Lectures in Finite Element Method

Ph. D. Students Course/ Structures Engineering

2hours/week

15 weeks course

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Syllabus

Ch.1: Finite element method: Description and application

Ch.2: Finite element formulation

Ch.3: FEM for trusses

Ch.4: FEM for beams

Ch.5: FEM for frames

Ch.6: FEM for plates

Ch.7: FEM for flow in coastal water bodies

Ch.8: FEM for species transport

Ch.9: FEM for heat transfer

References

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2- Fish, J. and Belytschko, T. (2007) " A First Course in Finite Elements", JohnWiley & Sons, Ltd.

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Chapter 1

Finite Element Method: Description and Application

1.1 Introduction

The behavior of a phenomenon in a system depends upon the geometry or domain of the system, the property of the material or medium, and the boundary, initial and loading conditions. For an engineering system, the geometry or domain can be very complex. Further, the boundary and initial conditions can also be complicated. It is therefore, in general, very difficult to solve the governing differential equation via analytical means. In practice, most of the problems are solved using numerical methods. These methods include; finite element method (FEM), finite difference method, finite volume method, and boundary element method.

1.2 General Description of FEM

It is one of numerical methods that can be used to solve ordinary and partial differential equations. In FEM, the actual continuous body of matter like solid, liquid or gas is represented as an assemblage of subdivisions called finite elements. These elements are considered to be interconnected at specified joints which are called nodes or nodal points. The nodes usually lay on the element boundaries where elements are connected. Since the actual variation of the field variable (like stress, displacement, temperature, pressure, or velocity) inside the continuum is not known, it is assumed that the variation of the field variable inside a finite element can be approximated by simple function. These approximating functions (also called interpolation models) are defined in terms of the values of the field variables at the nodes. When field equations for the whole body are written, the unknowns will be the nodal values of the field variable. By solving the field equations, which are generally in the form of matrix equations, the nodal values of the field variable will be known.

1.3 Steps of FEM Application

The steps of FEM application include:

1. Discretization of the domain.
2. Selection of interpolation function.
3. Finite element formulation (derivation of finite element equations).
4. Assemblage of element equations to obtain the overall equilibrium equations (or final global equations).
5. Solution of finite matrix equations.
6. Computation of field functions and other variables.

1.3.1 Domain discretization

The discretization of the domain or solution region into sub regions (finite element) is the first step in the FEM. By this step, the domain having an infinite number of degrees of freedom is replaced by a system having finite number of degrees of freedom. Here, the finite element mesh (or finite element grid map) is formed. In discretization, the followings must be considered:

- a- Type and order of the elements.
- b- Size of the elements.
- c- Number of the elements.
- d- Location of nodes.
- e- Nodes numbering.

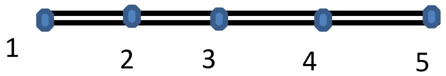
Why do we discretize???

A- Type and order of elements

The type of the elements depends on the number of the independent variables of the problem. It may be one-, two- or three-dimensional element.

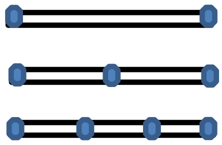
One- dimensional element

For example if we have a beam, the beam is usually discretized into a number of one-dimensional elements.



The order of one-dimensional elements may be;

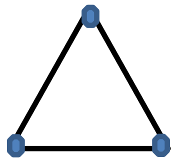
- 1-D linear element (number of nodes=2)
- 1-D quadratic element (number of nodes=3)
- 1-D cubic element (number of nodes=4)



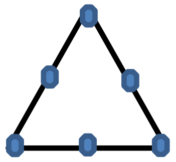
Two-dimensional elements

2-D elements can be triangular or quadrilateral (e.g., rectangular or parallelogram) elements.

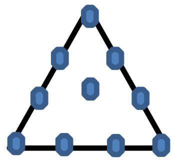
<u>Order of triangular elements:</u>



Linear element
(3 nodes)

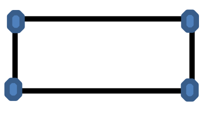


Quadratic element
(6 nodes)

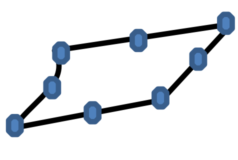


Cubic element
(10 nodes)

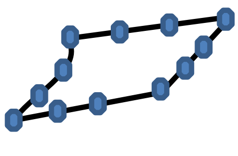
<u>Order of quadrilateral elements</u>



Linear element
(4-nodes)



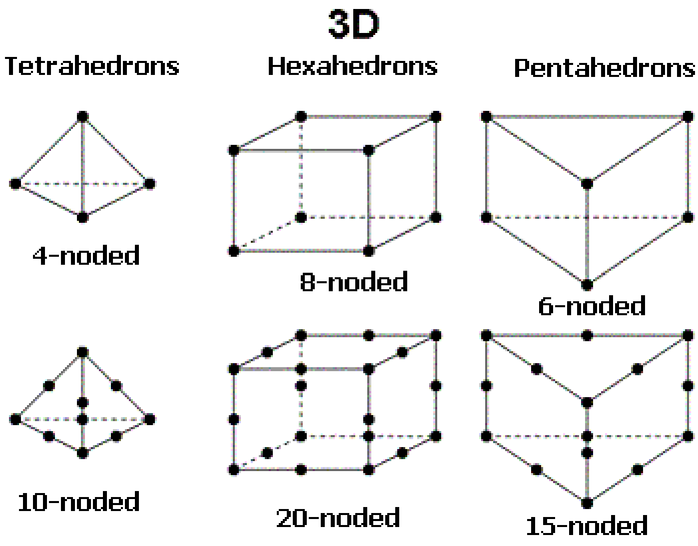
Quadratic element
(8-nodes)



Cubic element
(12-nodes)

Three- dimensional elements

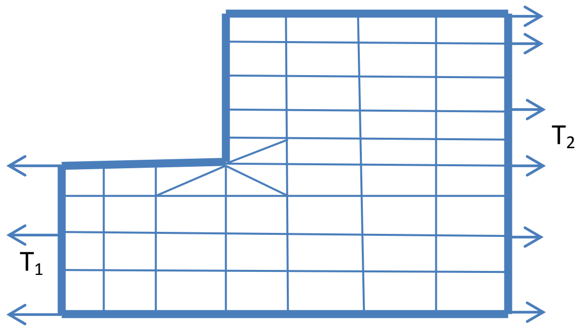
3-D elements include tetrahedron, hexahedron and pentahedron elements. They may be linear, quadratic or cubic.



Note: as the order of elements increases, the accuracy of FEM increases

B- Size of elements

The size of elements influences the convergence of the solution and hence it has to be chosen with care. If the size of elements is small, the final solution is expected to be more accurate. However, the use of small elements means more computational time. Different sizes of elements can be used with the same problem domain. Generally, small size of elements must be used where high gradients of field variables are expected.

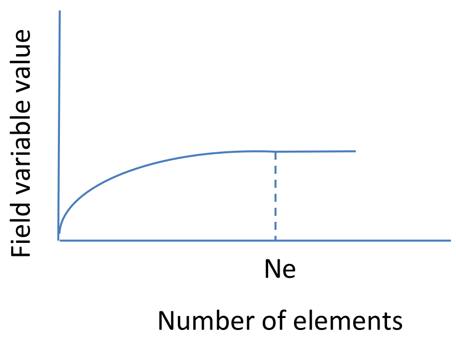


C- Number of elements

The number of elements is related to;

- Accuracy desired.
- Element size.
- Number of degrees of freedom.

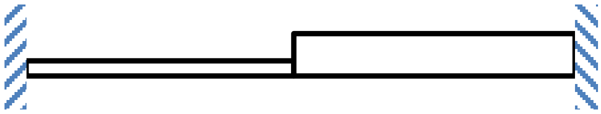
Generally, at first a number of elements is chosen then solution is obtained. After that, the number of elements is increased. If the results are not the same, then, there is a need for increasing the elements number. This operation is called mesh refinement.



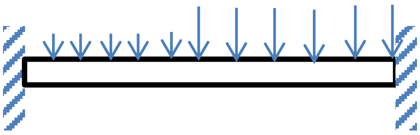
No significant improvement beyond Ne → Use Ne

D- Location of nodes

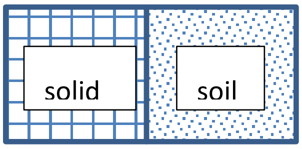
If the body has no abrupt changes in geometry, material properties and external conditions (like load, temp., etc..), the body can be divided into equal elements and hence the spacing of nodes can be uniform. On the other hand if there are any discontinuities in the problem, nodes have to be introduced at these discontinuities.



A beam with different



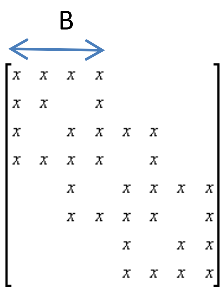
The applied load is varied abruptly



Soil- solid interaction

E- Nodes numbering

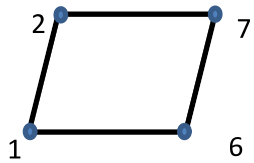
Node numbering influences the computer storage requires to solve the final global matrix. The band width (B) of the overall or global matrix depends on node numbering scheme and the number of degrees of freedom per node.



$B=(R+1)\times f$

Where R is the max largest difference in the node numbers occurring for all elements of the assemblage and f is the number of degrees of freedom at each node.

<u>e.g.</u>



if f=3

B= 7×3=21

Total storage= B× number of equations

Note: number the nodes in short direction at first

F- Mesh generation

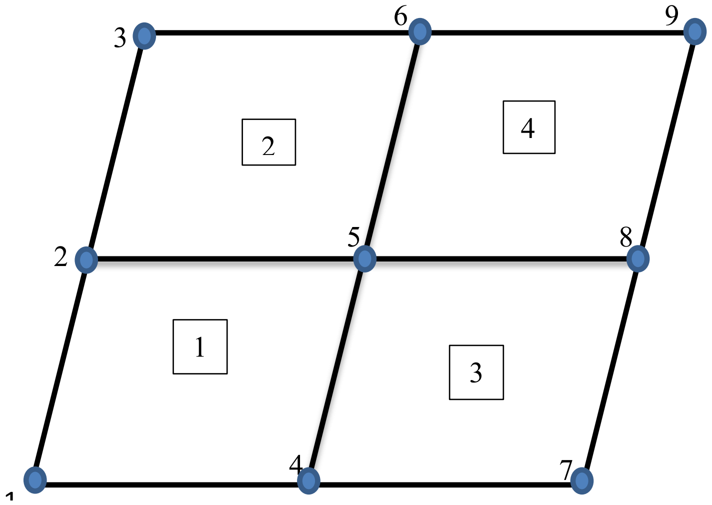
The output of discretization step is mesh generation, which includes:

- Nodes generation
- Elements generation

Nodes generation produces a list contains the number and coordinates of each node in the finite element mesh. In 2-D (Cartesian coordinates system) FE mesh it will include;

<u>Node number</u>   <u>x- coordinate</u>   <u>y- coordinate</u>

Elements generation produces a list contains element numbers of the mesh and the numbers of nodes associated with each element. For example, if the problem domain is discretized using linear quadrilateral elements as shown below;



Elements generation list shall be;

Element No.	Node numbers			
1	1	4	5	2
2	2	5	6	3
3	4	7	8	5
4	5	8	9	6

1.3.2 Interpolation functions

After the division of the problem domain into subdomains or elements, the variation of the variable is defined approximately over the element in terms of an interpolation function. The most used function type is polynomial interpolation function. For example if we have a 1-D problem and one variable say Ø, then;

Variation of Ø over the element $=Ø^e$

If linear elements are used;

$Ø^e = \alpha_1 + \alpha_2 x$ …(1.1)

Quadratic elements;

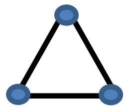
$Ø^e = \alpha_1 + \alpha_2 x + \alpha_3 x^2$ … (1.2)

Cubic elements;

$\varnothing^e = \alpha_1 + \alpha_2\,x + \alpha_3\,x^2 + \alpha_4\,x^3$ …(1.3)

For 2-D problems;

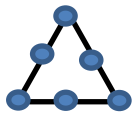
Linear triangle elements;



$\varnothing^e = \alpha_1 + \alpha_2\,x + \alpha_3\,y$ …(1.4)

Quadratic triangle elements;

$\varnothing^e = \alpha_1 + \alpha_2\,x + \alpha_3\,y + \alpha_4 x^2 + \alpha_5\,xy + \alpha_6\,y^2$ …(1.5)



Where; α_1, α_2 , α_3 …and α_6 are constants (or the coefficient for the monomials)

Cubic triangular elements (10 nodes)

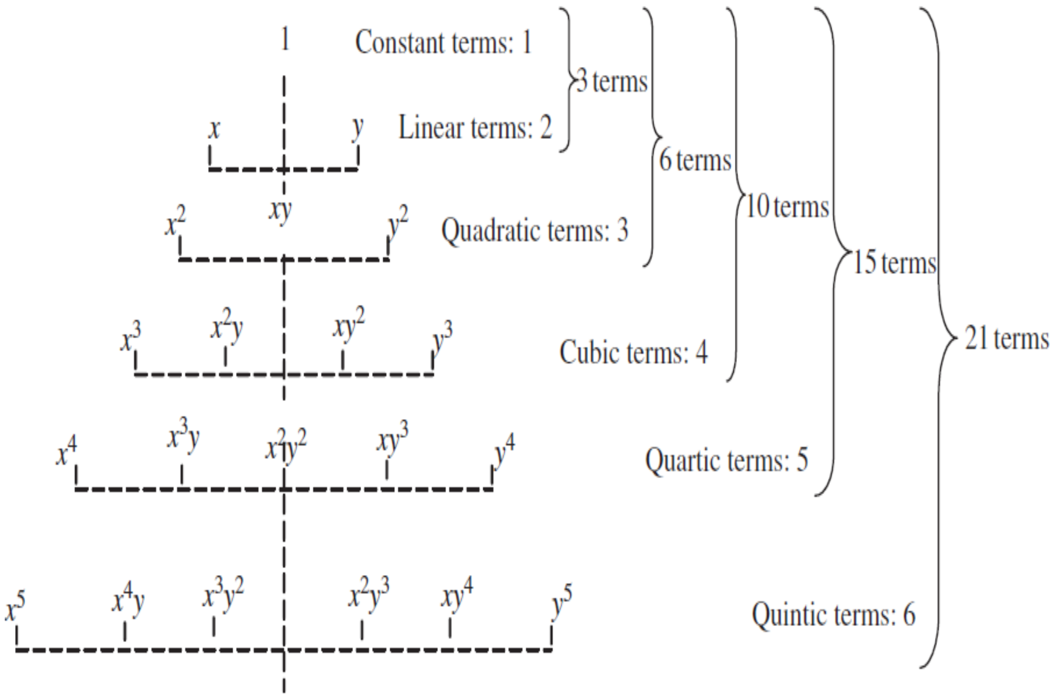


Fig.(1.1): Pascal triangle of monomials (two- dimensional case)

1.3.2.1 Interpolation functions in terms of nodal values

Let $\O^e = [C]\ \{\alpha\}$ …(1.6)

Where; $[C] = [1\ \ x\ \ y\ \ x^2\ \ xy\ \ y^2]$ for quadratic triangular elements

$$\{\alpha\}=\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\end{Bmatrix}$$

In terms of nodal values:

$$\O_1 = \alpha_1 + \alpha_2 x_1 + +\alpha_3 y_1 + \alpha_4 x_1^2 + \cdots$$

$$\O_2 = \alpha_1 + \alpha_2 x_2 + +\alpha_3 y_2 + \alpha_4 x_2^2 + \cdots$$

$$\O_3 = \alpha_1 + \alpha_2 x_3 + +\alpha_3 y_3 + \alpha_4 x_3^2 + \cdots$$

.

.

$$\O_n = \alpha_1 + \alpha_2 x_n + +\alpha_3 y_n + \alpha_4 x_n^2 + \cdots$$

where; $\O_n$ is Ø value at node n.

$\therefore\ \{\O\} = [D]\{\alpha\}$; {Ø} = nodal values vector

where; [D]= matrix of nodal points coordinates

$$[D]=\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & . & . & \\ 1 & x_2 & y_2 & x_2^2 & . & . & \\ 1 & x_1 & y_3 & x_3^2 & . & . & \\ . & . & & & & & \\ . & & & & & & \\ . & . & & & & & \\ . & & & & & & \\ 1 & x_n & y_n & x_3^2 & . & . & y_n^2 \end{bmatrix} \quad \text{…(1.7)}$$

We want the values of α_1, α_2, …….and α_6

$$\therefore \{\alpha\} = [D]^{-1}\{\emptyset\} \qquad \ldots(1.8)$$

Substituting Eq.(1.8) into Eq.(1.6);

$$\emptyset^e = [C][D]^{-1}\{\emptyset\} \qquad \ldots(1.9)$$

Let $[N] = [C][D]^{-1}$; N= shape function in terms of x and y

$$\therefore \emptyset^e = [N]\{\emptyset\} \qquad \ldots(1.10)$$

or;

$$\emptyset^e = [N_1 \quad N_2 \quad N_3 \quad \cdot \quad \cdot \quad \cdot \quad N_n]\begin{Bmatrix} \emptyset_1 \\ \emptyset_2 \\ \emptyset_3 \\ \\ \\ \emptyset_n \end{Bmatrix}$$

$$\therefore \emptyset^e = N_1\emptyset_1 + N_2\emptyset_2 + N_3\emptyset_3 + \cdots + N_4\emptyset_4$$

∴ The variation of the field variable Ø over the element e is approximated as;

$$\emptyset^e = \sum_{i=1}^{n} N_i\emptyset_i \qquad \ldots(1.11)$$

Where;

N_i= shape function of node i

n= number of nodes in element e

If the case of two field variables is considered;

Let u and v are displacements in x- and y-direction, respectively.

The variations of u and v over the element (e) are approximated as;

$$\mathrm{u}^e = \sum_{i=1}^{n} N_i \mathrm{u}_i \qquad \& \qquad \mathrm{v}^e = \sum_{i=1}^{n} N_i \mathrm{v}_i$$

Hint: u and v may be defined using different types of shape functions.

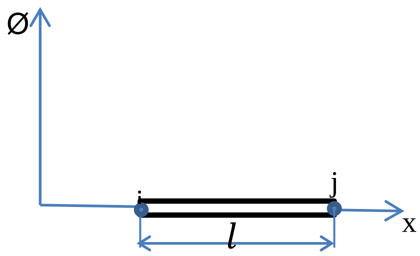
u and v are connected into a vector d^e which is called displacement vector;

$$d^e = \begin{Bmatrix} \mathrm{u}^e \\ \mathrm{v}^e \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & . & . & N_n & 0 \\ 0 & N_1 & 0 & N_2 & . & . & 0 & N_n \end{bmatrix} \begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{v}_1 \\ \mathrm{u}_2 \\ \mathrm{v}_2 \\ \vdots \\ \mathrm{u}_n \\ \mathrm{v}_n \end{Bmatrix} \quad \ldots (1.12)$$

Linear interpolation polynomials in terms of global coordinates

<u>1-D linear elements</u>

For 1-D linear elements;



$\emptyset^e = \alpha_1 + \alpha_2 x \qquad \ldots(1.13)$

$$\therefore \emptyset_i = \alpha_1 + \alpha_2 x_i$$

$$\emptyset_j = \alpha_1 + \alpha_2 x_j$$

The solution of the above two equations gives;

$$\alpha_1 = \frac{\emptyset_i x_j - \emptyset_j x_i}{x_j - x_i}$$

$$\alpha_2 = \frac{\emptyset_j - \emptyset_i}{x_j - x_i}$$

Substituting the values of α_1 and α_2 into Eq.(1.13) gives;

$$\emptyset^e = \frac{x_j - x}{l}\emptyset_i + \frac{x - x_i}{l}\emptyset_j \qquad \ldots (1.14)$$

$$\emptyset^e = N_i \emptyset_i + N_j \emptyset_j \qquad \ldots(1.15)$$

Where; N_i and N_j are shape functions of nodes *i* and *j*, respectively.

$$\emptyset^e = [N]\{\emptyset\}$$

Where;

[N] = matrix of shape functions (1×2)

{Ø}=vector of nodal unknowns of element e

Hints:

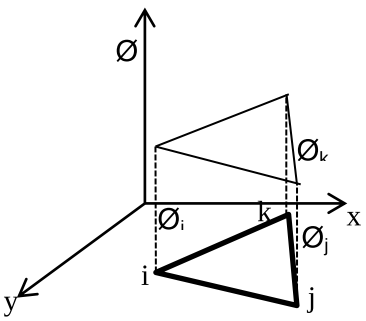
- N_i =1 at node i and N_i =0 at the other nodes.
- $\frac{\partial \emptyset}{\partial x} = \frac{\partial N_i}{\partial x}\emptyset_i + \frac{\partial N_j}{\partial x}\emptyset_j$

2-D Triangular element

For linear elements;

$\emptyset^e = \alpha_1 + \alpha_2 x + \alpha_3 y$ …(1.16)

$\therefore \emptyset_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$
$\emptyset_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j$
$\emptyset_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k$



Solving the above equations for α gives;

$$\alpha_1 = \frac{1}{2A}\left[(x_j y_k - x_k y_j)\emptyset_i + (x_k y_i - x_i y_k)\emptyset_j + (x_i y_j - x_j y_i)\emptyset_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(y_j - y_k)\emptyset_i + (y_k - y_i)\emptyset_j + (y_i - y_j)\emptyset_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[(x_k - x_j)\emptyset_i + (x_i - x_k)\emptyset_j + (x_j - x_i)\emptyset_k\right]$$

Where; A=area of the triangle i j k given by;

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \quad \ldots(1.17)$$

By substituting the values of α into Eq.(1.16) and arranging the results;

$$\emptyset^e = N_i\emptyset_i + N_j\emptyset_j + N_k\emptyset_k$$

Where;

$N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$

$a_i = x_j y_k - x_k y_j$

$b_i = y_j - y_k$

$c_i = x_k - x_j$

$N_j = \frac{1}{2A}\left(a_j + b_j x + c_j y\right)$

$a_j = x_k y_i - x_i y_k$

$b_j = y_k - y_i$

$c_j = x_i - x_k$

$N_k = \frac{1}{2A}(a_k + b_k x + c_k y)$

$a_k = x_i y_j - x_j y_i$

$b_k = y_i - y_j$

$c_k = x_j - x_i$

Hints;

- N_i =1 at node i and N_i =0 at the other nodes and so on.
- $\frac{\partial\emptyset}{\partial x} = \frac{\partial N_i}{\partial x}\emptyset_i + \frac{\partial N_j}{\partial x}\emptyset_j + \frac{\partial N_k}{\partial x}\emptyset_k = constant$
- $\frac{\partial\emptyset}{\partial y} = \frac{\partial N_i}{\partial y}\emptyset_i + \frac{\partial N_j}{\partial y}\emptyset_j + \frac{\partial N_k}{\partial y}\emptyset_k = constant$

Where; $\emptyset_i$, $\emptyset_j$, and $\emptyset_k$ are the nodal values of Ø and they are independent on x or y. A constant value of $\frac{\partial\emptyset}{\partial x}$ and $\frac{\partial\emptyset}{\partial y}$ over an element means that many small elements have to be used in locations where rapid changes of Ø are expected.

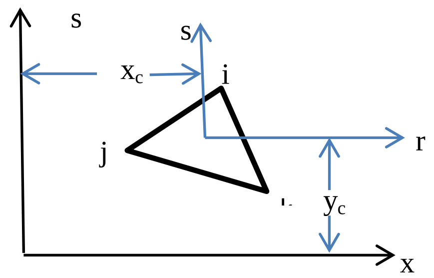
Linear interpolation polynomials in terms of local coordinates

The derivation of element matrices and vectors involves the integration of the shape functions or their derivatives or both over the element. These integrals can be evaluated easily if the interpolation functions are written in terms of a local coordinates system. A local coordinates system is one which is located on or within the boundaries of the element and it is different for different elements.

r-s ⇒ local coordinates system
x-y ⇒ global coordinates system

$$x = x_c + r$$
$$y = y_c + s$$

Where x_c and y_c denotes the centroidal coordinates given by;



$x_c = (x_i + x_j + x_k)/3$

$y_c = (y_i + y_j + y_k)/3$

H.W.1 Show that;

$$N_i(r,s) = \frac{1}{2A}(\bar{a}_i + \bar{b}_i x + \bar{c}_i y)$$
$$N_j(r,s) = \frac{1}{2A}(\bar{a}_j + \bar{b}_j x + \bar{c}_j y)$$
$$N_k(r,s) = \frac{1}{2A}(\bar{a}_k + \bar{b}_k x + \bar{c}_k y)$$

where;

$\bar{a}_i = \bar{a}_j = \bar{a}_k = 2A/3$

$\bar{b}_i = b_i;\ \bar{b}_j = b_j;\ \bar{b}_k = b_k$

$\bar{c}_i = c_i;\ \bar{c}_j = c_j;\ \bar{c}_k = c_k$

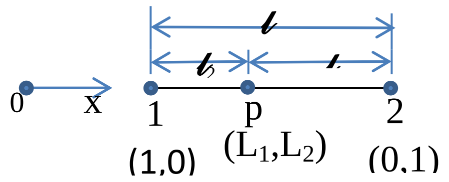
H.W.2 Show that the integral of shape function in global coordinates system is the same as the integral of shape function in local coordinates system.

Natural coordinates system

It is a local coordinates system that permits the specification of any point inside the element by a set of non-dimensional numbers, which have magnitude lies between 0 and 1.

1-D linear element

Any point in the element (p) is identified by two natural coordinates L_1 and L_2 which are defined as;

$L_1 = \frac{l_1}{l}$ and $L_2 = \frac{l_2}{l}$ or; $L_1 = \frac{x_2 - x}{x_2 - x_1}$ and $L_2 = \frac{x - x_1}{x_2 - x_1}$ where; $L_1 + L_2 = 1$	

From above it can be shown that;

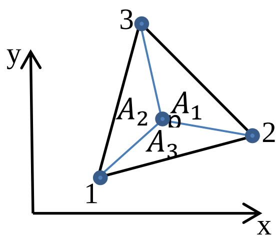
$N_1 = L_1$ and $N_2 = L_2$

Integration of polynomial terms in natural coordinates system can be done using; $\int_{x_1}^{x_2} L_1^{\alpha}\, L_2^{\beta} dx = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} l$

2-D Linear triangular element

A natural coordinate system for a triangular element (also known as the triangular coordinate system) is shown in the figure below. Although three coordinates L_1, L_2, and L_3 are used to define a point P, only two of them are independent. The natural coordinates are defined as;

$$L_1 = \frac{A_1}{A}; \quad L_2 = \frac{A_2}{A}\ ; \quad L_3 = \frac{A_3}{A}$$



where A_1 is the area of the triangle formed by the points P, 2 and 3; A_2 is the area of the triangle formed by the points P, 1 and 3; A_3 is the area of the triangle formed by the points P, 1 and 2; and A is the area of the triangle 123. Because L_i are defined in terms of areas, they are also known as area coordinates. Since

$A_1 + A_2 + A_3 = A$, then;

$$L_1 + L_2 + L_3 = 1 \quad \ldots(1.17)$$

A study of the properties of L_1, L_2, and L_3 shows that they are also the shape functions for the 2-D triangular element:

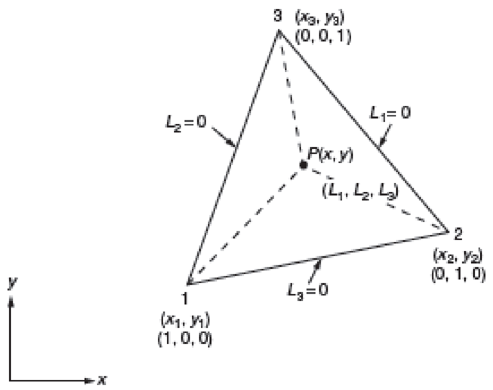
$N_1 = L_1$; $N_2 = L_2$; $N_3 = L_3$

To every set of natural coordinates (L_1, L_2, L_3), there corresponds a unique set of Cartesian coordinates (x, y). The relation between the natural and Cartesian coordinates is given by;

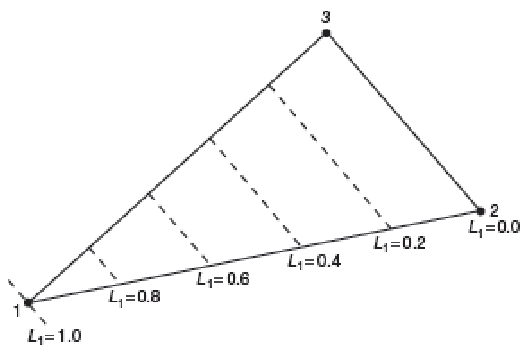
$$x = x_1L_1 + x_2L_2 + x_3L_3 \quad \ldots(1.18)$$

$$y = y_1L_1 + y_2L_2 + y_3L_3 \quad \ldots(1.19)$$

At node 1, L_1 = l and $L_2 = L_3 = 0$, and so on. The linear relationship between L_i (i = 1, 2, 3) and (x, y) implies that the contours of L_1 are equally placed straight lines parallel to the side 2, 3 of the triangle (on which $L_1 = 0$), and so on.



(a)



(b)

Equations (1.17), (1.18) and (1.19) can be written in a matrix form as;

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad \ldots(1.20)$$

Then;

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad \ldots(1.21)$$

Where A is the area of the triangle 1, 2, 3 given by;

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad \ldots(1.22)$$

Or;

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}$$

Thus at any point (p) in the element there is a value of N_1, N_2 and N_3. If f is a function of L_1, L_2 and L_3, the differentiation with respect to x and y can be performed as;

$$\left. \begin{aligned} \frac{\partial f}{\partial x} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i} \frac{\partial L_i}{\partial x} \\ \frac{\partial f}{\partial y} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i} \frac{\partial L_i}{\partial y} \end{aligned} \right\}$$

Derive $\frac{\partial L_1}{x}$ as an example

By using natural coordinates system, there is an exact expressions for integrals in terms of natural coordinates.

⇒ Integration over an edge

$$\int_{\Gamma} L_1^{\alpha} L_2^{\beta} d\Gamma = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} \cdot \Gamma \quad \ldots(1.23)$$

Where; Γ= length of the edge where the integration is carried out.

Note: any point locates on the edge between points 1 and 3, as an example, has coordinates in terms of L_1 *and* L_3*. While a point inside the triangle has three coordinates* L_1*,* L_2 *and* L_3*.*

⇒ Integration over an area

$$\int_{A} L_1^{\alpha} L_2^{\beta} L_3^{\gamma} dA = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} \cdot 2A \quad \ldots(1.24)$$

e.g.; if we have an integration

$$\int_A N_i N_j dA = \int_A L_1^1 L_2^1 L_3^0 dA = \frac{1!\,1!\,0!}{(1+1+0+2)!} \cdot 2A = \frac{A}{12}$$

3- D Tetrahedron linear element

The natural coordinates for a tetrahedron element can be defined analogous to those of a triangular element. Thus, four coordinates L_1, L_2, L_3, and L_4 will be used to define a point P, although only three of them are independent. These natural coordinates are defined as;

$$L_1 = \frac{V_1}{V} \;\; ; \;\; L_2 = \frac{V_2}{V} \;\; ; \;\; L_3 = \frac{V_3}{V} \;\; ; \;\; L_4 = \frac{V_4}{V}$$

where V_i is the volume of the tetrahedron formed by the points P and the vertices other than the vertex i (i = 1, 2, 3, 4), and V is the volume of the tetrahedron element defined by the vertices 1, 2, 3, and 4 (Figure 3.15). Because the natural coordinates are defined in terms of volumes, they are also known as volume or tetrahedral coordinates. Since;

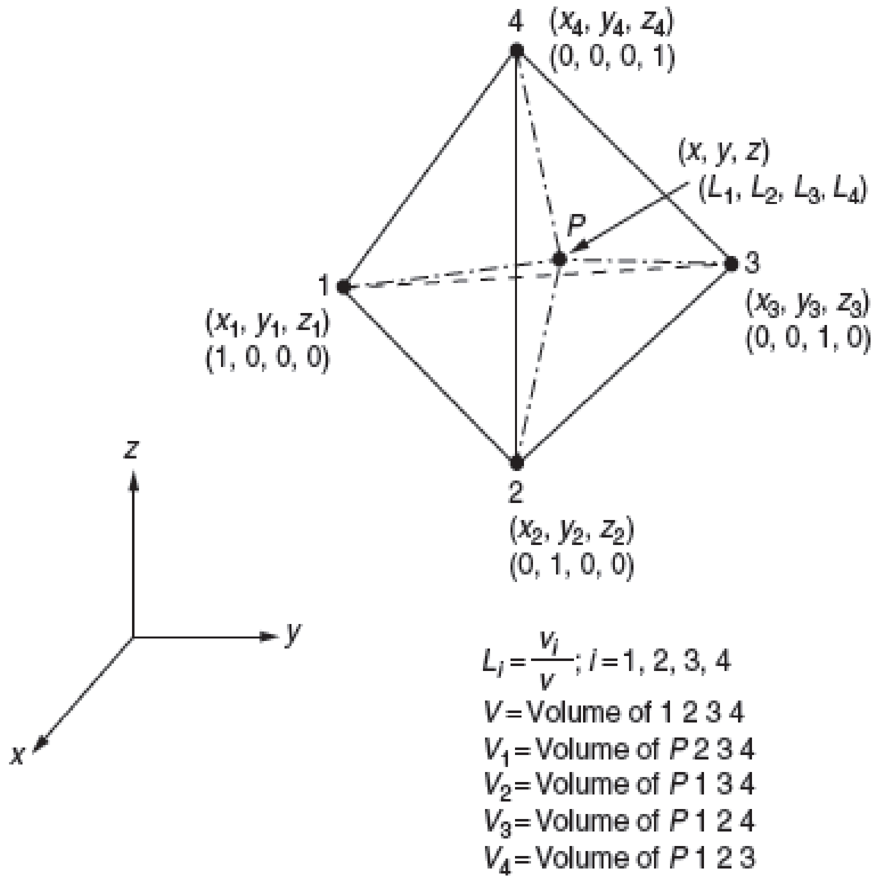
$V_1 + V_2 + V_3 + V_4 = V$

Then;

$L_1 + L_2 + L_3 + L_4 = 1$

The volume coordinates L_1, L_2, L_3, and L_4 are also the shape functions for a three-dimensional simplex element:

$N_1 = L_1$; $N_2 = L_2$; $N_3 = L_3$; $N_4 = L_4$



The Cartesian and natural coordinates are related as;

$$\left.\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 + L_4 y_4 \\ z &= L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4 \end{aligned}\right\}$$

Then;

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix}$$

The inverse relation can be expressed as;

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

Where;

$$V = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \text{volume of the tetrahedron } 1, 2, 3, 4$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$$

$$b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$c_1 = -\begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix}$$

$$d_1 = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Also;

$$\int_V L_1^{\alpha} L_2^{\beta} L_3^{\gamma} L_4^{\varepsilon} dV = \frac{\alpha!\,\beta!\,\gamma!\,\varepsilon!}{(\alpha+\beta+\gamma+\varepsilon+3)!} \cdot 6V \quad \ldots (1.25)$$

Higher Order Elements in Terms of Natural Coordinates

H.W

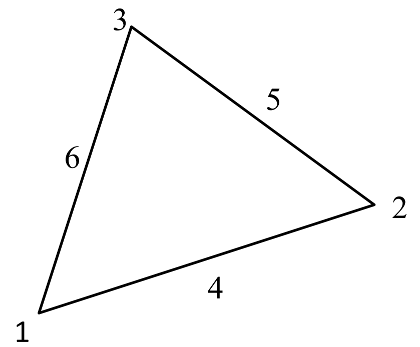
1- Show that for 1-D quadratic elements;

$N_1 = L_1\,(2L_1 - 1)$; $N_2 = 4L_1L_2$; $N_3 = L_2\,(2L_2 - 1)$

2- Show that for 2-D quadratic triangular elements;

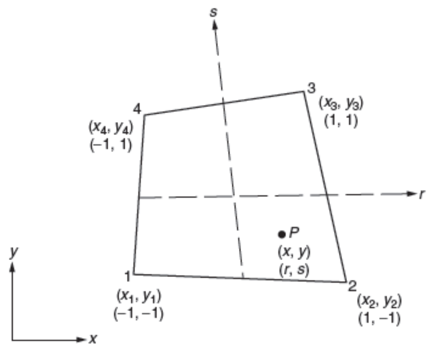
$$N_1 = L_1(2L_1 - 1) \;;\quad N_2 = L_2(2L_2 - 1) \;;\quad N_3 = L_3(2L_3 - 1)$$

$$N_4 = 4L_1L_2 \;;\quad N_5 = 4L_2L_3 \;;\quad N_6 = 4L_1L_3$$



2-D Quadrilateral elements

A different type of natural coordinate system can be established for a quadrilateral element in two dimensions as shown in the figure below. For the local r, s (natural) coordinate system, the origin is taken as the intersection of lines joining the midpoints of opposite sides and the sides are defined by r = ±1 and s = ±1.



1- Linear quadrilateral elements

$N_i\ (r,s) = \frac{1}{4}(1 + rr_i)(1 + ss_i)$; i=1, 2, 3, 4 …(1.26)

2- Quadratic quadrilateral element

For nodes at r=±1 , s=±1

$N_i\ (r,s) = \frac{1}{4}(1 + rr_i)(1 + ss_i)(rr_i + ss_i)$ ... (1.27)

For nodes at r=0 , s=±1

$N_i\ (r,s) = \frac{1}{2}(1 - r^2)(1 + ss_i)$ …(1.28)

For nodes at r =±1 , s=0

$$N_i\ (r,s) = \frac{1}{2}(1 + rr_i)(1 - s^2) \quad ...\ (1.29)$$

Nodes Mapping

The nodes in r-s plane may be mapped into corresponding nodes in x- y plain using the following relations;

$x = \sum_{i=1}^{n} N_i\ (r,s)x_i = [N_i]\{x_i\}$ …(a)

$y = \sum_{i=1}^{n} N_i\ (r,s)y_i = [N_i]\{y_i\}$ …(b)

Where $\{x_i\}$ and $\{y_i\}$ are list the nodal coordinates x and y and N_i (i=1, 2, … n) are shape functions in terms of natural coordinates.

Based on the order of shape function used to describe the geometry, the elements are classified into three types:

1. Isoparametric element: In this type, the geometry and field variable are defined using the same order of shape function.
2. Subparametric element: In this type, the geometry is described by a lower order shape function than that used for describing the field variable.
3. Superparametric element: In this type, the geometry is described by a higher order shape function.

Matrix transformation

The finite element formulations usually contain terms of N, $\frac{\partial N}{\partial x}$ and $\frac{\partial N}{\partial y}$. Since N is expressed as a function of local coordinates r and s, it is necessary to express $\frac{\partial N}{\partial x}$, $\frac{\partial N}{\partial y}$ and dxdy in terms of r and s, also. This can be done as;

$$\frac{\partial N_i}{\partial r} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial N_i}{\partial s} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s}$$

Or;

$$\begin{Bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

Let [J] = $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$

Then;

$$\begin{Bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

Where [J] is defined as a Jacobian matrix and evaluated by using equations (a) and (b) for each element. The desired derivatives are obtained by inverting the above equation and this involves the finding of Jacobian matrix inverse;

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{Bmatrix} \qquad \ldots(1.30)$$

Similarly;

$dxdy = |J|drds$

Where $|J| = determinate\ of\ [J]$

Chapter two

Finite Element Formulation

2.1 Methods of Finite Element Formulation

Consider a domain Ω described by a set of independent variables $x_1, x_2, x_3, \ldots, x_n$ and typically say x_j. Let the dependent variables be $\emptyset_1$, $\emptyset_2$, $\emptyset_3$, …., $\emptyset_m$ and typically say $\emptyset_i$. In the domain, the relationships between $\emptyset_i$ and x_j are expressed by one or more of field equations of the form;

$$F_\Omega(\emptyset_1, \emptyset_2, \emptyset_3, \ldots \ldots, x_1, x_2, x_3, \ldots \ldots, x_n) = 0 \quad in \quad \Omega \quad \ldots (2.1)$$

There will be also one or more boundary equations;

$$F_S(\emptyset_1, \emptyset_2, \emptyset_3, \ldots \ldots, x_1, x_2, x_3, \ldots \ldots, x_n) = 0 \quad in \quad S \quad \ldots (2.2)$$

The set of the equations (2.1) and (2.2) are the governing equations of the system.

In the finite element procedure, once the mesh for the solution domain is decided, the behavior of the field variables over the domain is approximated as;

$$\widehat{\emptyset}_i^e = \sum_{k=1}^{l} N_{ik}^e \emptyset_{ik}^e \qquad i = 1, m \qquad \ldots (2.3)$$

Where the circumflex distinguishes the approximation $\widehat{\emptyset}_i$ from the true solution $\emptyset_i$ and $\emptyset_{ik}^e$ are the nodal values of $\emptyset_i^e$.

or;

$$\widehat{\emptyset}_i^e = [N_{ik}^e]\{\emptyset_{ik}^e\} \qquad k=1, l \qquad \ldots (2.4)$$

Over the domain, $\widehat{\emptyset}_i$ values are given by;

$\{\widehat{\emptyset}_i\} = \sum_{e=1}^{ne} [N_{ik}^e]\{\emptyset_{ik}^e\}$ …(2.5)

The finite element method is subdivided according to the procedure by which the equations in the nodal values are formulated into;

1- Weighted residual method.
2- Variational principle method.
3- Direct finite element method.

2.2 Weighted Residual Method

The method of weighted residuals is a technique for obtaining approximate solutions to linear and nonlinear partial differential equations.

When $\widehat{\emptyset}_i$ is substituted into Eq.2.1, the equation will not be satisfied, that is;

$$F_\Omega(\widehat{\emptyset}_i,, x_{j.}) = R \neq 0 \quad in \quad \Omega \quad ...(2.6)$$

where R is the residual or error that results from approximating $\emptyset_i$ by $\widehat{\emptyset}_i$. The weighted residual method seeks to determine the nodal unknowns in such a way that the error R over the entire solution domain is small.

In order to make R identically zero, a set of weighting functions W are employed such that;

For the domain;

$$\int_\Omega W.R.d\Omega = 0 \quad ...(2.7)$$

Where W is any function. If the number of unknowns is nn and if nn independent weighting functions W_k are chosen, Eq.(2.7) can be written as;

$$\int_{\Omega} W_k.R.d\Omega = \int_{\Omega} W_k.F_{\Omega}(\widehat{\emptyset}_i,,x_{j.}).d\Omega = 0 \qquad K = 1,2,\ldots nn \qquad \ldots(2.8)$$

There are a variety of weighted residual techniques because of the board of w_k or error distribution principle. These techniques include:

- Collection method.
- Subdomain collection method.
- Galerkin method.
- Least squares method.

Galerkin method is generally the most used one and gives the best approximate solution.

<u>H.W</u>:
Write a brief report about collection, subdomain collection and least squares methods.

2.2.1 Galerkin WRM

According to this method the weighting functions are chosen to be the same as the shape functions used to represent $\emptyset_i$, that is $W_k=N_k$ for k=1, 2, 3,…,nn. Thus Eq.(2.8) becomes;

$$\int_{\Omega} N_k.F_{\Omega}(\widehat{\emptyset}_i,,x_{j.}).d\Omega = 0 \qquad K = 1,2,\ldots nn. \qquad \ldots(2.9)$$

In the finite element method, for each subdomain or element, the contribution would be;

$$\int_{\Omega^e} N_k^e.F_{\Omega}(\widehat{\emptyset}_i^e,x_{j.}).d\Omega_e = 0 \qquad K = 1,2,\ldots l \qquad \ldots(2.10)$$

such that for the whole domain;

$\sum_{e=1}^{ne} \int_{\Omega^e} N_k^e . F_\Omega(\widehat{\emptyset}_i^e, x_{j.}). d\Omega_e = 0 \quad ...(2.11)$

Where Ω^e is element domain and ne is the total number of elements.

The choice of shape functions ensures continuity of Ø at the element boundaries as well as continuity of its partial derivatives up to one order less than the highest order derivatives appearing in the expression to be integrated. For example when the highest order derivative is second order ($\frac{\partial^2 \emptyset}{\partial x^2}$), then, C^1 and C^0 continuity are required, i.e., Ø and $\frac{\partial \emptyset}{\partial x}$ must be continuous. To escape C^1 continuity requirement, integration by parts is applied to the second order derivative terms. Integration by part offers a convenient way to introduce the natural boundary conditions that must be satisfied on some portion of the boundary.

Then, if second order derivative is existing in the governing equations, the following procedure is adopted;

1- Use the product rule for differentiation first which produces the following expression:

$$\frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \phi}{\partial x}\right) = \mathbf{N}^T \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \phi}{\partial x} \quad ...(2.12)$$

2- Use Gauss's divergence theorem which can be stated mathematically as;

$$\int_{A_e} \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \phi}{\partial x}\right) \mathrm{d}A = \int_{\Gamma_e} \mathbf{N}^T \frac{\partial \phi}{\partial x} \cos\theta \, \mathrm{d}\Gamma \quad ...(2.13)$$

where θ is the angle of the outwards normal on the boundary Γ_e of the element with respect to the x-axis.

2.3 Variational Principle Method

In this method the variation of the dependent variable over the whole domain is represented in terms of an integral functional (I). The term "functional" means function of group of functions and I usually has a clear physical meaning in most of the applications.

$$I = \int_V F\left(\emptyset, \frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \frac{\partial^2\emptyset}{\partial x^2}, \frac{\partial^2\emptyset}{\partial x\partial y}, \frac{\partial^2\emptyset}{y^2}\right) dV + \int_S g\left(\emptyset, \frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \ldots.\right) ds \quad \ldots(2.14)$$

For example, in structural and solid mechanics, the potential energy (π) plays the role of functional (π is a function of the displacement vector Ø, whose components are u, v and w, which is a function of coordinates x, y and z).

Here, we are looking for an exact solution to satisfy the boundary conditions and make the integral functional minimum. From mathematical point of view, the solution that satisfies the boundary conditions and minimizes the integral functional will satisfy the partial differential equation (i.e., solution to the P.D.E.).

The steps of solution are:

1- Write the integral functional for each element;

$$I^e = \int_{V^e} F\left(\emptyset, \frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \ldots\right) dV + \int_{S^e} g\left(\emptyset, \frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \ldots.\right) ds \quad \ldots(2.15)$$

2- Approximate $Ø^e$ ($\emptyset^e = \sum N_i \emptyset_i$) and substitute it into Eq.(2.15)

3- Write I for the whole domain;

$$I = \sum_{1}^{ne} I^e$$

where I is in term of nodal values for the whole domain.

4- The solution is that makes I minimum;

$$\frac{\partial I}{\partial \emptyset_i} = 0 \quad \ldots(2.16)$$

where i=1, 2, ……, n and n=total number of degrees of freedom for the whole domain.

That means;

$$\frac{\partial I}{\partial \emptyset_1} = \frac{\partial \sum_1^{ne} I^e}{\partial \emptyset_1} = \sum_1^{ne} \frac{\partial I^e}{\partial \emptyset_1} = 0 \quad \ldots(1)$$

$$\sum_1^{ne} \frac{\partial I^e}{\partial \emptyset_2} = 0 \quad \ldots(2)$$

$$\sum_1^{ne} \frac{\partial I^e}{\partial \emptyset_3} = 0 \quad \ldots(3)$$

.
.
.

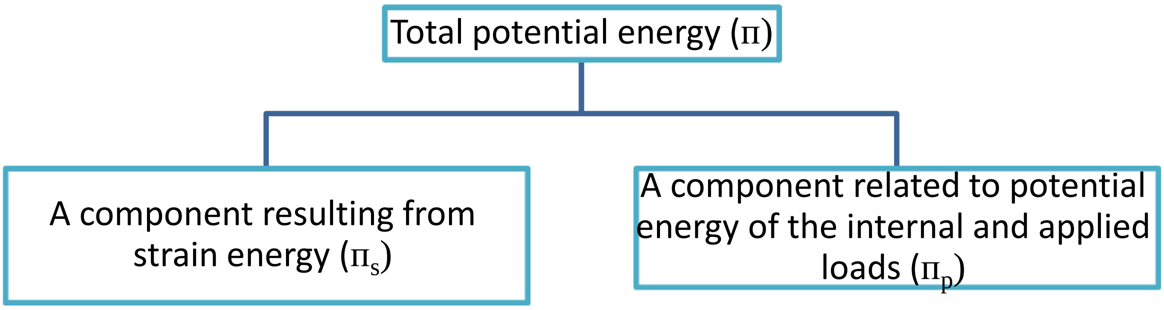
$$\sum_1^{ne} \frac{\partial I^e}{\partial \emptyset_n} = 0 \quad \ldots(n)$$

The unknowns are the nodal values so we get the matrix equation for the whole domain.

2.3.1Variational Principle in Elasticity Problems

Theorem of Potential Energy

Of all displacements satisfying the given boundary conditions, those satisfy the equations of equilibrium are distinguished by a stationary (extreme) value of potential energy.



$$\pi = \pi_s + \pi_p$$

$$\pi_p = -w_p$$

where w_p= work done by internal and applied loads.

$\therefore\ \pi = \pi_s - w_p$

<u>Strain energy</u>

Strain energy ($п_s$) for an elastic body of volume dv under an initial strain of ε_o is;

$d\pi_s = \frac{1}{2}\{\varepsilon\}^T\{\sigma\}$-$\frac{1}{2}\{\varepsilon_o\}^T\{\sigma\}$

where; for 2-D elasticity problems

$\{\varepsilon\}^T = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{xy}\}$

$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$

$$\pi_s = \int_V \frac{1}{2}[\{\varepsilon\}^T\{\sigma\} - \{\varepsilon_o\}^T\{\sigma\}]dv$$

or;

$\pi_s = \int_V \frac{1}{2}[\{\varepsilon\}^T\{\sigma\} - \{\sigma\}^T\{\varepsilon_o\}]dv$ …(2.17)

From Hook's law;

$\{\sigma\} = [D]\{\varepsilon\} - [D]\{\varepsilon_0\}$ …(2.18)

Where [D] is material properties matrix and it depends on the considered problem, whether it is plane strain or plain stress problem. For 2-D problems, [D] is 3×3 matrix. For plane stress problem;

$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$; E=modulus of elasticity and ν = Poisson's ratio

For plane strain problems;

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

$\pi_s = \sum_1^{ne} \pi_s^e$

$$\pi_s^e = \int_{V^e} f(displacements)dV$$

For 2-D problems, we have displacement in x-dir. (u) and displacement in y-dir. (v). Approximate u^e and v^e using an interpolation function;

$u^e = [N]\{u\}$ & $v^e = [N]\{v\}$

$\varepsilon_{xx}^e = \frac{\partial u^e}{\partial x} = \frac{\partial}{\partial x}\left[[N]\{u\}\right] = [B]\{u\}$

$$\varepsilon_{\mathrm{yy}}^{e} = \frac{\partial \mathrm{v}^{e}}{\partial y} = \frac{\partial}{\partial y}\big[[N]\{\mathrm{v}\}\big] = [B]\{\mathrm{v}\}$$

$$\varepsilon_{\mathrm{xy}}^{e} = \frac{\partial \mathrm{u}^{e}}{\partial y} + \frac{\partial \mathrm{v}^{e}}{\partial x} = \frac{\partial}{\partial y}\big[[N]\{\mathrm{u}\}\big] + \frac{\partial}{\partial x}\big[[N]\{\mathrm{v}\}\big]$$

Generally,

$$\varepsilon^{e} = [B]\{\mathrm{U}\}$$

Where; [B]= matrix contains derivative of N. if N is linear, then, [B]=constant

$$\pi_{s}^{e} = \int_{V^{e}} \frac{1}{2}[\{\mathrm{U}\}^{T}[B^{e}]^{T}.[D^{e}][B^{e}]\{\mathrm{U}\} - 2\{\mathrm{U}\}^{T}[B^{e}]^{T}.[D^{e}]\{\varepsilon_{o}^{e}\}$$

$$+ \{\varepsilon_{o}^{e}\}^{T}[D^{e}]\{\varepsilon_{o}^{e}\}]dV$$

Properties

(1) $\{\varepsilon\} = [B]\{\mathrm{U}\}$
$\{\varepsilon\}^{T} = \{\mathrm{U}\}^{T}[B]^{T}$

(2) $\{\sigma\}^{T}\{\varepsilon_{o}\} = \{\varepsilon_{o}\}^{T}\{\sigma\}$

Term-2 W_p

There are three types of loads;

1- Self-weight (internal or body force)
2- Concentrated external load.
3- Distributed external load.

1-Self weight

Let X, Y, and Z are body forces in x-, y-, and z-directions, respectively, then;

$$W_{p}^{e} = \int_{V^{e}} (\mathrm{u}X + \mathrm{v}Y + \mathrm{w}Z)\, dV$$

$$W_{p}^{e} = \int_{V^{e}} \{\mathrm{U}\}^{T}\, [N]^{T} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} dV$$

Since the body forces are variable for each element;

$$W_p^e = \int_{V^e} \{\mathrm{U}\}^T\,[N]^T \begin{Bmatrix} X^e \\ Y^e \\ Z^e \end{Bmatrix} dV$$

2- Concentrated external load

Usually a node is located at the location of concentrated load;

$W_p = \{U\}^T\{\mathrm{P}\} = \{P\}^T\{\mathrm{U}\}$

> Note: if we have a distributed load p acting on the surface of the element (S), analyze it into P_x, P_y, and P_z and
>
> $W_x = P_x\, u$; $W_y = P_y\, v$; $W_z = P_z\, w$

For the whole domain;

$$\pi = \sum_1^{ne} \pi^e$$

By minimizing π (for the whole domain);

$$\frac{\partial \pi}{\partial \{U\}} = \sum_1^{ne} \frac{\partial \pi^e}{\partial \{U\}} = 0$$

Where {U} is displacement nodal values vector. For 3-D problems;

$$\{U\} = \begin{Bmatrix} \mathrm{u} \\ \mathrm{v} \\ \mathrm{w} \end{Bmatrix}$$

By substituting π_s and W_p;

$$\frac{\partial\pi}{\partial\{U\}} = \sum_{1}^{ne}\left[\int_{V^e} [B^e]^T.[D^e][B^e]dV\{\mathrm{U}\} - \int_{V^e} [B^e]^T.[D^e]\{\varepsilon_o^e\}\,dV \right.$$

$$\left. - \int_{V^e} [N^e]^T \begin{Bmatrix} X^e \\ Y^e \\ Z^e \end{Bmatrix} dV - \int_{s^e} [N^e]^T \begin{Bmatrix} P_x{}^e \\ P_y{}^e \\ P_z{}^e \end{Bmatrix} ds \right] - \{P\} = 0$$

The above formulation gives;

$[K]\{U\} + \{F\} + \{G\} = 0$

Where;

$$[K] = \sum_{1}^{ne} \int_{V^e} [B^e]^T.[D^e][B^e]dV$$

But

$$[K] = \sum_{1}^{ne} [k^e]$$

$$[K^e] = \int_{V^e} [B^e]^T.[D^e][B^e]dV$$

$[K^e]$ = element stiffness matrix

$\{F\} = \sum_{1}^{ne}\{F^e\}$

$$\{F^e\} = -\int_{V^e} [B^e]^T.[D^e]\{\varepsilon_o^e\}\,dV - \int_{V^e} [N^e]^T \begin{Bmatrix} X^e \\ Y^e \\ Z^e \end{Bmatrix} dV - \int_{s^e} [N^e]^T \begin{Bmatrix} P_x{}^e \\ P_y{}^e \\ P_z{}^e \end{Bmatrix} ds$$

$$\{G\} = -\{P\}$$

Notes: For non-linear elements, use numerical integration.

$$\frac{\partial}{\{\emptyset\}}[\{\emptyset\}^T[A]\{\emptyset\}] = 2[A]\{\emptyset\}$$

Equations For Two-Dimensional Solids

Stress and Strain

Three-dimensional problems can be simplified if they can be treated as 2-D solid. For representation as a 2D solid, one coordinate (usually the z-axis) is usually removed, and hence assume that all the dependent variables are independent of the z-axis, and all the external loads are independent of the z coordinate, and applied only in the x–y plane. Therefore, a 2-D system has only the x and y coordinates. There are primarily two types of 2D solids; plane stress solid and plane strain solid.

Plane stress solids are solids whose thickness in the z direction is very small compared with dimensions in the x and y directions. All the external forces are applied within the x–y plane, and hence the displacements are functions of x and y only. Herein, the stresses in the z direction (σ_{zz}, σ_{xz} and σ_{yz}) are all zero, as shown in Fig. (2.1)

Plane strain solids are those solids whose thickness in the z direction is very large compared with the dimensions in the x and y directions and the cross-section and the external forces do not vary in the *z* direction. External forces are applied evenly along the z axis, and the movement in the z direction at any point is constrained. The strain components in the z direction (ε_{zz}, ε_{xz} and ε_{yz}) are, therefore, all zero, as shown in Fig.(2.2). Note that for the plane stress problems, the strains ε_{xz} and ε_{yz} are zero, but ε_{zz} will not be zero.

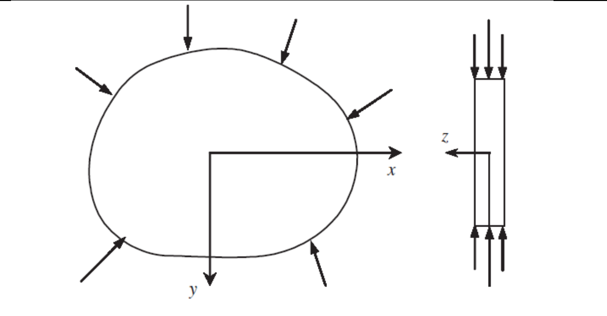


Fig.(2.1) Plane stress problem.

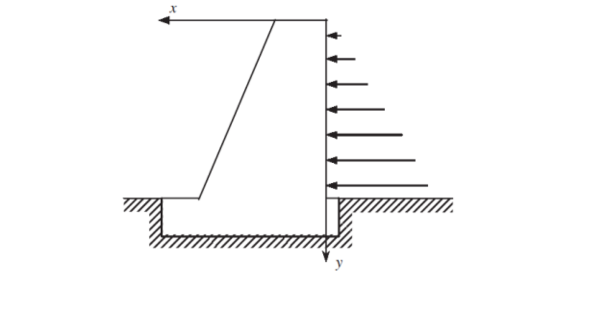


Fig.(2.2): Plane strain problem.

For 1-D elasticity problems

$$\pi_s = \int_V \frac{1}{2}\{\varepsilon_x\}^T\{\sigma_x\}d\mathrm{v}$$

The potential energy of the external forces, being opposite in sign from the external work expression because the potential energy of external forces is lost when the work is done by the external forces, is given by;

$$W_p = \int_V Xud\mathrm{v} + \int_{S_1} T_x u_s ds + \sum_{i=1}^{m} f_{ix}u_{ix}$$

Where; the first, second, and third terms on the right side of the above equation represent the potential energy of (1) body forces, typically from the self-weight of the bar (in units of force per unit volume) moving through displacement function u, (2) surface loading or traction T_x, typically from distributed loading acting along the surface of the element (in units of force per unit surface area) moving through displacements u_s, where u_s are the displacements occurring over surface S, and (3) nodal concentrated forces f_{ix} moving through nodal displacements u_{ix}. The three forces are considered to act in x- direction of the bar as shown in Fig.(2.3). V is the volume of the body and S_1 is the part of the surface S on which surface loading acts. For a bar element with two nodes and one degree of freedom per node, m =2.

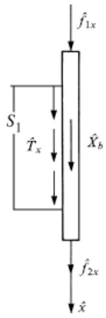


Fig.(2.3): General forces acting on a one-dimensional bar

From Hook's law;

$\{\sigma_x\} = [E]\{\varepsilon_X\}$

$$\frac{\partial \pi}{\partial \{u\}} = \sum_1^{ne} \left[\int_{V^e} [B^e]^T . [E^e][B^e] dV \{\mathrm{u}\} - \int_{V^e} [N^e]^T \{X^e\} dV \right.$$

$$\left. - \int_{S^e} [N^e]^T \{T_x{}^e\} ds \right] - \{f\} = 0$$

The above formulation gives;

$[K]\{U\} + \{F\} + \{G\} = 0$

Where;

$$[K] = \sum_1^{ne} \int_{V^e} [B^e]^T . [E^e][B^e] dV$$

But

$$[K] = \sum_1^{ne} [k^e]$$

$$[K^e] = \int_{V^e} [B^e]^T . [E^e][B^e] dV$$

$[K^e]$ = element stiffness matrix

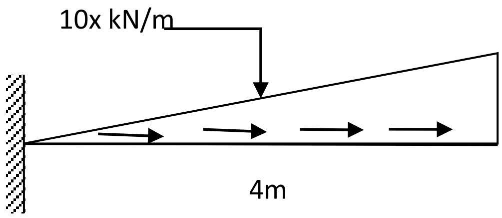
$\{F\} = \sum_1^{ne}\{F^e\}$

$$\{F^e\} = -\int_{V^e} [N^e]^T\{X^e\}dV - \int_{s^e} [N^e]^T\{T_x{}^e\}ds$$

$$\{G\} = -\{f\}$$

H.W.1: Solve the following problem using Galerkin weighted residual method and variational principle method.

For the bar subjected to the linear varying axial load shown below, determine the nodal displacements and axial stress distribution using (a) two equal-length elements and (b) four equal-length elements. Take E=210 GPa and A=4×10^{-4} m^2



Hint: for the simple two-noded bar element subjected to a linearly varying load (triangular loading), place one-third of the total load at the node where the distributed loading begins (zero end of the load) and two-thirds of the total load at the node where the peak value of the distributed load ends.

Chapter Three

FEM for Trusses

3.1 General

A truss is one of the simplest and most widely used structural members. It is a straight bar that is designed to take only axial forces, therefore it deforms only in its axial direction. The cross-section of the bar can be arbitrary, but the dimensions of the cross-section should be much smaller than that in the axial direction. Finite element equations for such truss members will be developed in this chapter. The element developed is commonly known as the *truss element* or *bar element*. Such elements are applicable for analysis of the skeletal type of truss structural systems both in two-dimensional planes and in three-dimensional space.

In planar trusses there are two components in the x and y directions for the displacements as well as for the forces. For space trusses, however, there will be three components in the x, y and z directions for both displacements and forces. In skeletal structures consisting of truss members, the truss members are joined together by pins or hinges (not by welding), so that there are only forces (not moments) transmitted between the bars.

3.2 FEM Equations

Consider a structure consisting of a number of trusses or bar members. Each of the members can be considered as a truss/bar element of uniform cross-section bounded by two nodes. Consider a bar element with nodes 1 and 2 at each end of the element, as shown in Fig.(3.1). The length of the element is *le*. The local *x*- axis is taken in the axial direction of the element with the origin at node 1. In the local coordinate system, there is only one DOF at each node of the element, and that is the axial displacement (u). Therefore, there is a total of two DOFs for the element.

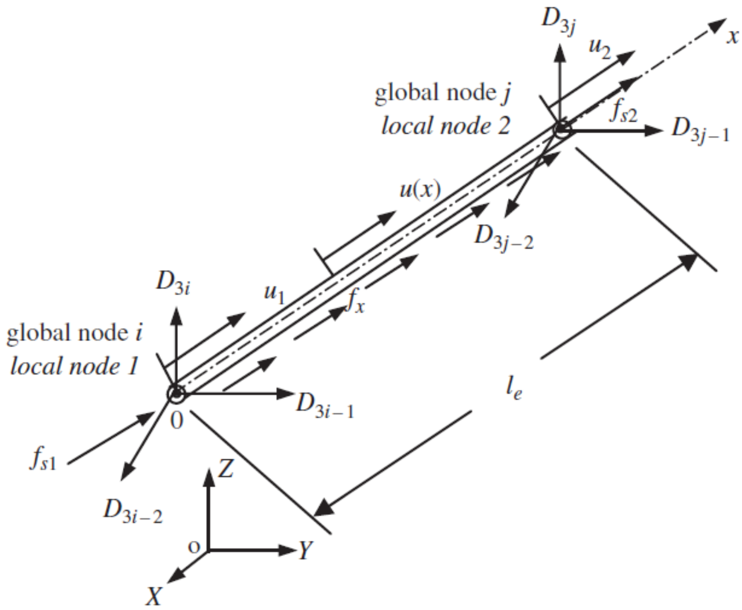


Fig.(3.1): Truss element and the coordinate system.

For 1-D linear element, the matrix of shape functions is;

[N]= $[N_1\ N_2]$

where the shape functions for a truss element can be written as;

$$N_1 = 1 - \frac{x}{l_e} \qquad\qquad N_2 = \frac{x}{l_e}$$

3.2.1 Element Matrices in the Local Coordinate System

The stiffness matrix for truss elements can be written as;

$$[k^e] = \int_{V^e} [B^e]^T.[E^e][B^e]dV = A_e \int_0^{le} \begin{bmatrix} -1/l_e \\ 1/l_e \end{bmatrix} E[-1/l_e \quad 1/l_e]dx = \frac{AE}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad ...(3.1)$$

where A_e is the area of the cross-section of the truss element.

The mass matrix for truss elements can be obtained as:

$$[m_e] = \int_{V_e} \rho[N]^T[N]dV$$

$$= A\rho \int_0^{l_e} \begin{bmatrix} N_1N_1 & N_1N_2 \\ N_2N_1 & N_2N_2 \end{bmatrix} dx = \frac{A\rho l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad ...(3.2)$$

To find the nodal force vector for truss elements, suppose the element is loaded by an evenly distributed force f_x along the x-axis, and two concentrated forces $fs_1$and fs_2, respectively, at two nodes 1 and 2, as shown in Fig.1; the total nodal force vector becomes;

$$\{f^e\} = \int_s [N]^T f_x ds + \begin{Bmatrix} f_{s1} \\ f_{s2} \end{Bmatrix} = f_x \int_0^{l_e} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx + \begin{Bmatrix} f_{s1} \\ f_{s2} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_x l_e /2 + f_{s1} \\ f_x l_e /2 + f_{s2} \end{Bmatrix} \quad \text{... (3.3)}$$

3.2.2 Element Matrices in the Global Coordinate System

Element matrices in Eqs. (3.1), (3.2) and (3.3) were formulated based on the *local coordinate system*, where the *x*-axis coincides with the mid axis of the bar 1–2, shown in Fig.(3.1). In practical trusses, there are many bars of different orientations and at different locations. To assemble all the element matrices to form the global system matrices, a coordinate transformation has to be performed for each element to formulate its element matrix based on the *global coordinate system* for the whole truss structure. The following performs the transformation for both spatial and planar trusses.

A- Spatial trusses

Assume that the local nodes 1 and 2 of the element correspond to the global nodes *i* and *j*, respectively, as shown in Fig.(3.1). The displacement at a global node in space should have three components in the *X*, *Y* and *Z* directions, and numbered sequentially. For example, these three components at the i^{th} node are denoted by D_{3i-2}, D_{3i-1} and D_{3i}. The coordinate transformation gives the relationship between the displacement vector **{U}** based on the local coordinate system and the displacement vector {D} for the same element, but based on the global coordinate system *XYZ*:

$$\{U\} = [T]\{D\} \quad \ldots(3.4)$$

Where;

$$\{D\} = \begin{Bmatrix} D_{3i-2} \\ D_{3i-1} \\ D_{3i} \\ D_{3j-2} \\ D_{3j-1} \\ D_{3i} \end{Bmatrix}$$

and [T] is the transformation matrix for the truss element, given by;

$$= \begin{bmatrix} l_{ij} & m_{ij} & n_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ij} & m_{ij} & n_{ij} \end{bmatrix}_e$$

In which;

$$l_{ij} = \cos(x, X) = \frac{X_j - X_i}{l_e}$$
$$m_{ij} = \cos(x, Y) = \frac{Y_j - Y_i}{l_e}$$
$$n_{ij} = \cos(x, Z) = \frac{Z_j - Z_i}{l_e}$$

are the *direction cosines* of the axial axis of the element. The length of the element, *le*, can be calculated using the global coordinates of the two nodes of the element by;

$$l_e = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}$$

The matrix [T] for a truss element transforms a 6 × 1 vector in the global coordinate system into a 2 × 1 vector in the local coordinate system. The transformation matrix is, also, applied to the force vectors between the local and global coordinate systems:

$$\{f_e\} = [T]\{F_e\}$$

$$\{F_e\} = \begin{Bmatrix} F_{3i-2} \\ F_{3i-1} \\ F_{3i} \\ F_{3j-2} \\ F_{3j-1} \\ F_{3i} \end{Bmatrix}$$

in which F_{3i-2}, F_{3i-1} and F_{3i} stand for the three components of the force vector at node *i* based on the global coordinate system.

The element equation based on the global coordinate system can be written as:

$$[k_e][T]\{D_e\} = [T]\{F_e\}$$

Multiply both side by $[T]^T$;

$$[T]^T[k_e][T]\{D_e\} = [T]^T[T]\{F_e\}$$

Let $[K_e]$ = element stiffness matrix in global coordinates system, then;

$[K_e] = [T]^T[k_e][T]$

Since $[T]^T[T] = I$ = identity matrix ; *Prove*?

then;

$$[K_e]\{D_e\} = \{F_e\} \quad \ldots(3.5)$$

Where $[K_e]$=

Where $[K_e]$=
$$\frac{AE}{l_e}\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} \\ l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} & -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} \\ l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 & -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 \\ -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} & l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} \\ -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} & l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} \\ -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 & l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 \end{bmatrix}$$

and $\{F_e\}=$ $$\begin{Bmatrix} (f_x l_e/2 + f_{s1})\, l_{ij} \\ (f_x l_e/2 + f_{s1})\, m_{ij} \\ (f_x l_e/2 + f_{s1})\, n_{ij} \\ \left(f_y l_e/2 + f_{s1}\right) l_{ij} \\ \left(f_y l_e/2 + f_{s1}\right) m_{ij} \\ \left(f_y l_e/2 + f_{s1}\right) n_{ij} \end{Bmatrix}$$

Note that the element stiffness matrix $[K_e]$ has a dimension of 6 × 6 in the three-dimensional global coordinate system, and the displacement {D} and the force vector {F*e*} have a dimension of 6 × 1.

B- Planar trusses

For a planar truss, the global coordinates *X–Y* can be employed to represent the plane of the truss. All the formulations of coordinate transformation can be obtained from the counterpart of those for spatial trusses by simply removing the rows and/or columns corresponding to the *z*- (or *Z*-) axis. The displacement at the global node *i* should have two components in the *X* and *Y* directions only: D_{2i-1} and D_{2i} . The coordinate transformation, which gives the relationship between the displacement vector **{U}** based on the local coordinate system and the displacement vector **{D***e*}, has the same form as Eq. (3.4), except that {De} and {Fe} are 4×1 vectors and [T] is 2×4 matrix.

3.3 Boundary Conditions

The stiffness matrix $[K_e]$ in Eq. (3.5) is usually singular, because the whole structure can perform rigid body movements. There are two DOFs of rigid movement for planer trusses and three DOFs for space trusses. These rigid body movements are constrained by supports or displacement constraints. In practice, truss structures are fixed somehow to the ground or to a fixed main structure at a number of the nodes. When a node is fixed, the displacement at the node must be zero. This fixed displacement boundary condition can be imposed on Eq.

(3.5). The imposition leads to a cancellation of the corresponding rows and columns in the stiffness matrix.

3.4 Determination of Stress and Strain

Eq. (3.5) can be solved using standard routines and the displacements at all the nodes can be obtained after imposing the boundary conditions. The displacements at any position other than the nodal positions can also be obtained using interpolation by the shape functions. The stress in a truss element can also be determined using the following equation:

$$\sigma_x = E\mathbf{[B]\{U\}} = E\mathbf{[B][T]\{D}_e\}$$

In deriving the above equation, Hooke's law in the form of $\sigma = E\varepsilon$ is used.

Chapter Four

FEM for Beams

4.1 General

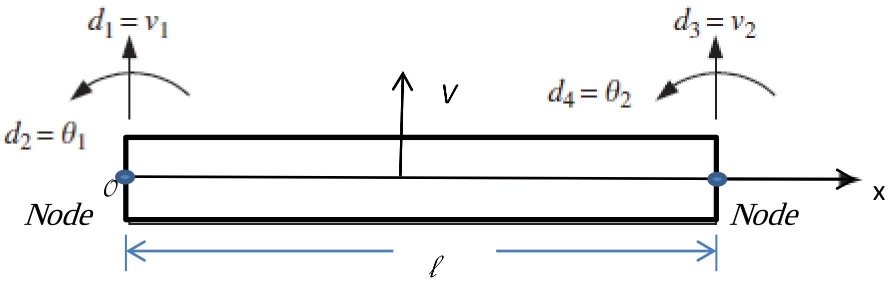
A *beam* is another simple but commonly used structural component. It is also geometrically a straight *bar* of an arbitrary cross-section, but it deforms only in directions perpendicular to its axis. Note that the main difference between the beam and the truss is the type of load they carry. Beams are subjected to transverse loading, including transverse forces and moments that result in transverse deformation. Herein, the cross-section of the beam structure is assumed uniform. If a beam has a varying cross-section, the beam should be divided into shorter beams, where each can be treated as a beam with a uniform cross-section.

4.2 FEM Equations

In planar beam elements there are two degrees of freedom (DOFs) at a node in its local coordinate system. They are deflection in the y direction, v, and rotation in the x–y plane, θ_z with respect to the z-axis. Therefore, each beam element has a total of four DOFs.

4.3 Shape Function Construction

Consider a beam element of length l with nodes 1 and 2 at each end of the element, as shown in Fig.1. The local x-axis is taken in the axial direction of the element with its origin at the beginning of the beam. Similar to all other structures, to develop the FEM equations, shape functions for the interpolation of the variables from the nodal variables would first have to be developed. The transverse displacement model for the beam element is assumed to be a cubic polynomial involving four constants.



The cubic interpolation function for the displacement of a beam element is expressed as;

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

with the nodal degrees of freedom defined as $v_1 = v(x = x_1)$, $\vartheta_1 = (dv/dx)(x = x_1)$, $v_2 = v(x = x_2)$, and $\vartheta_2 = (dv/dx)(x = x_2)$, where x_1 and x_2 denote the x coordinates of nodes 1 and 2 of the element. The shape functions are defined in local coordinates system as;

$$N_1 = \frac{1}{l^3}(2x^3 - 3lx^2 + l^3)$$

$$N_2 = \frac{1}{l^3}(lx^3 - 2l^2x^2 + l^3x)$$

$$N_3 = \frac{1}{l^3}(-2x^3 + 3lx^2)$$

$$N_4 = \frac{1}{l^3}(lx^3 - l^2x^2)$$

And the displacement function for the element e can be written as;

$$v^e = [N]\{d\} \quad \ldots(4.1)$$

Where {d} is displacement vector defined as;

$$\{d\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

4.4 Strain Matrix

After the obtaining of the shape functions, the next step would be to obtain the element strain matrix. The relationship between the strain (ε_{XX}) and the deflection (v) is;

$$\varepsilon_{xx} = -y\frac{\partial^2 v}{\partial x^2} \quad \ldots(4.2)$$

Substituting Eq. (4.1) into Eq. (4.2), which gives the relationship between the strain and the deflection;

$$\varepsilon_{xx} = [B]\{d\}$$

where the strain matrix **B** is given by;

$$[B] = -y\frac{\partial^2}{\partial x^2}[N] \quad \dots (4.3)$$

$$[B] = -y[\frac{\partial^2 N_1}{\partial x^2} \quad \frac{\partial^2 N_2}{\partial x^2} \quad \frac{\partial^2 N_3}{\partial x^2} \quad \frac{\partial^2 N_4}{\partial x^2}]$$

4.5 Element Matrix

The element stiffness matrix is defined as;

$$[k^e] = \int_{V^e}[B^e]^T.[E^e][B^e]dV \quad \dots(4.4)$$

By substituting Eq. (4.3) into Eq. (4.4), the stiffness matrix can be obtained as;

$$[k^e] = \int_{V^e}[B^e]^T.[E^e][B^e]dV = E^e\int_{A^e} y^2 dA \int_0^l \left[\frac{\partial^2 N}{\partial x^2}\right]^T.\left[\frac{\partial^2 N}{\partial x^2}\right]dx =$$

$$E^e I_z \int_0^l \left[\frac{\partial^2 N}{\partial x^2}\right]^T.\left[\frac{\partial^2 N}{\partial x^2}\right]dx \quad \dots(4.5)$$

where I_z ($= \int_A \; y^2$dA) is the second moment of area (or moment of inertia) of the cross section of the beam with respect to the z-axis. Eq.(4.5) can be rewritten as;

$$[k^e] = E^e I_z \int_0^l \begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2}\frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2}\frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2}\frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_1}{\partial x^2}\frac{\partial^2 N_4}{\partial x^2} \\ \frac{\partial^2 N_2}{\partial x^2}\frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2}\frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2}\frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2}\frac{\partial^2 N_4}{\partial x^2} \\ \frac{\partial^2 N_3}{\partial x^2}\frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2}\frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2}\frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2}\frac{\partial^2 N_4}{\partial x^2} \\ \frac{\partial^2 N_4}{\partial x^2}\frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2}\frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2}\frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2}\frac{\partial^2 N_4}{\partial x^2} \end{bmatrix} dx$$

Evaluating the integrals in the above equation leads to;

$$\left[K^e\right] = \frac{2EI_Z}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \qquad \ldots(4.6)$$

4.6 Force Vector

Suppose the element is loaded by an external distributed force f_y along the x-axis, two concentrated forces f_{s1} and f_{s2}, and concentrated moments m_{s1} and m_{s2}, respectively, at nodes 1 and 2; the total nodal force vector becomes;

$$\{f^e\} = \int_s [N]^T f_y ds + \begin{Bmatrix} f_{s1} \\ m_{s1} \\ f_{s2} \\ m_{s2} \end{Bmatrix} = f_y \int_0^{l_e} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} dx + \begin{Bmatrix} f_{s1} \\ m_{s1} \\ f_{s2} \\ m_{s2} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_y l_e/2 + f_{s1} \\ f_y {l_e}^2/12 + m_{s1} \\ f_y l_e/2 + f_{s2} \\ -f_y {l_e}^2/12 + m_{s1} \end{Bmatrix} \qquad \ldots(4.7)$$

4.7 Beam Finite Element Equation

The final finite element equation of a beam is defined as;

$$[k_e]\{d_e\} = \{f_e\} \qquad \ldots(4.8)$$

Where [k_e] and $\{f_e\}$ are defined using Eqs. (4.6) and (4.7), respectively.

Chapter Five

FEM for Frames

5.1 Frame Element

A frame element is formulated to model a straight *bar* of an arbitrary cross-section, which can deform not only in the axial direction but also in the directions perpendicular to the axis of the bar. The bar is capable of carrying both axial and transverse forces, as well as moments. Therefore, a frame element is seen to possess the properties of both truss and beam elements.

Frame elements are applicable for the analysis of skeletal type systems of both *planar frames* (two-dimensional frames) and *space frames* (three-dimensional frames). Frame members in a frame structure are joined together by welding so that both forces and moments can be transmitted between members. Herein, it is assumed that the frame elements have a uniform cross sectional area. If a structure of varying cross-section is to be modeled using the formulation given below, then the structure must be divided into smaller elements of different constant cross-sectional area so as to simulate the varying cross-section. If the variation in cross-section is too severe for accurate approximation, then the equations for a varying cross-sectional area can also be formulated without much difficulty using the same concepts and procedure.

5.2 FEM Equations for Planar Frames

Consider a frame structure whereby the structure is divided into frame elements connected by nodes. Each element is of length $l_e = 2a$, and has two nodes at its two ends. The elements and nodes are numbered separately in a convenient manner. In a planar frame element, there are three degrees of freedom (DOFs) at one node in its local coordinate system, as shown in

Fig.(5.1). They are the axial deformation in the x direction, u; deflection in the y direction, v; and the rotation in the x–y plane and with respect to the z-axis, θ_z. Therefore, each element with two nodes will have a total of six DOFs.

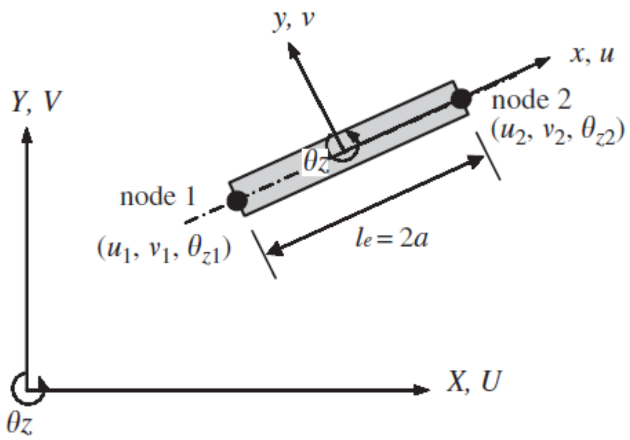


Fig.(5.1). Planar frame element and the DOFs.

5.2.1 Equations in Local Coordinate System

Considering the frame element shown in Fig.(5.1) with nodes labeled 1 and 2 at each end of the element, it can be seen that the local x-axis is taken as the axial direction of the element with its origin at the middle of the element. As mentioned, a frame element contains both the properties of the truss element and the beam element. Therefore, the element matrices for a frame element can be simply formulated by combining element matrices for truss and beam elements. Recall that the truss element has only one degree of freedom at each node (axial deformation), and the beam element has two degrees of freedom at each node (transverse deformation and rotation). Combining these will give the degrees of freedom of a frame element, and the element displacement vector for a frame element can thus be written as;

$$
\mathbf{d}_e = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \end{Bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} u_1 \\ v_1 \\ \theta_{z1} \end{matrix}}\right\} \text{displacement components at node 1} \\ \left.\vphantom{\begin{matrix} u_2 \\ v_2 \\ \theta_{z2} \end{matrix}}\right\} \text{displacement components at node 2} \end{matrix} \qquad \ldots(5.1)
$$

To construct the stiffness matrix, the stiffness matrix for truss elements is extended at first to a 6×6 matrix corresponding to the order of the degrees of freedom of the frame element in the element displacement vector in Eq. (5.1):

$$
\mathbf{k}_e^{\text{truss}} = \begin{array}{c} \begin{array}{cccccc} d_1 = u_1 & & & d_4 = u_2 & & \\ \uparrow & & & \uparrow & & \end{array} \\ \begin{bmatrix} AE/(2a) & 0 & 0 & -AE/(2a) & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & AE/(2a) & 0 & 0 \\ & sy. & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \rightarrow d_1 = u_1 \\ \\ \\ \rightarrow d_4 = u_2 \\ \\ \\ \end{array} \qquad \ldots(5.2)
$$

Next, the stiffness matrix for the beam element is also extended to a 6×6 matrix corresponding to the order of the degrees of freedom of the frame element in Eq. (5.1):

$$
\mathbf{k}_e^{\text{beam}} = \begin{array}{c} \begin{array}{cccccc} & d_2(v_1) & d_3(\theta_{z1}) & & d_5(v_2) & d_6(\theta_{z2}) \\ & \uparrow & \uparrow & & \uparrow & \uparrow \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \dfrac{3EI_z}{2a^3} & \dfrac{3EI_z}{2a^2} & 0 & -\dfrac{3EI_z}{2a^3} & \dfrac{3EI_z}{2a^2} \\ & & \dfrac{2EI_z}{a} & 0 & -\dfrac{3EI_z}{2a^2} & \dfrac{EI_z}{a} \\ & & & 0 & 0 & 0 \\ & sy. & & & \dfrac{3EI_z}{2a^3} & -\dfrac{3EI_z}{2a^2} \\ & & & & & \dfrac{2EI_z}{a} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \rightarrow d_2 = v_1 \\ \rightarrow d_3 = \theta_{z1} \\ \\ \rightarrow d_5 = v_2 \\ \rightarrow d_6 = \theta_{z2} \end{array} \qquad \ldots(5.3)
$$

The two matrices in Eqs. (5.2) and (5.3) are now superimposed together to obtain the stiffness matrix for the frame element:

$$\mathbf{k}_e = \begin{bmatrix} \frac{AE}{2a} & 0 & 0 & -\frac{AE}{2a} & 0 & 0 \\ & \frac{3EI_z}{2a^3} & \frac{3EI_z}{2a^2} & 0 & -\frac{3EI_z}{2a^3} & \frac{3EI_z}{2a^2} \\ & & \frac{2EI_z}{a} & 0 & -\frac{3EI_z}{2a^2} & \frac{EI_z}{a} \\ & & & \frac{AE}{2a} & 0 & 0 \\ & sy. & & & \frac{3EI_z}{2a^3} & -\frac{3EI_z}{2a^2} \\ & & & & & \frac{2EI_z}{a} \end{bmatrix} \qquad \ldots(5.4)$$

The same simple procedure can be applied to the force vector as well. The element force vectors for the truss and beam elements are extended into 6×1 vectors corresponding to their respective DOFs and added together. If the element is loaded by external distributed forces f_x and f_y along the x-axis; concentrated forces f_{sx1}, f_{sx2}, f_{sy1} and f_{sy2}; and concentrated moments m_{s1} and m_{s2}, respectively, at nodes 1 and 2, the total nodal force vector becomes;

$$\mathbf{f}_e = \begin{Bmatrix} f_x a + f_{sx1} \\ f_y a + f_{sy1} \\ f_y a^2/3 + m_{s1} \\ f_x a + f_{sx2} \\ f_y a + f_{sy2} \\ -f_y a^2/3 + m_{s1} \end{Bmatrix} \qquad \ldots(5.5)$$

5.2.2 Equations in Global Coordinate System

The matrices formulated in the previous section are for a particular frame element in a specific orientation. A full frame structure usually comprises numerous frame elements of different orientations joined together. As such, their local coordinate system would vary from one orientation to another. To assemble the element matrices together, all the matrices must first be expressed in a common coordinate system, which is the global coordinate system. The coordinate transformation process is the same as that described for truss structures.

Assume that local nodes 1 and 2 correspond to the global nodes *i* and *j*, respectively. The displacement at a local node should have two translational components in the *x* and *y* directions and one rotational deformation. They are numbered sequentially by u, v and θ_z at each of the two nodes, as shown in Fig.(5.2). The displacement at a global node should also have two translational components in the *X* and *Y* directions and one rotational deformation. They are numbered sequentially by D_{3i-2}, D_{3i-1} and D_{3i} for the *i*th node, as shown in Fig.(5.2). The same sign convention also applies to node *j*. The coordinate transformation gives the relationship between the displacement vector $\mathbf{d}e$ based on the local coordinate system and the displacement vector $\mathbf{D}e$ for the same element, but based on the global coordinate system:

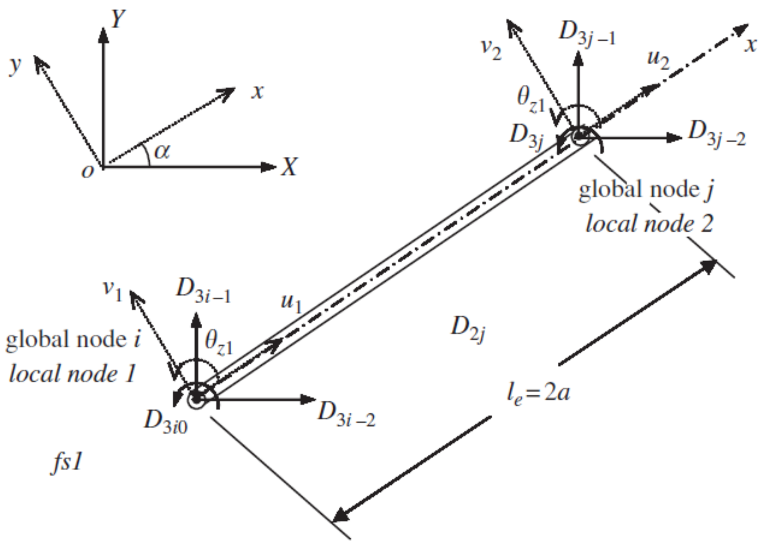


Fig.(5.2). Coordinate transformation for 2D frame elements.

$$d_e = TD_e \qquad \text{... (5.6)}$$

Where;

$$\mathbf{D}_e = \begin{Bmatrix} D_{3i-2} \\ D_{3i-1} \\ D_{3i} \\ D_{3j-2} \\ D_{3j-1} \\ D_{3j} \end{Bmatrix} \qquad \text{...(5.7)}$$

and **T** is the transformation matrix for the frame element given by;

$$\mathbf{T} = \begin{bmatrix} l_x & m_x & 0 & 0 & 0 & 0 \\ l_y & m_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & m_x & 0 \\ 0 & 0 & 0 & l_y & m_y & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{...(5.8)}$$

in which;

$$\begin{aligned} l_x &= \cos(x, X) = \cos\alpha = \frac{X_j - X_i}{l_e} \\ m_x &= \cos(x, Y) = \sin\alpha = \frac{Y_j - Y_i}{l_e} \end{aligned} \qquad \text{...(5.9)}$$

where α is the angle between the x-axis and the X-axis, as shown in Fig.(5.2), and;

$$\begin{aligned} l_y &= \cos(y, X) = \cos(90^\circ + \alpha) = -\sin\alpha = -\frac{Y_j - Y_i}{l_e} \\ m_y &= \cos(y, Y) = \cos\alpha = \frac{X_j - X_i}{l_e} \end{aligned} \qquad \text{...(5.10)}$$

Note that the coordinate transformation in the *X*–*Y* plane does not affect the rotational DOF, as its direction is in the *z* direction (normal to the *x*–*y* plane), which still remains the same as the Z direction in the global coordinate system. The length of the element, le, can be calculated by;

$$l_e = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2} \qquad \ldots(5.11)$$

The matrix **T** for a frame element transforms a 6 × 6 matrix into another 6 × 6 matrix. Using the transformation matrix, **T**, the matrices for the frame element in the global coordinate system become;

$$K_e = T^T k_e T \qquad \ldots(5.12)$$

$$F_e = T^T f_e \qquad \ldots(5.13)$$

Note that there is no change in dimension between the matrices and vectors in the local and global coordinate systems.

Chapter Six

FEM for Plates

6.1 Plate Elements

A plate structure is geometrically similar to the structure of the 2D plane stress problem, but it usually carries only transversal loads that lead to bending deformation in the plate. For example, higher floors of a building are a typical plate structure that carries most of us every day. The plate structure can be schematically represented by its middle plane laying on the *x*–*y* plane, as shown in Fig.(6.1). The deformation caused by the transverse loading on a plate is represented by the deflection and rotation of the normals of the middle plane of the plate, and they will be independent of *z* and a function of only *x* and *y*. The element to be developed to model such plate structures is known as the *plate element*.

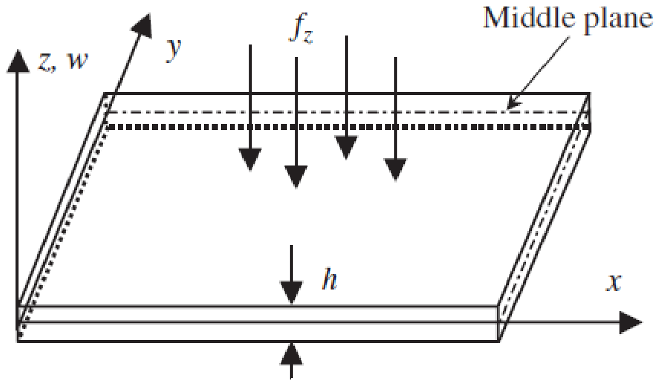


Fig.(6.1): A plate and its coordinates system.

Plate elements are normally used to analyze the bending deformation of plate structures and the resulting forces such as shear forces and moments. In this aspect, it is similar to the beam element, except that the plate element is two-dimensional whereas the beam element is one-dimensional. A plate element can also be triangular, rectangular or quadrilateral in shape. In this section, we the development of the rectangular element is covered only, as it is often used. Matrices for the triangular element can also be developed easily using similar procedures.

There are a number of theories that govern the deformation of plates. In this section, rectangular elements based on the Reissner–Mindlin plate theory that works for thick plates will be developed. There are a number of higher order plate theories that can be used for the development of finite elements. It is assumed that the element has a uniform thickness h. If the plate structure has a varying thickness, the structure has to be divided into small elements that can be treated as uniform elements. However, the formulation of plate elements with a varying thickness can also be done, as the procedure is similar to that of a uniform element. Consider now a plate that is represented by a two-dimensional domain in the x–y plane, as shown in Fig.(6.1). The plate is divided in a proper manner into a number of rectangular (linear quadrilateral) elements of four nodes, as shown in Fig.(6.2). At a node, the degrees of freedom include the deflection w, the rotation about x axis θ_x , and the rotation about y axis θ_y, making the total DOF of each node three. Hence, for a rectangular element with four nodes, the total DOF of the element would be twelve. Following the Reissner–Mindlin plate theory, its shear deformation will force the cross-section of the plate to rotate in the way shown in Fig.(6.3). Any straight fiber that is perpendicular to the middle plane of the plate before the deformation rotates, but remains straight after the deformation. The two displacement components that are parallel to the middle surface (u and v) at a distance z from the centroidal axis can be expressed by;

$$u(x,y,z) = z\theta_y(x,y) \qquad \ldots(6.1)$$

$$v(x,y,z) = -z\theta_x(x,y) \qquad \ldots(6.2)$$

where ϑ_x and ϑ_y are, respectively, the rotations of the fiber of the plate with respect to the x and y axes. The in-plane strains can then be given as;

$$\{\varepsilon\} = -z\{\boldsymbol{\chi}\} \qquad \dots(6.3)$$

where $\{\boldsymbol{\chi}\}$ is the curvature vector, given as;

$$\{\boldsymbol{\chi}\} = \begin{Bmatrix} -\partial\theta_y/\partial x \\ \partial\theta_x/\partial y \\ \partial\theta_x/\partial x - \partial\theta_y/\partial y \end{Bmatrix} \qquad \dots(6.4)$$

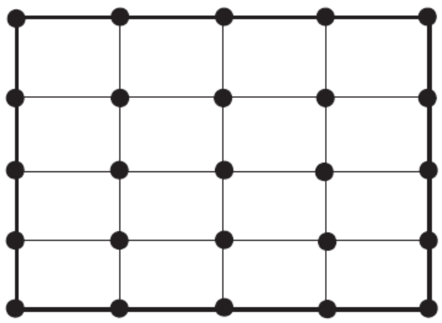


Fig.(6.2): 2D domain of a plate meshed by rectangular elements.

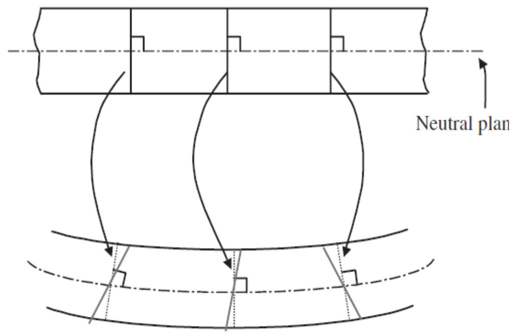


Fig.(6.3): Shear deformation in a plate.

The off-plane shear strain is then given as;

$$\{\gamma\} = \begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_y + \dfrac{\partial w}{\partial x} \\ -\theta_x + \dfrac{\partial w}{\partial y} \end{Bmatrix} \qquad \dots(6.5)$$

The potential (strain) energy expression for a thick plate element is;

$$\pi_s = \int_V \frac{1}{2}[\{\varepsilon\}^T\{\sigma\} + \{\gamma\}^T\{\tau\}]d\mathrm{v} \qquad \dots(6.6)$$

or;

$$\pi_s = \frac{1}{2}\int_{A^e}\int_0^h \{\varepsilon\}^T\{\sigma\}dAdz + \frac{1}{2}\int_{A^e}\int_0^h \{\gamma\}^T\{\tau\}dAdz \quad \ldots(6.8)$$

The first term on the right-hand side of the above equation is for the in-plane stresses and strains, whereas the second term is for the transverse stresses and strains. $\{\tau\}$ is the average shear stresses vector relating to the shear strain in the form**;**

$$\{\tau\} = \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = k\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\{\gamma\} = k[D_s]\{\gamma\} \quad \ldots(6.9)$$

where G is the shear modulus, and κ is a constant that is usually taken to be $\pi^2/12$ or 5/6. Substituting Eqs. (6.3) and (6.9) into Eq. (6.8), the potential (strain) energy becomes**;**

$$\pi_s = \frac{1}{2}\int_{A^e}\frac{h^3}{12}\{\chi\}^T[D]\{\chi\}dA + \frac{1}{2}\int_{A^e} kh\{\gamma\}^T[D_s]\{\gamma\}dA \quad \ldots(6.10)$$

6.2 Shape Functions

For four-node rectangular thick plate elements, the deflection and rotations can be approximated as;

$$w^e = \sum_{i=1}^4 N_i w_i \quad \ldots(6.11\text{-a})$$

$$\theta_x{}^e = \sum_{i=1}^4 N_i\theta_{xi} \quad \ldots(6.11\text{-b})$$

$$\theta_y{}^e = \sum_{i=1}^4 N_i\theta_{yi} \quad \ldots(6.11\text{-c})$$

where the shape function *Ni* in natural coordinates system is;

$$N_i\,(r,s) = \frac{1}{4}(1 + rr_i)(1 + ss_i) \quad \ldots(6.12)$$

Rewriting Eqs. (6.11a-c) into a single matrix equation;

$$\begin{Bmatrix} w^e \\ {\theta_x}^e \\ {\theta_y}^e \end{Bmatrix} = [N]\{d^e\} \quad \text{…(6.13)}$$

where {d} is the displacement vector for all the nodes in the element, arranged in the order;

$$\{d^e\} = \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \theta_{x_2} \\ \theta_{y_2} \\ w_3 \\ \theta_{x_3} \\ \theta_{y_3} \\ w_4 \\ \theta_{x_4} \\ \theta_{y_4} \end{Bmatrix}$$

6.3 Element Matrices

To obtain the stiffness matrix **[k**^{e}], Eq. (6.13) is substituted into Eq. (6.10);

$$[k^e] = \int_{A^e} \frac{h^3}{12} [\mathrm{B}^I]^T [D][B^I] dA + \frac{1}{2} \int_{A^e} kh [B^O]^T [D_s][\mathrm{B}^O] dA \quad \text{... (6.14)}$$

The first term in the above equation represents the strain energy associated with the in-plane stress and strains. The strain matrix **[B**$^{\mathrm{I}}$] has the form of ;

$$[B^I] = [B_1^I \quad B_2^I \quad B_3^I \quad B_4^I]$$

where;

$$B^I{}_j = \begin{bmatrix} 0 & 0 & -\partial N_j/\partial x \\ 0 & \partial N_j/\partial y & 0 \\ 0 & \partial N_j/\partial x & -\partial N_j/\partial y \end{bmatrix}, \qquad \text{j=1,…,4}$$

The strain matrix $[B^o]$ has the form;

$$[B^O] = [B_1^O \quad B_2^O \quad B_3^O \quad B_4^O]$$

where;

$$B^O{}_j = \begin{bmatrix} \partial N_j/\partial x & 0 & N_j \\ \partial N_j/\partial y & -N_j & 0 \end{bmatrix}, \qquad j=1,\ldots,4$$

The integration in the stiffness matrix **[k**^{e}], can be evaluated numerically using the Gauss integration scheme. The simplest and most practical means to solve this problem is to use 2 ×2 Gauss points for the integration of the first term, and use only one Gauss point for the second term in Eq. (6.14).

6.4 Force Vector

The force vector is written as;

$$\{f^e\} = \int_{A^e} [N]^T \begin{Bmatrix} f_z \\ 0 \\ 0 \end{Bmatrix} \mathrm{dA}$$