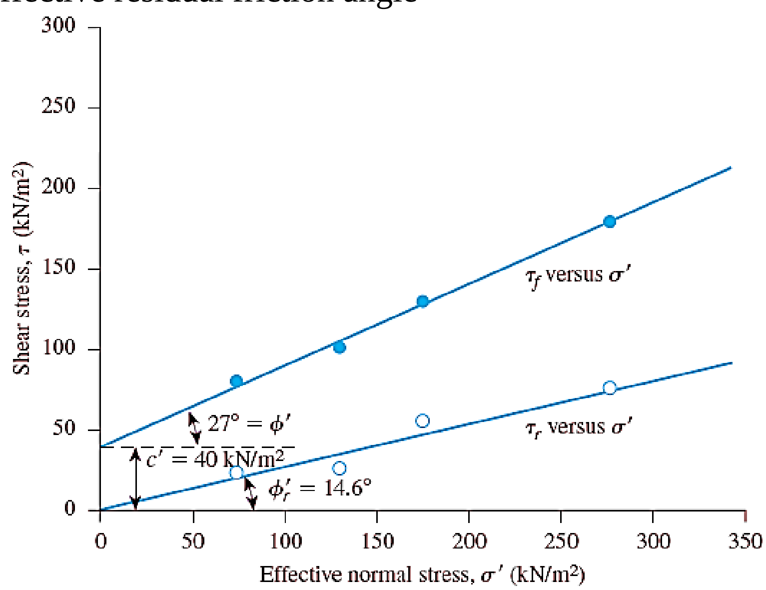

Figure 12.15 Variation of τ_f and τ_r with σ'

12.8 Triaxial Shear Test (General)

The triaxial shear test is one of the most reliable methods available for determining shear strength parameters. It is used widely for research and conventional testing. A diagram of the triaxial test layout is shown in Figure (12.16).

In this test, a soil specimen about 36mm (1.4 in.) in diameter and 76 mm (3 in.) long generally is used. The specimen is encased by a thin rubber membrane and placed inside a plastic cylindrical chamber that usually is filled with water or glycerine. The specimen is subjected to a confining pressure by compression of the fluid in the chamber. (*Note*: Air is sometimes used as a compression medium) To cause shear failure in the specimen, one must apply axial stress through a vertical loading ram (sometimes called *deviator stress*). The axial load applied by the loading ram corresponding to a given axial deformation is measured by a proving ring or load cell attached to the ram.

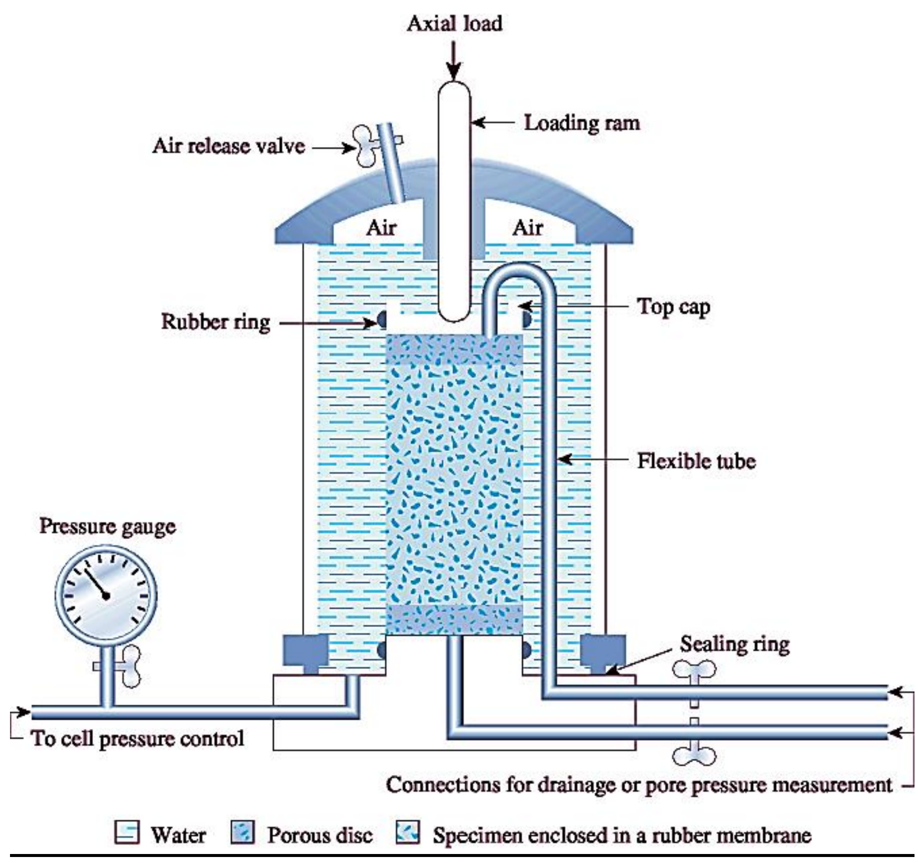

Figure 12.16 Diagram of triaxial test equipment (After Bishop and Bjerrum, 1960. With permission from ASCE.)

Connections to measure drainage into or out of the specimen, or to measure pressure in the pore water (as per the test conditions), also are provided. The following three standard types of triaxial tests generally are conducted:
1. Consolidated-drained test or drained test (CD test)
2. Consolidated-undrained test (CU test)
3. Unconsolidated-undrained test or undrained test (UU test)

The general procedures and implications for each of the tests in saturated soils are described in the following sections

12.9 Consolidated-Drained Triaxial Test (CD test)

In the CD test, the saturated specimen first is subjected to an all around confining pressure, σ_3 , by compression of the chamber fluid (Figure 12.17a). As confining pressure is applied, the pore water pressure of the specimen increases by u_c (if drainage is prevented).

Now, if the connection to drainage is opened, dissipation of the excess pore water pressure, and thus consolidation, will occur. With time, u_c will become equal to 0. In saturated soil, the change in the volume of the specimen (ΔVc) that takes place during consolidation can be obtained from the volume of pore water drained. Next, the deviator stress, $\Delta\sigma_d$, on the specimen is increased very slowly. The drainage connection is kept open, and the slow rate of deviator stress application allows complete dissipation of any pore water pressure that developed as a result ($\Delta u_d = 0$).

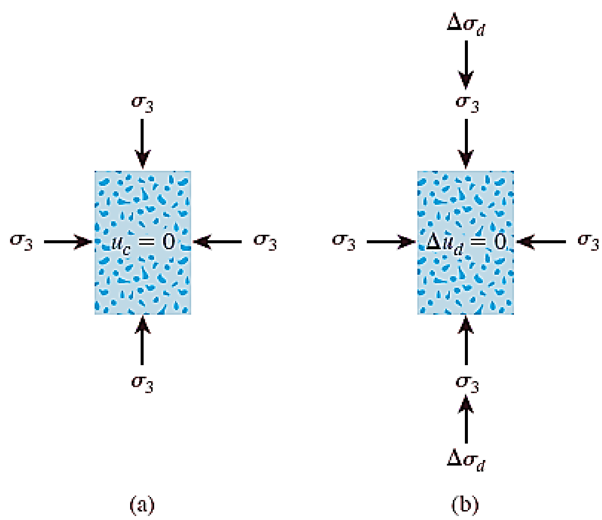

Figure 12.17 Consolidated-drained triaxial test: (a) specimen under chamber-confining pressure; (b) deviator stress application

Because the pore water pressure developed during the test is completely dissipated, we have

$$\text{Total and effective confining stress} = \sigma_3 = \sigma'_3$$

and

$$\text{Total and effective axial stress at failure} = \sigma_3 + (\Delta\sigma_d)_f = \sigma_1 = \sigma'_1$$

In a triaxial test, σ'_1 is the major principal effective stress at failure and σ'_3 is the minor principal effective stress at failure.

Several tests on similar specimens can be conducted by varying the confining pressure. With the major and minor principal stresses at failure for each test the Mohr's circles can be drawn and the failure envelopes can be obtained. Figure (12.18) shows the type of effective stress failure envelope obtained for tests on sand and normally consolidated clay. The coordinates of the point of tangency of the failure envelope with a Mohr's circle (that is, point A) give the stresses (normal and shear) on the failure plane of that test specimen.

For normally consolidated clay, referring to Figure 12.18

$$\sin\emptyset' = \frac{AO'}{OO'}$$

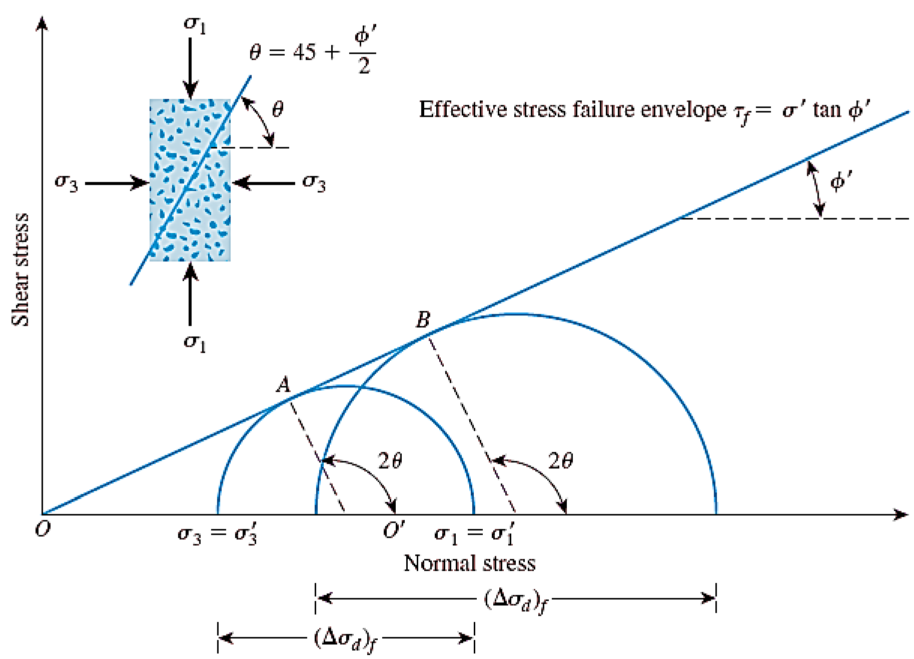

Figure 12.18 Effective stress failure envelope from drained tests on sand and normally consolidation clay

or

$$\sin\emptyset' = \frac{(\frac{\sigma'_1-\sigma'_3}{2})}{(\frac{\sigma'_1+\sigma'_3}{2})}$$

$$\emptyset' = \sin^{-1}(\frac{\sigma'_1-\sigma'_3}{\sigma'_1+\sigma'_3}) \qquad (12.16)$$

Also, the failure plane will be inclined at an angle of $\theta = 45 + \emptyset'/2$ to the major principal plane, as shown in Figure (12.18).

Overconsolidation results when a clay initially is consolidated under an all-around chamber pressure of $\sigma_c (= \sigma'_c)$ and is allowed to swell by reducing the chamber pressure to $\sigma_3 (= \sigma'_3)$. The failure envelope obtained from drained triaxial tests of such overconsolidated clay specimens shows two distinct branches (*ab* and *bc* in Figure 12.19). The portion *ab* has a flatter slope with a cohesion intercept, and the shear strength equation for this branch can be written as

$$\tau_f = c' + \sigma' \tan\emptyset'_1 \qquad (12.17)$$

The portion *bc* of the failure envelope represents a normally consolidated stage of soil and follows the equation $\tau_f = \sigma' \tan\emptyset'$.

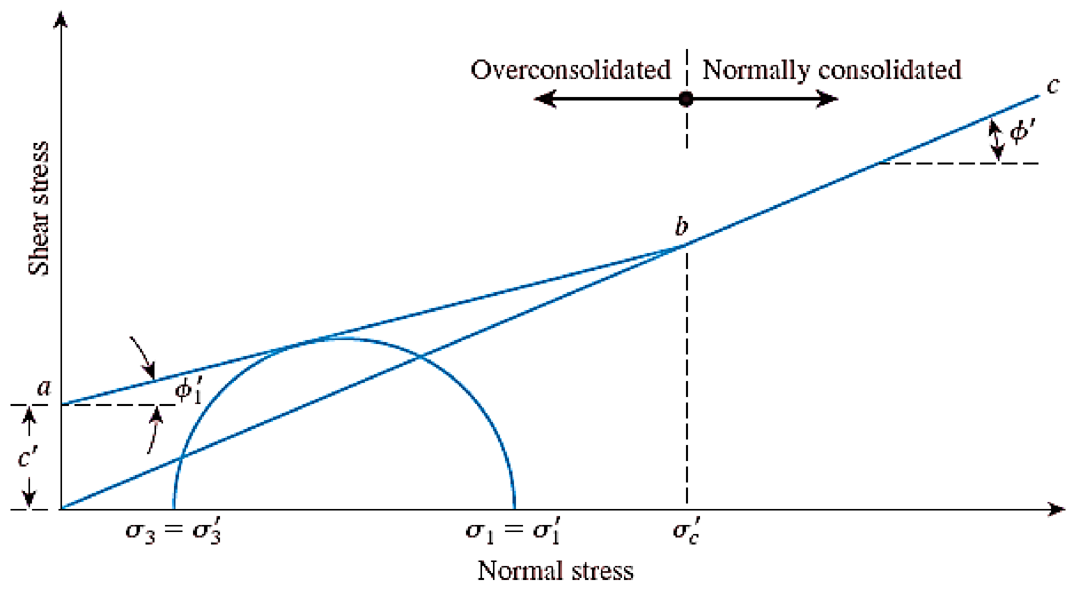

Figure 12.19 Effective stress failure envelope for overconsolidated clay

A consolidated-drained triaxial test on a clayey soil may take several days to complete. This amount of time is required because deviator stress must be applied very slowly to ensure full drainage from the soil specimen. For this reason, the CD type of triaxial test is uncommon.

Example 12.4

A consolidated-drained triaxial test was conducted on a normally consolidated clay. The results are as follows:

- $\sigma_3 = 110.4\ kN/m^2$
- $(\Delta\sigma_3)_f = 172.5 kN/m^2$

Determine
a. Angle of friction, $\emptyset'$
b. Angle θ that the failure plane makes with the major principal plane

Solution

For normally consolidated soil, the failure envelope equation is

$$\tau_f = \sigma' tan\emptyset' \qquad \text{(because } c'\text{=0)}$$

For the triaxial test, the effective major and minor principal stresses at failure are as follows:

$$\sigma'_1 = \sigma_1 = \sigma_3 + (\Delta\sigma_d)_f = 110.4 + 172.5 = 282.9\ kN/m^2$$

and

$$\sigma'_3 = \sigma_3 = 110.4\ kN/m^2$$

Part a

The Mohr's circle and the failure envelope are shown in Figure (12.20). From Eq. (12.16),

$$sin\emptyset' = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma_3} = \frac{282.9 - 110.4}{282.9 + 110.4} = 0.438$$

or

$$\emptyset' = 26^o$$

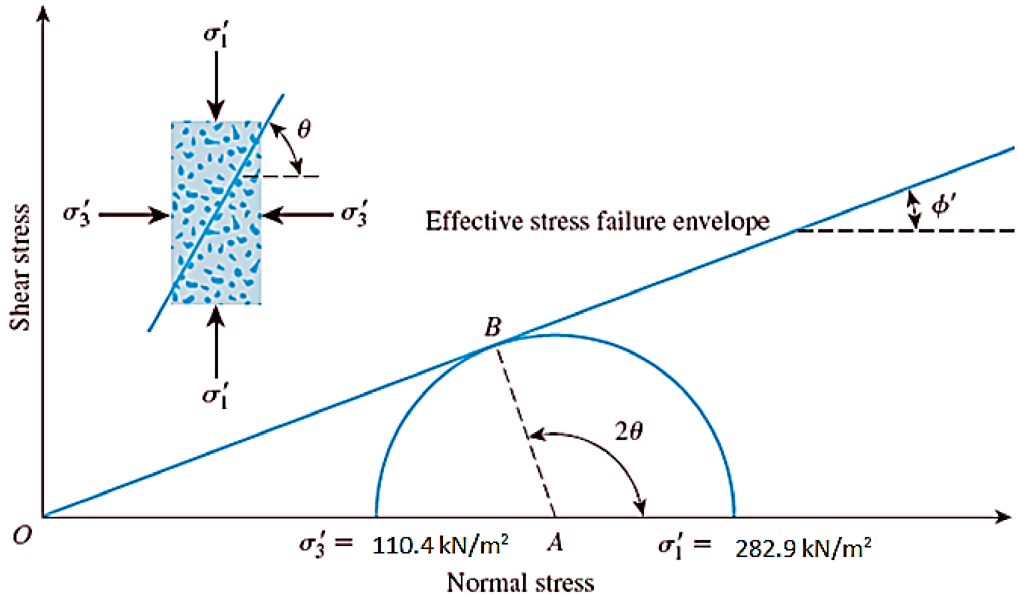

Figure 12.20 Mohr's circle and failure envelope for a normally consolidated clay

Part b

From Eq. (12.11),

$$\theta = 45 + \frac{\phi'}{2} = 45^o + \frac{26^o}{2} = 58^o$$

Example 12.5

Refer to Example 12.4.

a. Find the normal stress σ' and the shear stress τ_f on the failure plane.

b. Determine the effective normal stress on the plane of maximum shear stress.

Solution:

Part a

From Eqs. (12.6) and (12.7),

$$\sigma'(on the failure plane) = \frac{\sigma'_1 + \sigma'_3}{2} + \frac{\sigma'_1 - \sigma'_3}{2} cos2\theta$$

and

$$\tau_f = \frac{\sigma'_1 - \sigma'_3}{2} sin2\theta$$

Substituting the values of $\sigma'_1 = 282.9\ kN/m^2$, $\sigma'_3 = 110.4\ kN/m^2$, and $\theta = 58^o$ into the preceding equations, we get

$$\sigma' = \frac{282.9+110.4}{2} + \frac{282.9-110.4}{2}\cos(2 \times 58) = 158.84\ kN/m^2$$

$$\tau_f = \frac{282.9-110.4}{2}\sin(2 \times 58) = 77.52\ kN/m^2$$

Part b

From Eq. (12.7), it can be seen that the maximum shear stress will occur on the plane with $\theta = 45^o$. From Eq. (12.6),

$$\sigma' = \frac{\sigma'_1+\sigma'_3}{2} + \frac{\sigma'_1-\sigma'_3}{2} cos2\theta$$

Substituting $\theta = 45^o$ into the preceding equation gives

$$\sigma' = \frac{282.9+110.4}{2} + \frac{282.9-110.4}{2} cos90 = 196.65\ kN/m^2$$

Example 12.6

The results of two drained triaxial tests on a saturated clay follow:

Specimen I:

$$\sigma_3 = 70\ kN/m^2$$

$$(\Delta\sigma_d)_f = 130 kN/m^2$$

Specimen II:

$$\sigma_3 = 160\ kN/m^2$$

$$(\Delta\sigma_d)_f = 223.5\ kN/m^2$$

Determine the shear strength parameters.

Solution

Refer to Figure (12.21). For Specimen I, the principal stresses at failure are

$$\sigma'_3 = \sigma_3 = 70\ kN/m^2$$

and

$$\sigma'_1 = \sigma_1 = \sigma_3 + (\Delta\sigma_d)_f = 70 + 130 = 200 kN/m^2$$

Similarly, the principal stresses at failure for Specimen II are

$$\sigma'_3 = \sigma_3 = 160\ kN/m^2$$

and

$$\sigma'_1 = \sigma_1 = \sigma_3 + (\Delta\sigma_d)_f = 160 + 223.5 = 383.5\ kN/m^2$$

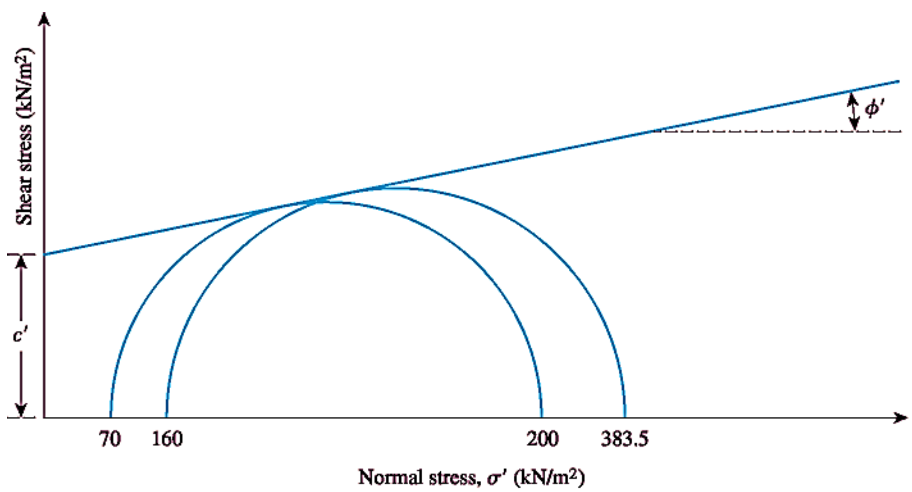

Figure 12.21 Effective stress failure envelope and Mohr's circles for Specimens I and II

From Figure (12.21), the values of $\emptyset'$ and c' are:

$$\emptyset' = 20^o, \qquad c' = 20kN/m^2$$

12.10 Consolidated-Undrained Triaxial Test

The consolidated-undrained test is the most common type of triaxial test. In this test, the saturated soil specimen is first consolidated by an all-around chamber fluid pressure, σ_3 , that results in drainage. After the pore water pressure generated by the application of confining pressure is dissipated, the deviator stress, $\Delta\sigma_d$, on the specimen is increased to cause shear failure. During this phase of the test, the drainage line from the specimen is kept closed. Because drainage is not permitted, the pore water pressure, Δu_d , will increase.

Unlike the consolidated-drained test, the total and effective principal stresses are not the same in the consolidated-undrained test. Because the pore water pressure at failure is measured in this test, the principal stresses may be analyzed as follows:

Major principal stress at failure (total): $\sigma_3 + (\Delta\sigma_d)_f = \sigma_1$

Major principal stress at failure (effective): $\sigma_1 - (\Delta u_d)_f = \sigma'_1$

Minor principal stress at failure (total): σ_3

Minor principal stress at failure (effective): $\sigma_3 - (\Delta u_d)_f = \sigma'_3$

In these equations, $(\Delta u_d)_f =$ pore water pressure at failure. The preceding derivations show that

$$\sigma_1 - \sigma_3 = \sigma'_1 - \sigma'_3$$

Tests on several similar specimens with varying confining pressures may be conducted to determine the shear strength parameters. Figure (12.22) shows the total and effective stress Mohr's circles at failure obtained from consolidated-undrained triaxial tests in sand and normally consolidated clay. Note that *A* and *B* are two total stress Mohr's circles obtained from two tests. *C* and *D* are the effective stress Mohr's circles corresponding to total stress circles *A* and *B*, respectively. The diameters of circles *A* and *C* are the same; similarly, the diameters of circles *B* and *D* are the same.

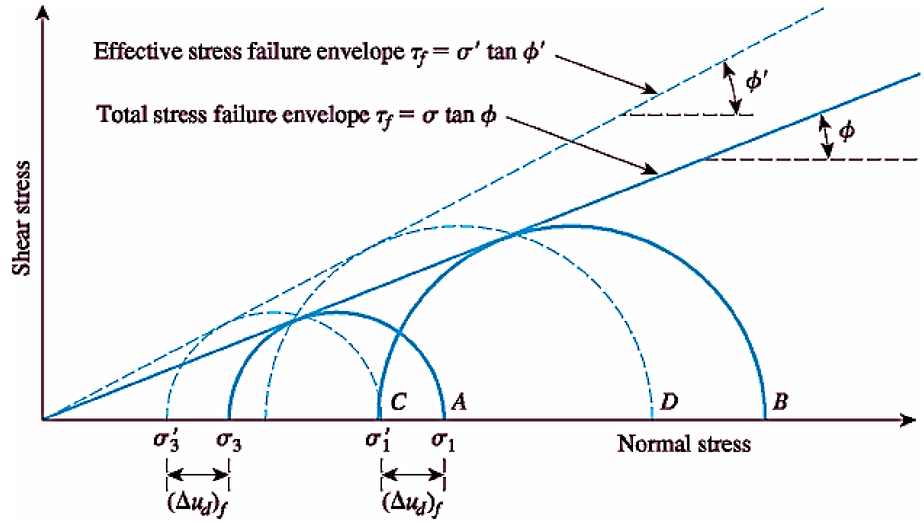

Figure 12.22 Total and effective stress failure envelopes for consolidated undrained triaxial tests. (Note: The figure assumes that no back pressure is applied.)

In Figure 12.27, the total stress failure envelope can be obtained by drawing a line that touches all the total stress Mohr's circles. For sand and normally consolidated clays, this will be approximately a straight line passing through the origin and may be expressed by the equation

$$\tau_f = \sigma \ tan\emptyset \qquad (12.18)$$

Where σ = total stress

$\emptyset$ = the angle that the total stress failure envelope makes with the normal stress axis, also known as the *consolidated-undrained angle of shearing resistance*

Equation (12.18) is seldom used for practical considerations. Similar to Eq. (12.16), for sand and normally consolidated clay, we can write

$$\emptyset = \sin^{-1}\left(\frac{\sigma_1 - \sigma_3}{\sigma_1 + \sigma_3}\right) \qquad (12.19)$$

and

$$\emptyset' = \sin^{-1}\left(\frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3}\right)$$

$$= \sin^{-1}\left\{\frac{[\sigma_1 - (\Delta u_d)_f] - [\sigma_3 - (\Delta u_d)_f]}{[\sigma_1 - (\Delta u_d)_f] + [\sigma_3 - (\Delta u_d)_f]}\right\}$$

$$= \sin^{-1}\left[\frac{\sigma_1-\sigma_3}{\sigma_1+\sigma_3-2(\Delta u_d)_f}\right] \qquad (12.20)$$

Again referring to Figure (12.22), we see that the failure envelope that is tangent to all the effective stress Mohr's circles can be represented by the equation $\tau_f = \sigma' tan\emptyset'$, which is the same as that obtained from consolidated-drained tests (see Figure 12.18).

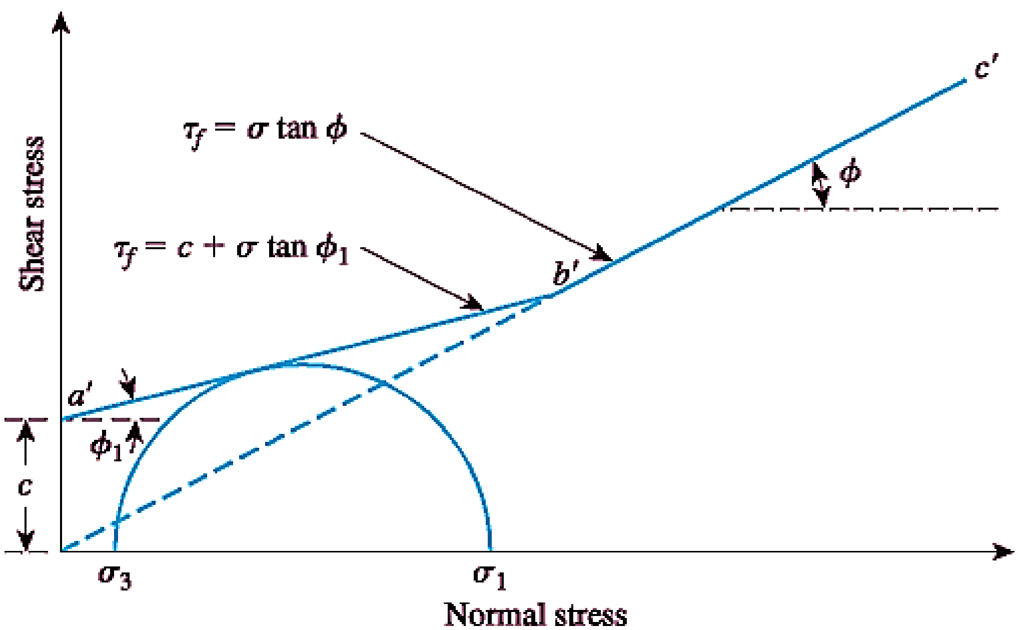

Figure 12.23 Total stress failure envelope obtained from consolidated-undrained tests in over-consolidated clay

In overconsolidated clays, the total stress failure envelope obtained from consolidated-undrained tests will take the shape shown in Figure (12.23). The straight line $a'b'$ is represented by the equation

$$\tau_f = c + \sigma \, tan\emptyset_1 \qquad (12.21)$$

and the straight line $a'b'$ follows the relationship given by Eq. (12.18). The effective stress failure envelope drawn from the effective stress Mohr's circles will be similar to that shown in Figure 12.19.

Consolidated-drained tests on clay soils take considerable time. For this reason, consolidated-undrained tests can be conducted on such soils with pore pressure measurements to obtain the drained shear strength parameters. Because drainage is not allowed in these tests during the application of deviator stress, they can be performed quickly.

Example 12.7

A specimen of saturated sand was consolidated under an all-around pressure of 120 kN/m^2. The axial stress was then increased and drainage was prevented. The specimen failed when the axial deviator stress reached 91 kN/m^2. The pore water pressure at failure was 68 kN/m^2. Determine

a. Consolidated-undrained angle of shearing resistance, $\emptyset$
b. Drained friction angle, $\emptyset'$

Solution

Part a

For this case, $\sigma_3 = 120\ kN/m^2$, $\sigma_1 = 120 + 91 = 211\ kN/m^2$, and $(\Delta u_d)_f = 68\ kN/m^2$. The total and effective stress failure envelopes are shown in Figure (12.24). From Eq. (12.19),

$$\emptyset = \sin^{-1}\left(\frac{\sigma_1-\sigma_3}{\sigma_1+\sigma_3}\right) = \sin^{-1}\left(\frac{211-120}{211+120}\right) \approx 16^o$$

Part b

From eq. (12.20),

$$\emptyset' = \sin^{-1}\left[\frac{\sigma_1-\sigma_3}{\sigma_1+\sigma_3-2(\Delta u_d)_f}\right] = \sin^{-1}\left[\frac{211-120}{211+120-2\times 68}\right] = 27.8^o$$

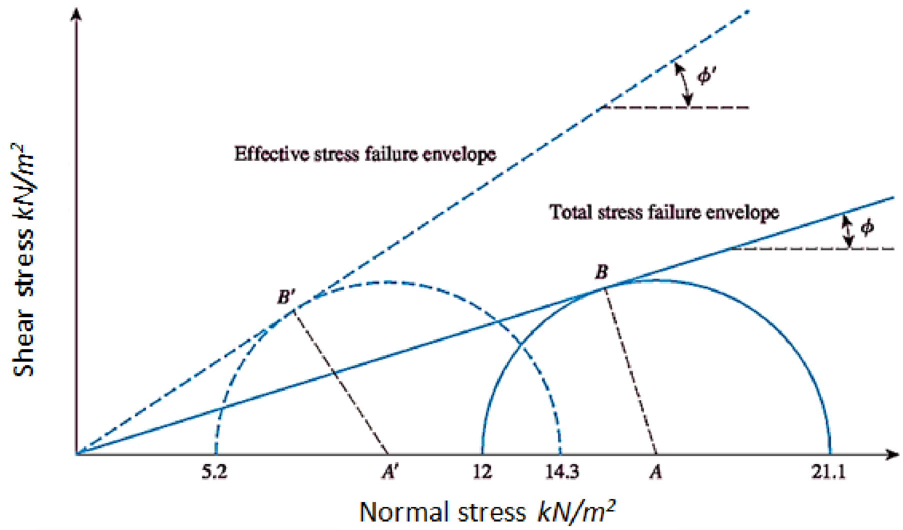

Figure 12.24 Failure envelopes and Mohr's circles for a saturated sand

12.11 Unconsolidated- Undrained Triaxial Test

In unconsolidated-undrained tests, drainage from the soil specimen is not permitted during the application of chamber pressure σ_3. The test specimen is sheared to failure by the application of deviator stress, $\Delta\sigma_d$, and drainage is prevented. Because drainage is not allowed at any stage, the test can be performed quickly. Because of the application of chamber confining pressure σ_3, the pore water pressure in the soil specimen will increase by u_c . A further increase in the pore water pressure (Δu_d) will occur because of the deviator stress application. Hence, the total pore water pressure u in the specimen at any stage of deviator stress application can be given as

$$u = u_c + \Delta u_d \tag{12.22}$$

This test usually is conducted on clay specimens and depends on a very important strength concept for cohesive soils if the soil is fully saturated. The added axial stress at failure $(\Delta\sigma_d)_f$ is practically the same regardless of the chamber confining pressure. This property is shown in Figure (12.25). The failure envelope for the total stress Mohr's circles becomes a horizontal line and hence is called a $\emptyset = 0$ condition. From Eq. (12.2) with $\emptyset = 0$, we get

$$\tau_f = c = c_u \tag{12.23}$$

where c_u is the undrained shear strength and is equal to the radius of the Mohr's circles. Note that the $\emptyset = 0$ concept is applicable to only saturated clays and silts.

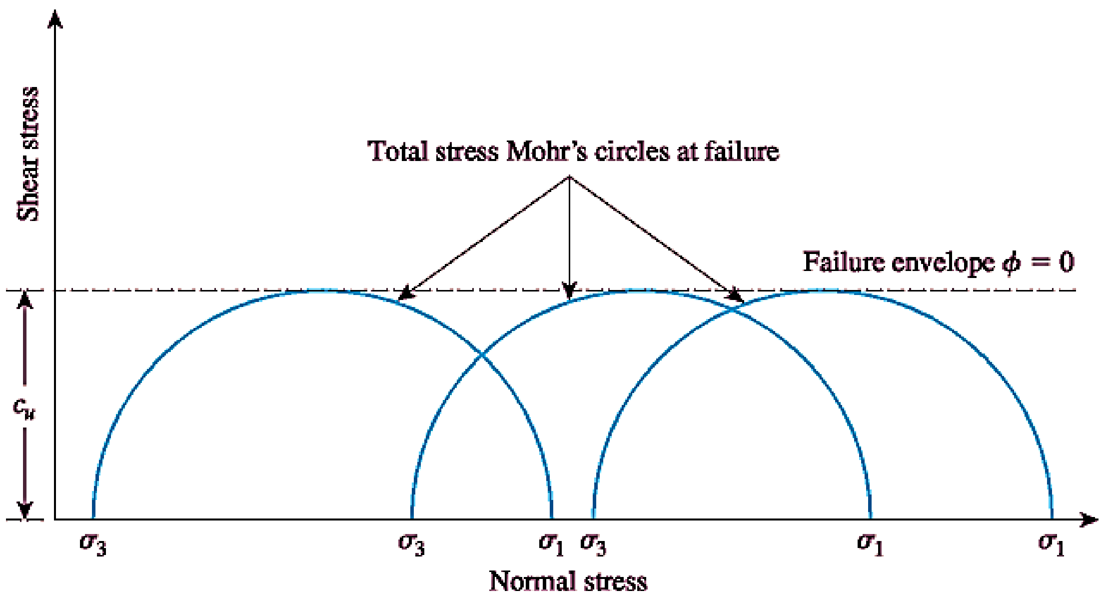

Figure 12.25 Total stress Mohr's circles and failure envelope ($\emptyset = 0$) obtained from unconsolidated-undrained triaxial tests on fully saturated cohesive soil

The reason for obtaining the same added axial stress $(\Delta\sigma_d)_f$ regardless of the confining pressure can be explained as follows. If a clay specimen (No.I) is consolidated at a chamber pressure σ_3 and then sheared to failure without drainage, the total stress conditions at failure can be represented by the Mohr's circle P in Figure (12.26). The pore pressure developed in the specimen at failure is equal to $(\Delta u_d)_f$. Thus, the major and minor principal effective stresses at failure are, respectively,

$$\sigma'_1 = [\sigma_3 + (\Delta\sigma_d)_f] - (\Delta u_d)_f = \sigma_1 - (\Delta u_d)_f$$

and

$$\sigma'_3 = \sigma_3 - (\Delta u_d)_f$$

Q is the effective stress Mohr's circle drawn with the preceding principal stresses. Note that the diameters of circles P and Q are the same.

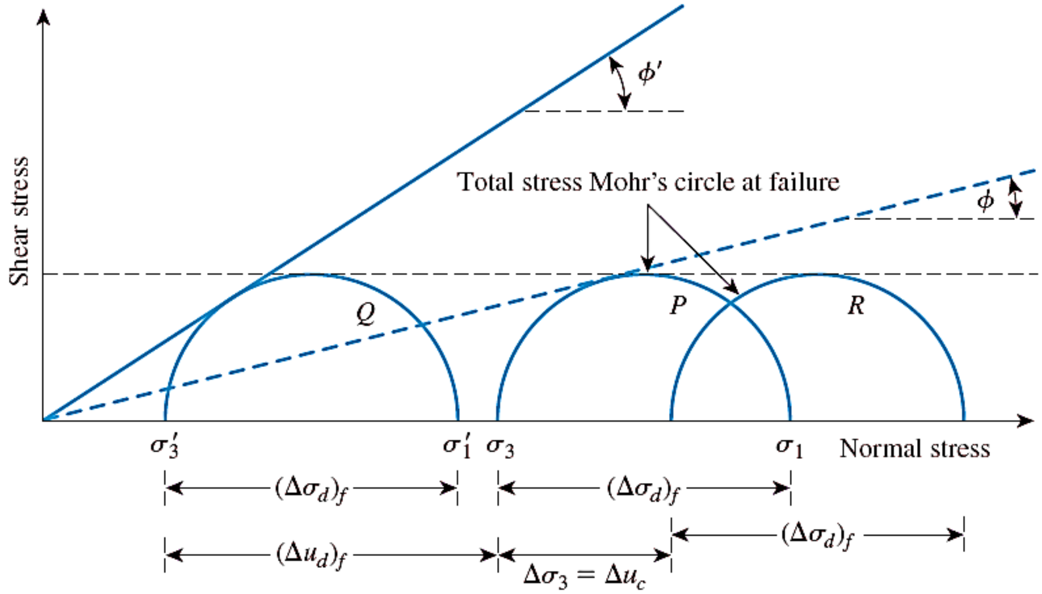

Figure 12.26 The $\emptyset = 0$ concept

Now let us consider another similar clay specimen (No. II) that has been consolidated under a chamber pressure σ_3 with initial pore pressure equal to zero. If the chamber pressure is increased by $\Delta\sigma_3$ without drainage, the pore water pressure will increase by an amount Δu_c. For saturated soils under isotropic stresses, the pore water pressure increase is equal to the total stress increase, so $\Delta u_c = \Delta\sigma_3$. At this time, the effective confining pressure is equal to $\sigma_3 + \Delta\sigma_3 - \Delta u_c = \sigma_3 + \Delta\sigma_3 - \Delta\sigma_3 = \sigma_3$. This is the same as the effective confining pressure of Specimen I before the application of deviator stress. Hence, if Specimen II is sheared to failure by increasing the axial stress, it should fail at the same deviator stress $(\Delta\sigma_d)_f$ that was obtained for Specimen I. The total stress Mohr's circle at failure will be R (see Figure 12.26). The added pore pressure increase caused by the application of $(\sigma_d)_f$ will be $(\Delta u_d)_f$.

At failure, the minor principal effective stress is

$$[(\sigma_3 + \Delta\sigma_3)] - \left[\Delta u_c + (\Delta u_d)_f\right] = \sigma_3 - (\Delta u_d)_f = \sigma'_3$$

and the major principal effective stress is

$$\left[\sigma_3 + \Delta\sigma_3 + (\Delta\sigma_d)_f\right] - \left[\Delta u_c + (\Delta u_d)_f\right] = \left[\sigma_3 + (\Delta\sigma_d)_f\right] - (\Delta u_d)_f = \sigma_1 - (\Delta u_d)_f = \sigma'_1$$

Thus, the effective stress Mohr's circle will still be Q because strength is a function of effective stress. Note that the diameters of circles P, Q, and R are all the same.

Any value of $\Delta\sigma_3$ could have been chosen for testing Specimen II. In any case, the deviator stress $(\Delta\sigma_d)_f$ to cause failure would have been the same as long as the soil was fully saturated and fully undrained during both stages of the test.

12.12 Unconfined Compression Test on Saturated Clay

The unconfined compression test is a special type of unconsolidated-undrained test that is commonly used for clay specimens. In this test, the confining pressure σ_3 is 0. An axial load is rapidly applied to the specimen to cause failure. At failure, the total minor principal stress is zero and the total major principal stress is σ_1 (Figure 12.27). Because the undrained shear strength is independent of the confining pressure as long as the soil is fully saturated and fully undrained, we have

$$\tau_f = \frac{\sigma_1}{2} = \frac{q_u}{2} = c_u \qquad (12.24)$$

where q_u is the *unconfined compression strength*. Table (12.2) gives the approximate consistencies of clays on the basis of their unconfined compression strength. A photograph of unconfined compression test equipment is shown in Figure (12.28). Figures (12.29) and (12.30) show the failure in two specimens—one by shear and one by bulging—at the end of unconfined compression tests.

Theoretically, for similar saturated clay specimens, the unconfined compression tests and the unconsolidated-undrained triaxial tests should yield the same values of c_u. In practice, however, unconfined compression tests on saturated clays yield slightly lower values of c_u than those obtained from unconsolidated-undrained tests.

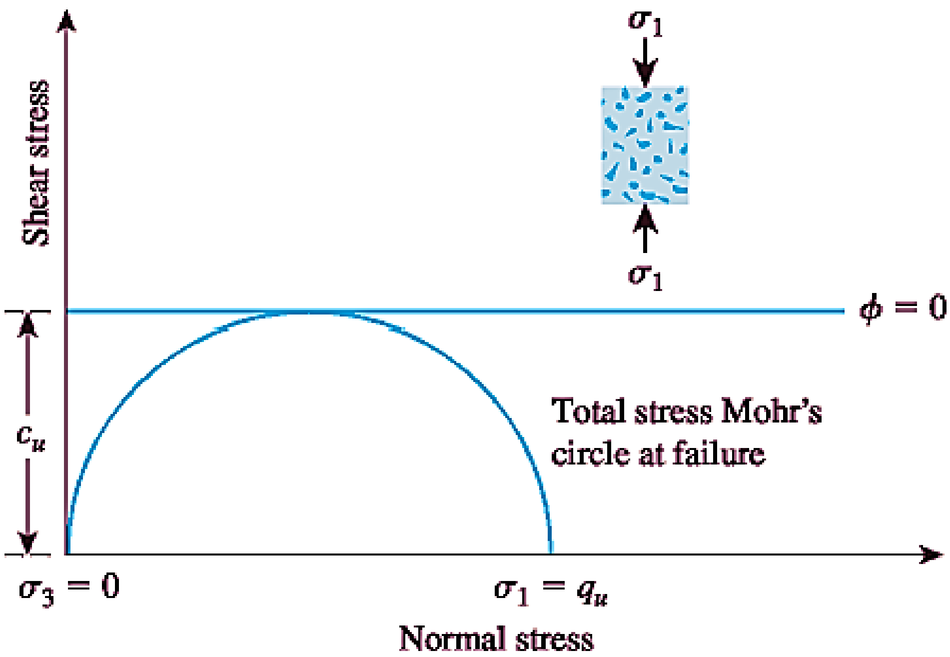

Figure 12.27 Unconfined compression test

Table 12.2 General Relationship of Consistency and Unconfined Compression Strength of Clays

	q_u	
Consistency	**kN/m²**	**ton/ft²**
Very soft	0–25	0–0.25
Soft	25–50	0.25–0.5
Medium	50–100	0.5–1
Stiff	100–200	1–2
Very stiff	200–400	2–4
Hard	>400	>4

Figure 12.28 Unconfined compression test equipment

Figure 12.29 Failure by shear of an unconfined compression test specimen

Figure (12.30) Failure by bulging of an unconfined compression test specimen

12.13 Vane Shear Test

Fairly reliable results for the undrained shear strength, c_u ($\emptyset = 0$ concept), of very soft to medium cohesive soils may be obtained directly from vane shear tests. The shear vane usually consists of four thin, equal-sized steel plates welded to a steel torque rod (Figure 12.31). First, the vane is pushed into the soil. Then torque is applied at the top of the torque rod to rotate the vane at a uniform speed. A cylinder of soil of height h and diameter d will resist the torque until the soil fails. The undrained shear strength of the soil can be calculated as follows.

If T is the maximum torque applied at the head of the torque rod to cause failure, it should be equal to the sum of the resisting moment of the shear force along the side surface of the soil cylinder (M_s) and the resisting moment of the shear force at each end (M_e) (Figure 12.32):

$$T = M_s + \underbrace{M_e + M_e}_{\text{Two ends}} \qquad (12.25)$$

The resisting moment can be given as

$$M_s = \underbrace{(\pi d h)_{c_u}}_{\text{Surface area}} \quad \underbrace{\left(\frac{d}{2}\right)}_{\text{Moment arem}}$$

where d = diameter of the shear vane
h = height of the shear vane

For the calculation of M_e, investigators have assumed several types of distribution of shear strength mobilization at the ends of the soil cylinder:

1. *Triangular*. Shear strength mobilization is c_u at the periphery of the soil cylinder and decreases linearly to zero at the center.
2. *Uniform*. Shear strength mobilization is constant (that is, c_u) from the periphery to the center of the soil cylinder.
3. *Parabolic*. Shear strength mobilization is c_u at the periphery of the soil cylinder and decreases parabolically to zero at the center.

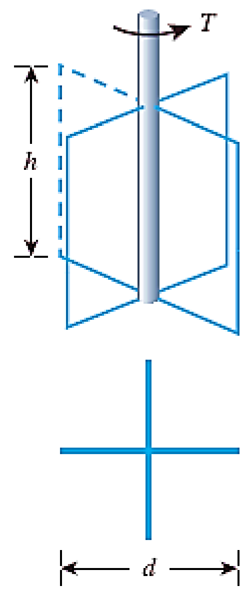

Figure 12.31 Diagram of vane shear test equipment

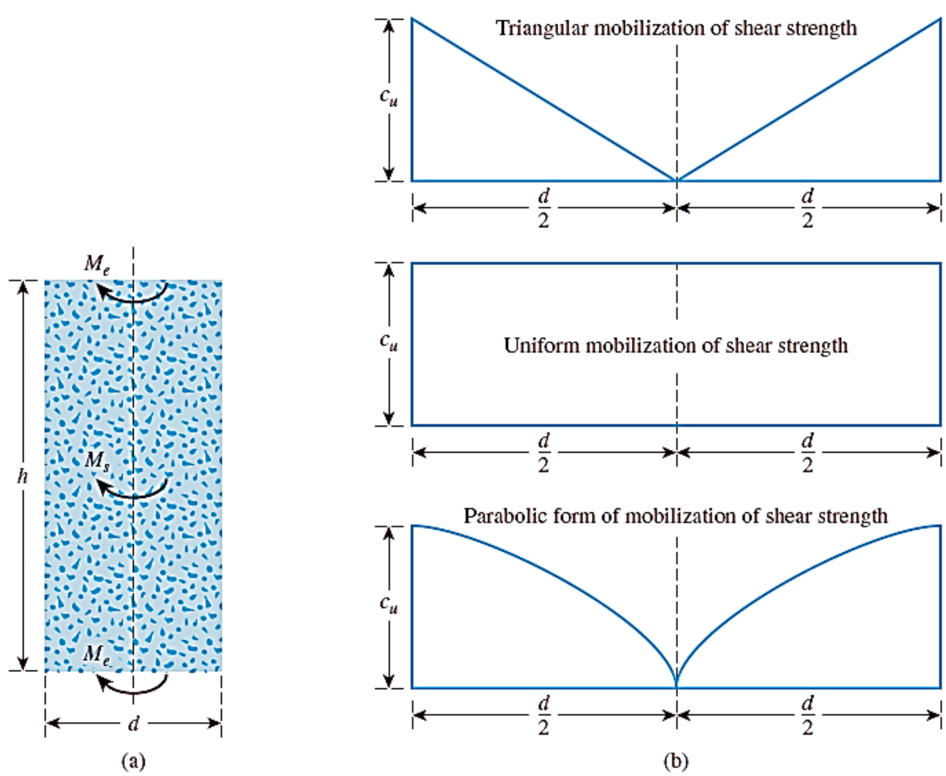

Figure 12.32 Derivation of Eq. (12.28): (a) resisting moment of shear force; (b) variations in shear strength-mobilization

These variations in shear strength mobilization are shown in Figure (12.32b). In general, the torque, T, at failure can be expressed as

$$T = \pi c_u \left[\frac{d^2 h}{2} + \beta \frac{d^3}{4}\right] \tag{12.27}$$

or

$$c_u = \frac{T}{\pi\left[\frac{d^2 h}{2} + \beta \frac{d^3}{4}\right]} \tag{12.28}$$

where β = 1/2 for triangular mobilization of undrained shear strength
β = 2/3 for uniform mobilization of undrained shear strength
β = 3/5 for parabolic mobilization of undrained shear strength

Note that Eq. (12.28) usually is referred to as Calding 's equation.

Vane shear tests can be conducted in the laboratory and in the field during soil exploration. The laboratory shear vane has dimensions of about 13 *mm* (1/2 in.) in diameter and 25 *mm* (1 in.) in height. Figure (12.33) shows a photograph of laboratory vane shear test equipment.

Figure 12.33 Laboratory vane shear test device (Courtesy of ELE International)

Problems

12.1 For a direct shear test on a dry sand, the following are given:

- Specimen size: 75 *mm* ×75 *mm* × 30 *mm* (height)
- Normal stress: 200 kN/m^2
- Shear stress at failure: 175 kN/m^2

a. Determine the angle of friction, $\emptyset'$

b. For a normal stress of 150 kN/m^2, what shear force is required to cause failure in the specimen?

Ans: a. 41.2º b. 0.739 *kN*

12.2 The following are the results of four drained, direct shear tests on a normally consolidated clay. Given:

- Size of specimen = 60 *mm* × 60 *mm*
- Height of specimen = 30 *mm*

Test no.	Normal force (N)	Shear force at failure (N)
1	200	155
2	300	230
3	400	310
4	500	385

Draw a graph for the shear stress at failure against the normal stress, and determine the drained angle of friction from the graph.

Ans: 37.5º

12.3 The equation of the effective stress failure envelope for a loose, sandy soil was obtained from a direct shear test at $\tau_f = \sigma' tan30^o$. A drained triaxial test was conducted with the same soil at a chamber confining pressure of 100 kN/m^2. Calculate the deviator stress at failure.

Ans: 200 kN/m^2

12.4 The relationship between the relative density, D_r, and the angle of friction, $\emptyset'$, of a sand can be given as $\emptyset' = 25 + 0.18D_r$ (D_r is in %). A drained triaxial test on the same sand was conducted with a chamber-confining pressure of 180 kN/m^2. The relative density of compaction was 60%. Calculate the major principal stress at failure.

Ans: 687 kN/m^2

12.5 For a normally consolidated clay, $\emptyset' = 24^o$. In a drained triaxial test, the specimen failed at a deviator stress of 175 *kN/m²*. What was the chamber confining pressure, σ'_3 ?

Ans: 127.7 *kN/m²*

12.6 A consolidated-drained triaxial test was conducted on a normally consolidated clay. The results were as follows:

$\sigma_3 = 250\ kN/m^2$

$(\Delta\sigma_d)_f = 275\ kN/m^2$

Determine:

a. Angle of friction, $\emptyset'$

b. Angle θ that the failure plane makes with the major principal plane

c. Normal stress, σ' , and shear stress, τ_f , on the failure plane

Ans: (a) 20.8º (b) 55.4º (c) $\boldsymbol{\sigma' = 338.7\frac{kN}{m^2}}$, $\boldsymbol{\tau = 128.7\ kN/m^2}$

12.7 The results of two drained triaxial tests on a saturated clay are given next:

Specimen I: Chamber confining pressure = 150 *kN/m²*

Deviator stress at failure = 314 *kN/m²*

Specimen II: Chamber-confining pressure = 250 *kN/m²*

Deviator stress at failure = 470 *kN/m²*

If the clay specimen described above is tested in a triaxial apparatus with a chamber-confining pressure of 250 *kN/m²*, what is the major principal stress at failure, σ'_1?

Ans: 720 *kN/m²*

12.8 A consolidated-undrained test on a normally consolidated clay yielded the following results:

- $\sigma_3 = 150\ kN/m^2$
- Deviator stress: $(\Delta\sigma_d)_f = 110\ kN/m^2$

Pore pressure: $(\Delta u_d)_f = 72\ kN/m^2$

Calculate the consolidated-undrained friction angle and the drained friction angle.

Ans: $\emptyset = \mathbf{15.6^o}$, $\emptyset' = \mathbf{24.4^o}$

12.9 The shear strength of a normally consolidated clay can be given by the equation $\tau_f = \sigma' \tan 31^o$. A consolidated-undrained triaxial test was conducted on the clay. Following are the results of the test:

- Chamber confining pressure = 112 kN/m^2
- Deviator stress at failure = 100 kN/m^2

Determine:

a. Consolidated-undrained friction angle

b. Pore water pressure developed in the clay specimen at failure

Ans: a. 18^o , b. 64.9 kN/m^2

12.10 For a normally consolidated clay soil, $\emptyset' = 32^o$ and $\emptyset = 22^o$. A consolidated undrained triaxial test was conducted on this clay soil with a chamber-confining pressure of 150 kN/m^2. Determine the deviator stress and the pore water pressure at failure.

Ans: $(\Delta\boldsymbol{\sigma_d})_f = \mathbf{179.7}\,\frac{kN}{m^2}$, $(\Delta\boldsymbol{u_d})_f = \mathbf{70.1}\,\frac{kN}{m^2}$

12.11 The friction angle, $\emptyset'$, of a normally consolidated clay specimen collected during field exploration was determined from drained triaxial tests to be 23^o. The unconfined compression strength, q_u, of a similar specimen was found to be 120 kN/m^2. Determine the pore water pressure at failure for the unconfined compression test.

Ans: -93.5 kN/m^2