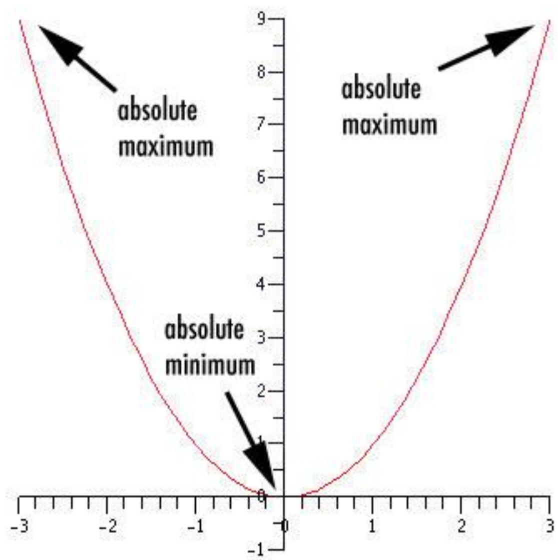
Lecture 6 (Applications of Differentiation)

Maximum and Minimum Values

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum value** of f on D if $f(c) \geq f(x)$ for any $x \in D$.
- **absolute minimum value** of f on D if $f(c) \leq f(x)$ for any $x \in D$.

Example: $f(x) = x^2, x \in [-3, 3]$



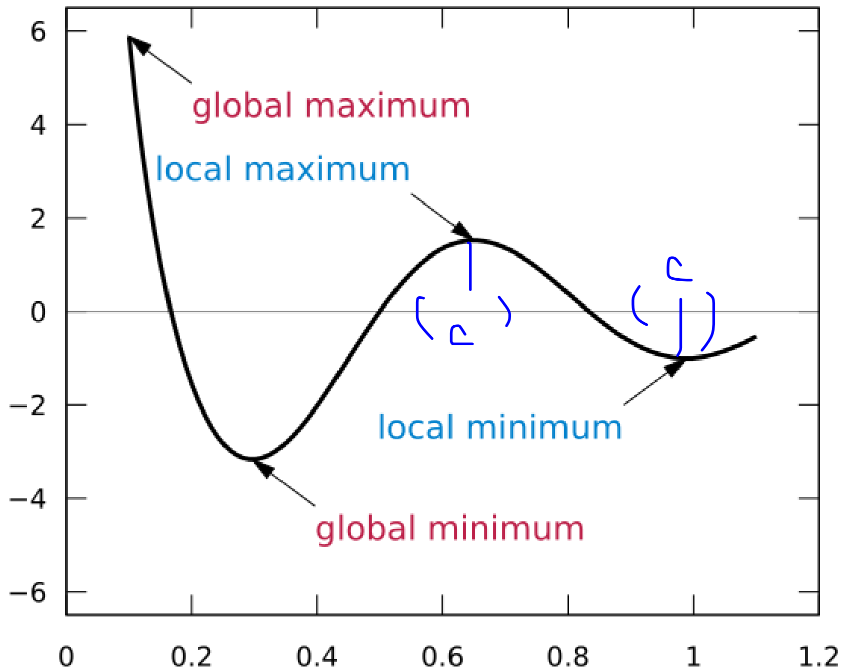
$c = 0$ - abs. min
$c = \pm 3$ - abs. max
$y = 0, 9$ are extreme values

Other names: an absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

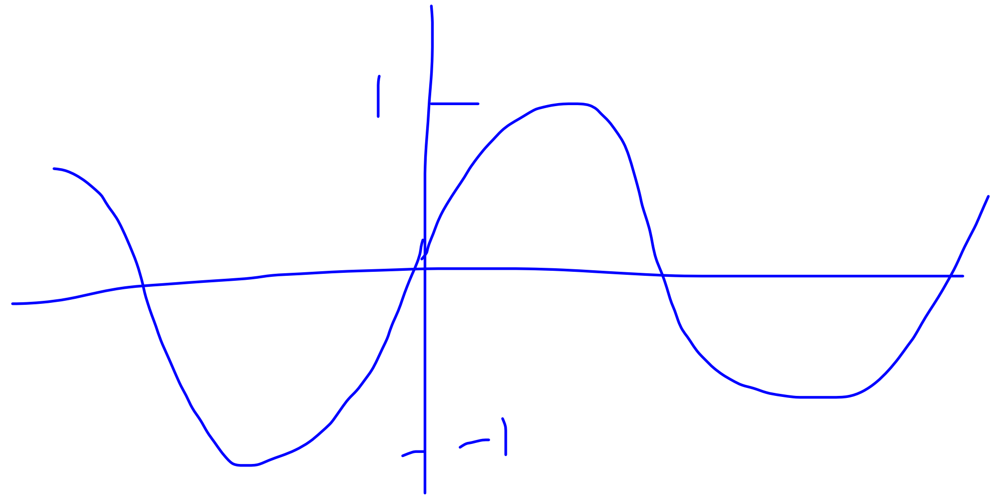
Definition: The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ for any x near c.
- **local minimum** value of f if $f(c) \leq f(x)$ for any x near c.

in open interval that contains c



<u>Example</u>: $f(x) = \sin x$



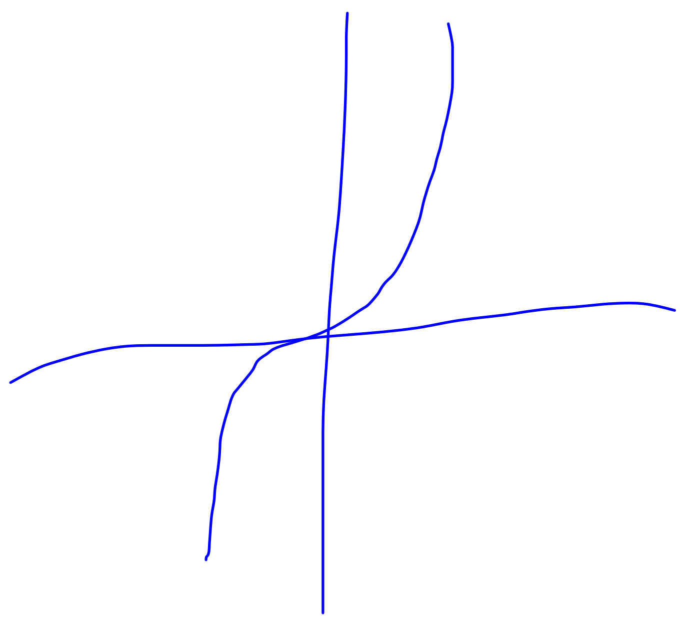
$x = \frac{\pi}{2} + 2k\pi$

↳ max

$x = -\frac{\pi}{2} + 2k\pi$

↳ min

<u>Example</u>: $y = x^3$

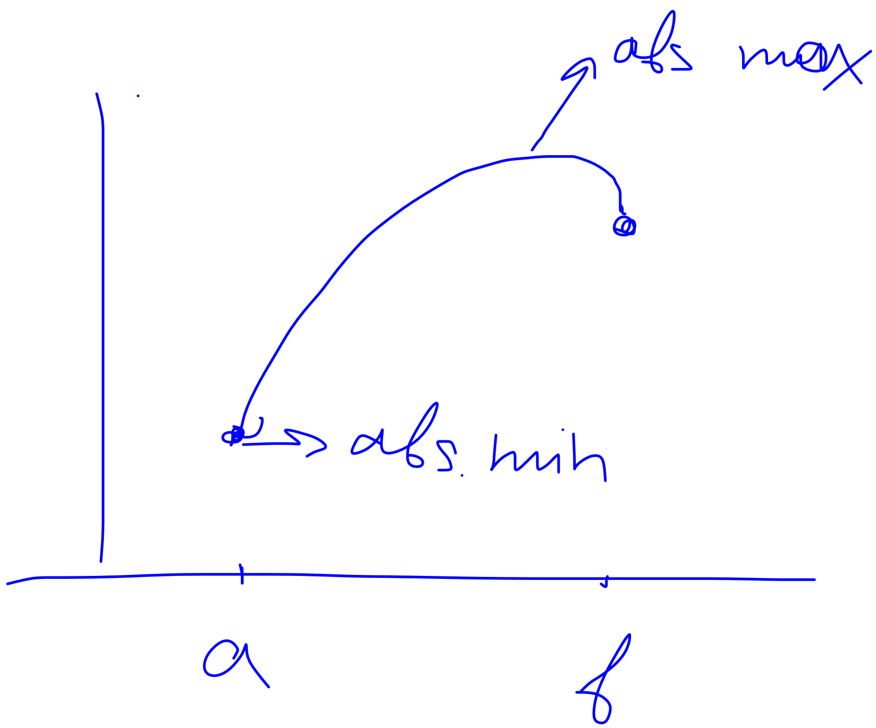


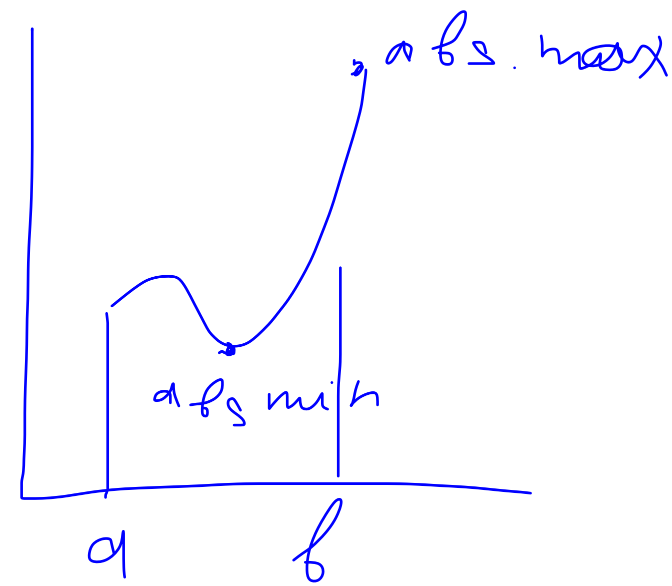
$f(x) \nearrow$ everywhere

no min/max

 When does a function have extreme values?

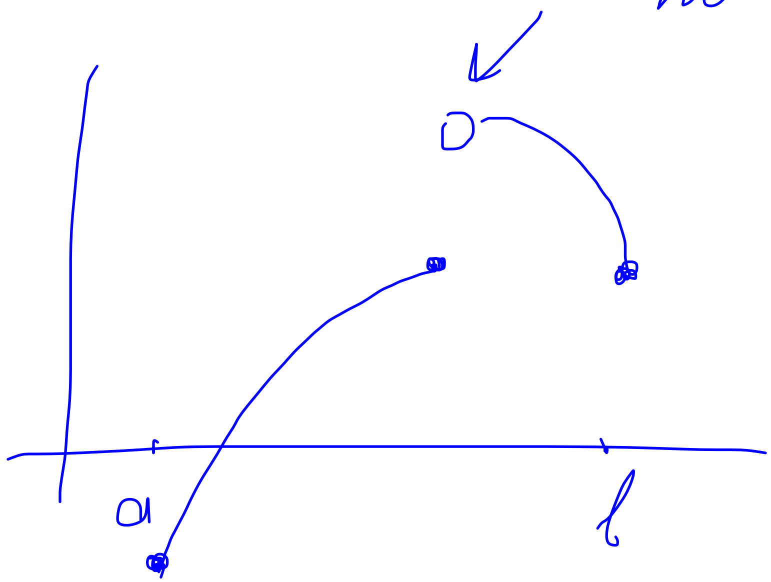
<u>Theorem</u>: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.





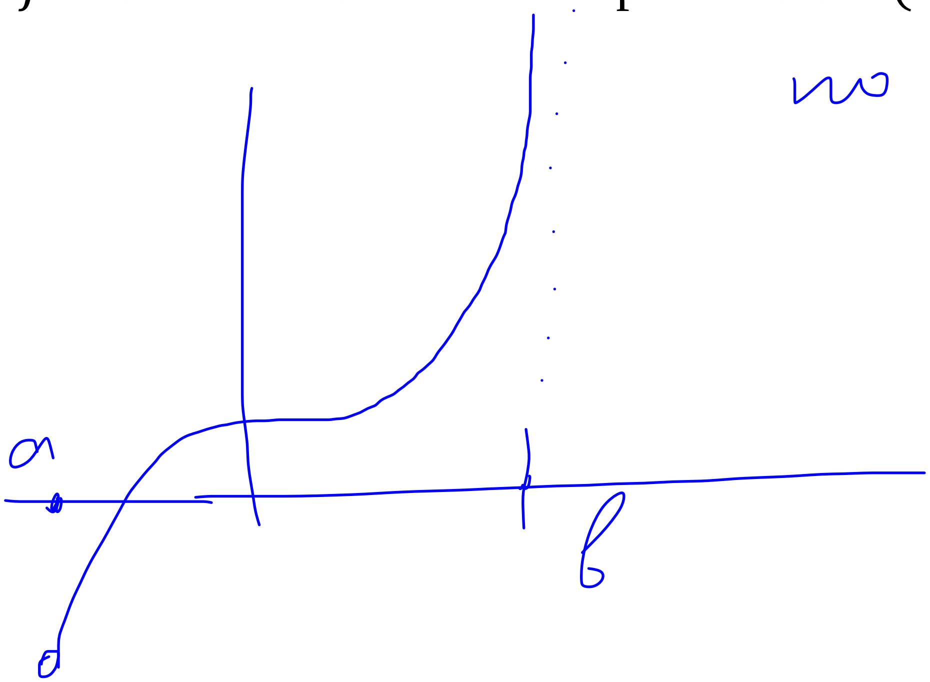
What if f is discontinuous?

no abs. max



What if f is continuous but on an open interval (a, b)?

no abs. max



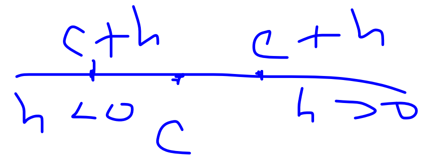
Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: Assume f has local max at c. Then

$f(c) \geq f(x)$ for x close to c

$f(c) \geq f(c+h)$, where h is close to 0

$f(c+h) - f(c) \leq 0$



Let $h > 0$

Then $\dfrac{f(c+h)-f(c)}{h} \leq 0$

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

Let $h < 0 \Rightarrow \dfrac{f(c+h)-f(c)}{h} \geq 0$

$$f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

Thus, $f'(c) = 0$ ∎

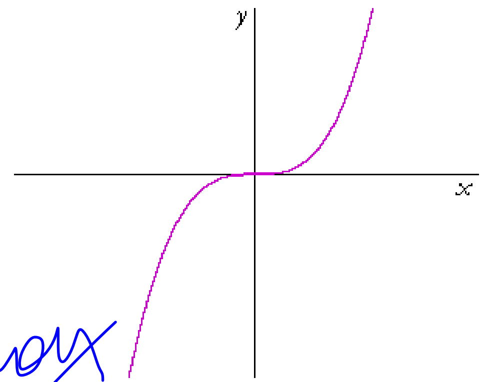
Caution: The converse of Fermat's Theorem is false in general.

Example: $f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

but no local min/max



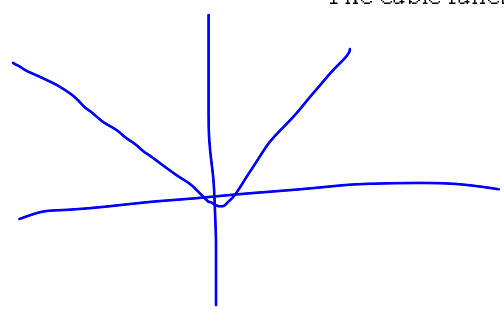
$y = x^3$

The cubic function

Example: $f(x) = |x|$

0 is local min

but $f'(0)$ dne



Note: However, Fermat's Theorem suggests that we should start looking for extreme values at numbers where the derivative is zero or does not exist.

Definition: A **critical number** of a function f is a number $c \in D$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example: Find the critical numbers of $f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{\underbrace{(1+x^2)^2}_{>0}}$$

$$1 - x^2 = 0$$

$$x = \pm 1 \longrightarrow \text{critical numbers}$$

Rephrase Fermat's Theorem: If f has a local maximum or minimum at c, then c is a critical number of f.

Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

- Find the values of f at the critical numbers of f in (a, b)
- Find the values of f at the endpoints of the interval.
- The largest of the values is the absolute maximum value; the smallest value is the absolute minimum value.

Example: Find the absolute maximum and absolute minimum values of

$f(x) = x^3 - 6x^2 + 5$ on $[-3, 5]$.

$f'(x) = 3x^2 - 12x = 0$

$x(x-4) = 0$

$x = 0$ or $x = 4$ $\in [-3, 5]$

$f(0) = 5$

$f(4) = 4^3 - 6 \cdot 4^2 + 5 = -27$

$f(-3) = -76$

$f(5) = -20$

$(0, 5)$ — abs. max

$(-3, -76)$ — abs. min

Example: $f(x)=e^{x}+e^{-2x}, 0\le x\le 1$

$$f'(x) = e^{x} + e^{-2x}(-2)$$

$$= e^{x} - 2e^{-2x}$$

$$= \underbrace{e^{-2x}}_{\neq 0}\underbrace{\left(e^{3x}-2\right)}_{} = 0$$

$$e^{3x} = 2$$

$$3x = \ln 2 \Rightarrow x = \frac{1}{3}\ln 2$$

$f\left(\frac{1}{3}\ln 2\right) = 1.89$ $\qquad$ $\left(\frac{1}{3}\ln 2, 1.89\right)$ - abs min

$f(0) = e^{0}+e^{0} = 2$ $\qquad$ $(1, 2.84)$ - abs max

$f(1) = e^{1}+e^{-2} = 2.84$

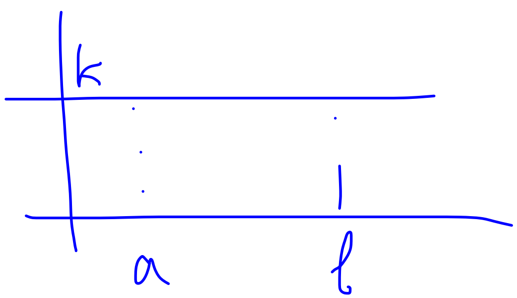
<u>Rolle's Theorem</u>: Let f be a function such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

<u>Proof</u>:

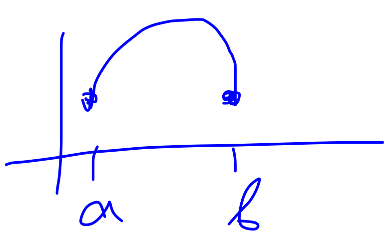
<u>Case 1</u>. $f(x) = \text{const} = k$, $f(a) = f(b) = k$

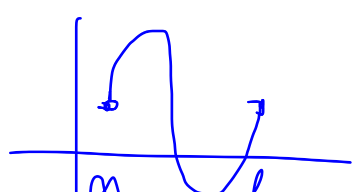


$f'(x) = 0$ for any $x \in (a, b)$

Take any c from (a, b)

<u>Case 2</u>. $f(x) > f(a)$ for some $x \in (a, b)$



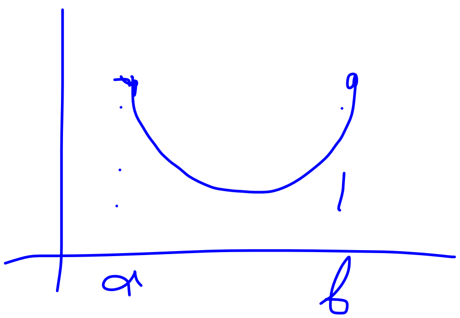


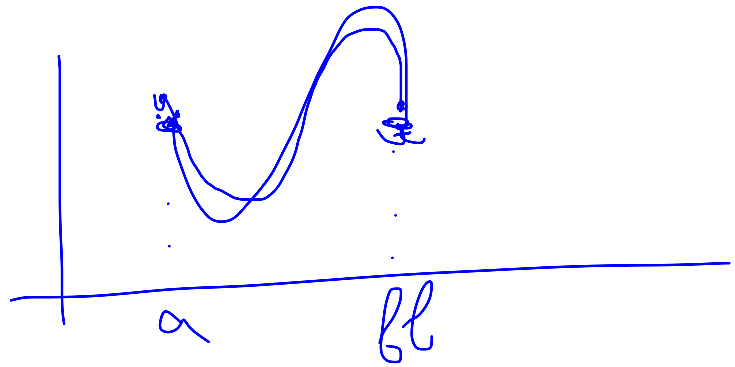
$f(x)$ must have max value for some $c \in (a, b)$

$\Rightarrow$ by Fermat's Thm, $f'(c) = 0$

Proof (continued):

Case 3 $f(x) < f(a)$ for some $x \in (a, b)$





$f(x)$ must have min value for some $c \in (a, b)$

$\Rightarrow$ by Fermat's Thm, $f'(c) = 0$

$\blacksquare$

Example: Prove that the equation $x^{13} + 7x - 5 = 0$ has exactly one (real) root.

$$f(x) = x^{13} + 7x - 5$$

$$f(0) = -5$$

$$f(1) = 1 + 7 - 5 = 3$$

$$-5 \leq 0 \leq 3$$

By IVT, there is $c \in (0,1)$ such that $f(c) = 0$.

Suppose there are two roots: a, b s.t. $f(a) = 0 = f(b)$

$\Rightarrow$ by Rolle's Thm, there is $d \in (a, b)$ s.t.

$$f'(d) = 0$$

$$f'(x) = 13x^{12} + 7 \geq 7 > 0 \Rightarrow \text{Contradiction} \quad \blacksquare$$

Mean Value Theorem (MVT): Let f be a function such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

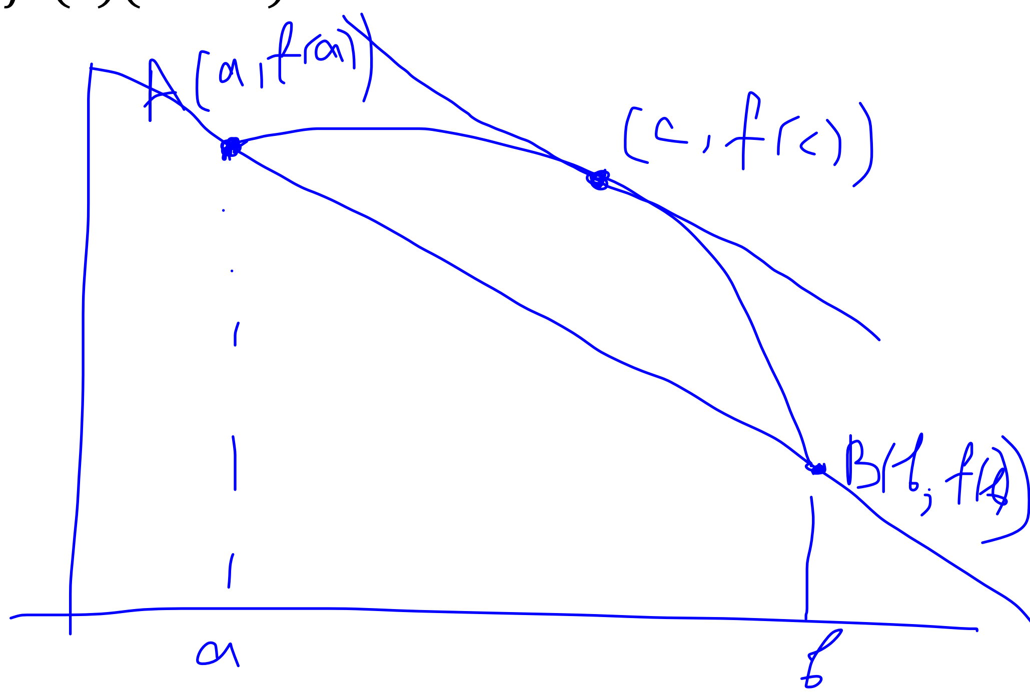
Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = m_{AB}$$

or

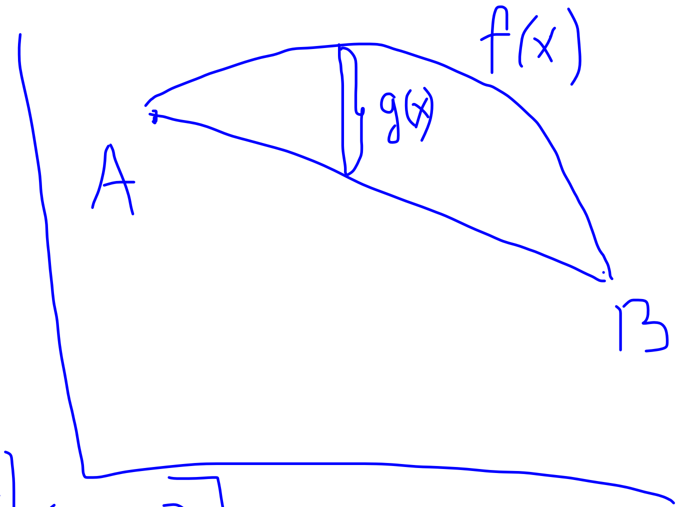
$$f(b) - f(a) = f'(c)(b - a)$$

Geometric interpretation: There is a point in (a, b) such that the tangent line at that point is parallel to the secant line going through points $(a, f(a))$ and $(b, f(b))$.



Proof: Equation of line AB:

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$



Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$

Check Rolle's Thm conditions:

1. $g(x)$ is continuous on $[a, b]$
2. $g(x)$ is diff. on (a, b)

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

3. $g(a) = f(a) - f(a) - \frac{f(b)-f(b)}{b-a}(a-a) = 0$

Proof (continued):

$$g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{\cancel{b-a}}\cancel{(b-a)} = 0$$

So, $g(a) = g(b)$

By Rolle's Thm, there is $c \in (a,b)$ s.t.

$$g'(c) = 0$$

So, $0 = g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\blacksquare$

Example: Suppose we know that $f(x)$ is continuous and differentiable on $[-7,0]$, $f(-7)=-3$, and $f'(x)\leq 2$. What is the largest possible value for $f(0)$? $= 11$

By MVT, there is $c \in [-7,0]$

s.t. $f'(c) = \dfrac{f(0)-f(-7)}{0-(-7)}$

$= \dfrac{f(0)-(-3)}{7}$

$7f'(c) = f(0)+3$

$f(0) = 7\cdot \underbrace{f'(c)}_{\leq 2} - 3 \leq 7\cdot 2 - 3$

$= 11$

Fact: If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Proof:

Let $x_1, x_2 \in (a, b)$ s.t. $x_1 < x_2$

$f(x)$ is diff. on $(a, b) \Rightarrow$ diff. on (x_1, x_2)

and, thus, continuous on $[x_1, x_2]$

By MVT, there is $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$$\Rightarrow f(x_1) = f(x_2)$$

Since x_1, x_2 are any numbers from (a, b), f is constant on (a, b).

$\blacksquare$

<u>Corollary</u>: If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on (a, b).

<u>Proof</u>:

$$h(x) = f(x) - g(x)$$

$$h'(x) = f'(x) - g'(x) = 0 \text{ for all } x \in (a, b)$$

By prev. fact, $h(x) = \text{const}$

$$\Rightarrow f - g = \text{const}$$

$\blacksquare$

How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test:

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval. ← similar

Proof: Let $x_1 < x_2$

Want: if $f'(x) > 0$ then $f(x_1) < f(x_2)$

$f(x)$ is diff. on (x_1, x_2)

By MVT, there is $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{>0}\underbrace{(x_2 - x_1)}_{>0} > 0$$

$f(x_2) > f(x_1) \Rightarrow f(x)$ is $\nearrow$ ∎

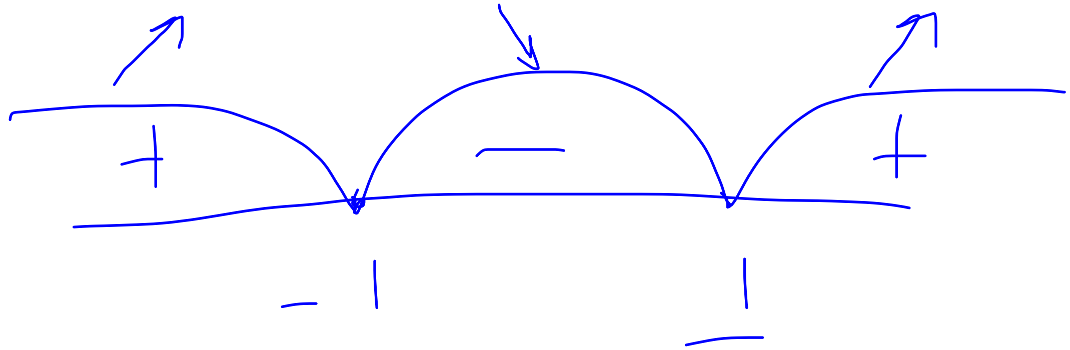
Example: Find the intervals where $f(x) = 4x^3 + 3x^2 - 6x + 1$ is inceasing or decreasing.

$$f'(x) = 12x^2 + 6x - 6 = 0$$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$x = \frac{1}{2}, -1$ are critical points



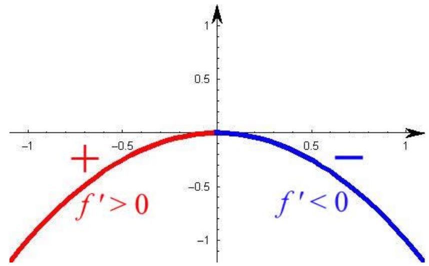
$f(x)$ is $\nearrow$ on $(-\infty, -1) \cup (\frac{1}{2}, \infty)$

$\searrow$ on $(-1, \frac{1}{2})$

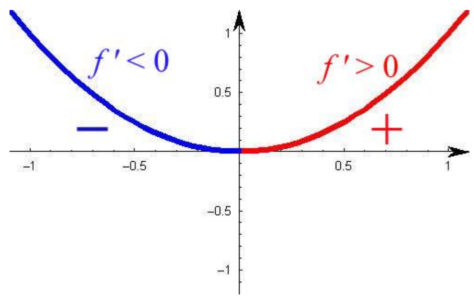
First Derivative Test:

Let c be a critical number of a continuous function f. Then,

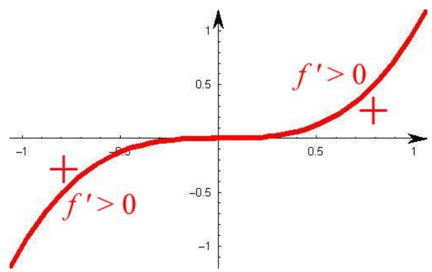
- If f' changes from positive to negative at c, then f has a local max at c.

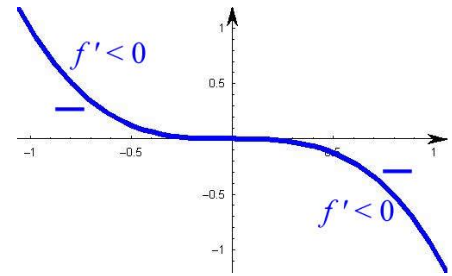


- If f' changes from negative to positive at c, then f has a local min at c.

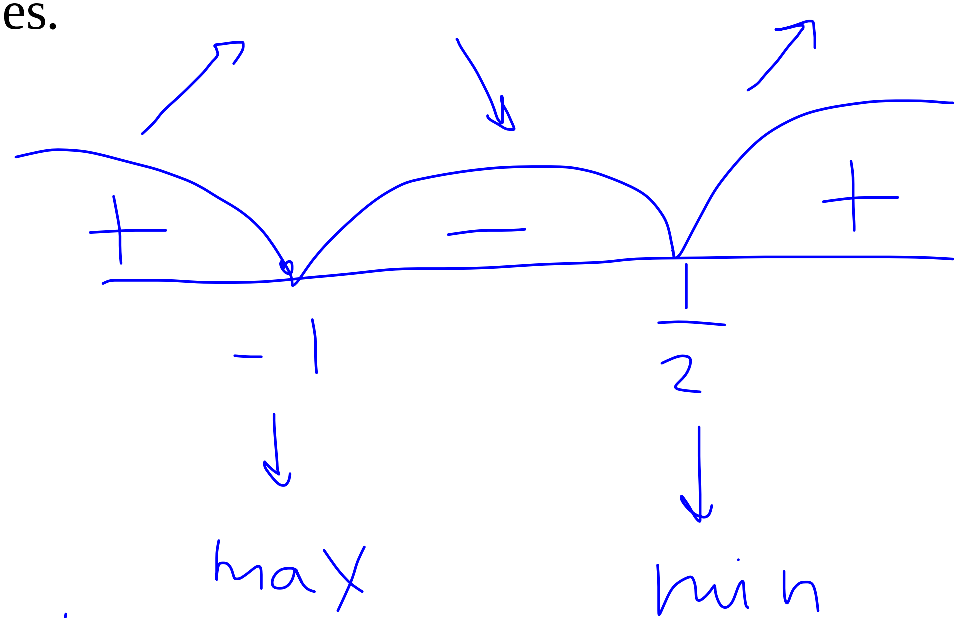


- If f' does not change sign at c, then f has no local max or min at c.





<u>Example</u>: Given $f(x) = 4x^3 + 3x^2 - 6x + 1$. Find the local minimum/maximum values.



$f(-1) = 6$

$f\left(\frac{1}{2}\right) = -0.75$

$(-1, 6)$ – local max

$\left(\frac{1}{2}, -0.75\right)$ – local min

Example: Given $f(x) = \cos^2 x - 2\sin x$, $0 \le x \le 2\pi$, find the local minimum/maximum values of f.

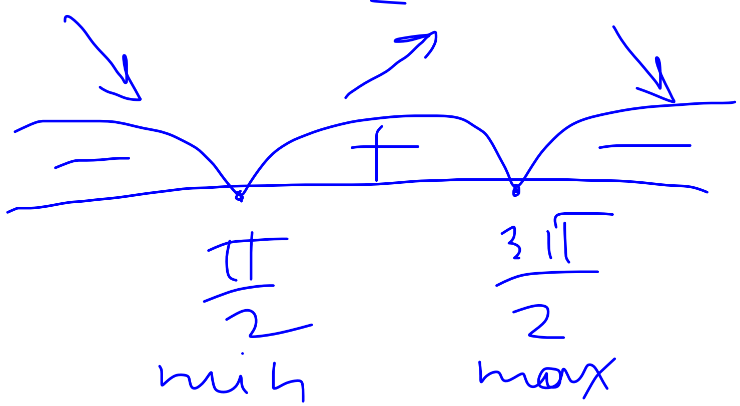
$f\left(\frac{\pi}{2}\right) = -2$, $f\left(\frac{3\pi}{2}\right) = 2$

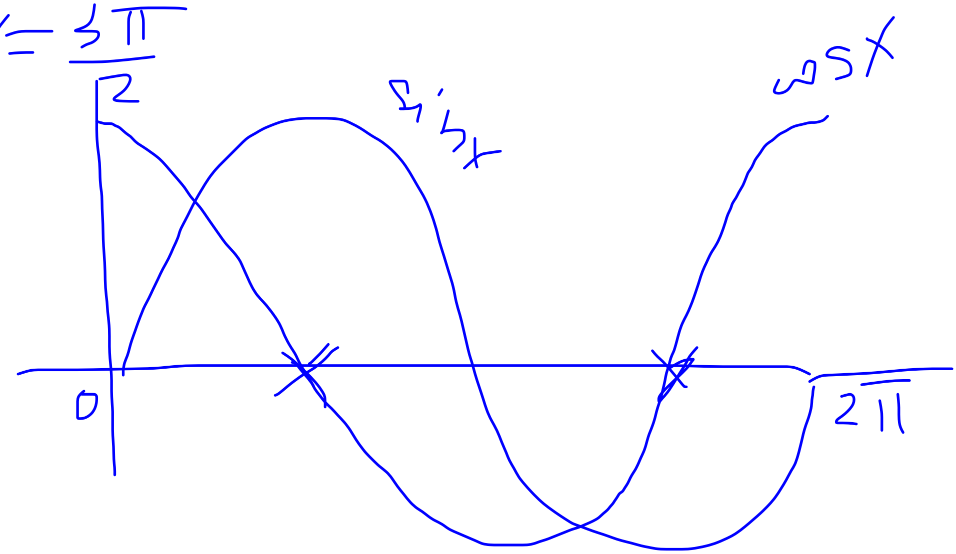
$f'(x) = 2\cos x \cdot (-\sin x) - 2\cos x = 0$

$-2\cos x(1 + \sin x) = 0$

$\cos x = 0$ $\qquad$ $\sin x = -1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $x = \frac{3\pi}{2}$



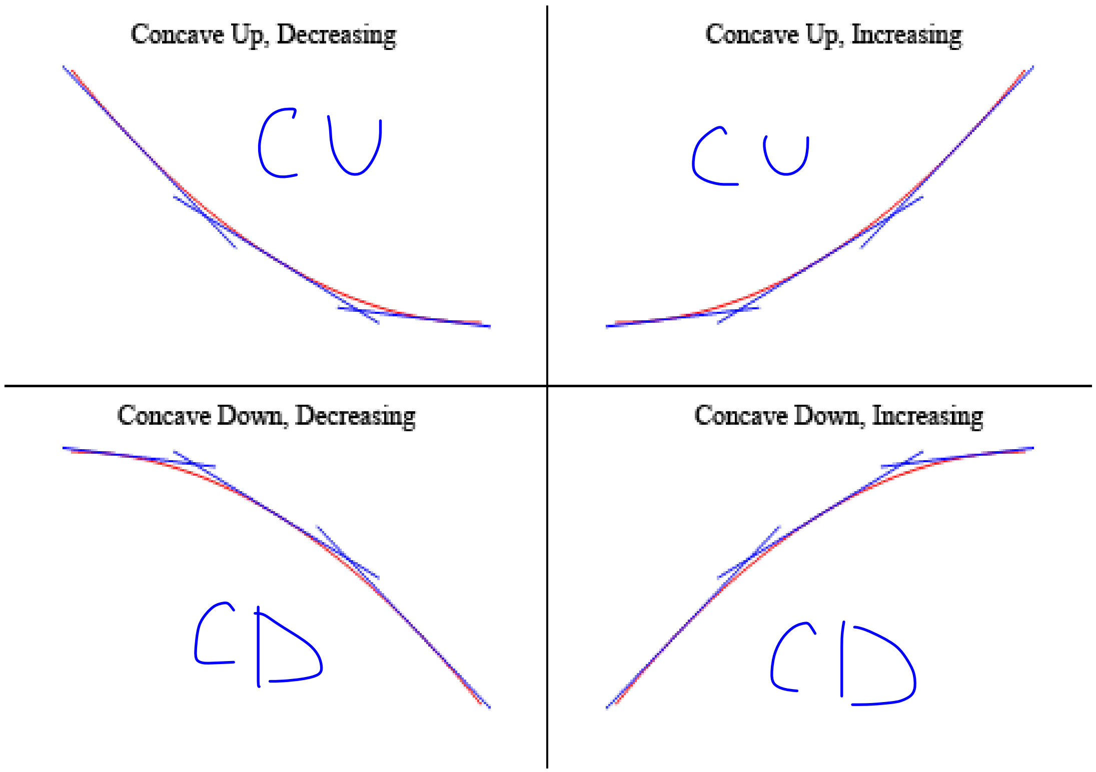


$\left(\frac{\pi}{2}, -2\right) \rightarrow$ local min

$\left(\frac{3\pi}{2}, 2\right) \rightarrow$ local max

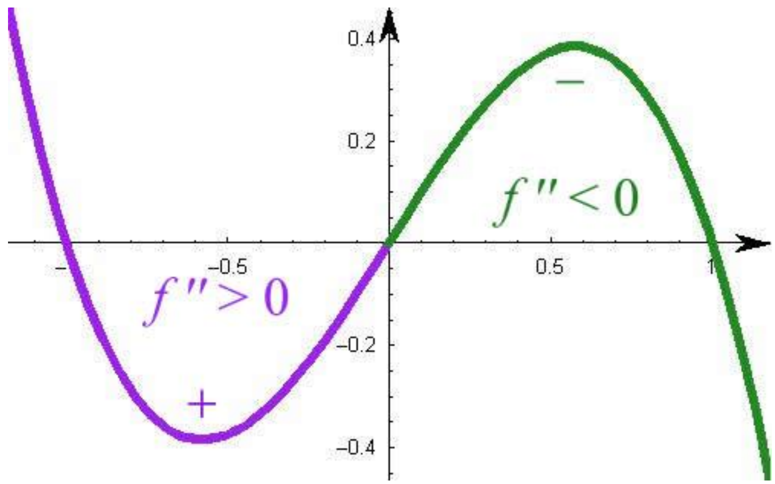
Concavities

Definition: If the graph of f lies above (below) all of its tangents on an interval, then we say, it is **concave up** (**concave down**) on that interval.



Concavity Test:

- If $f''(x) > 0$ on an interval, then the graph of f is concave up (CU) on that interval.
- If $f''(x) < 0$ on an interval, then the graph of f is concave down (CD) on that interval.



Definition: A point P on a curve $y = f(x)$ is called an **inflection point** (IP) if f is continuous there and the curve changes from CU to CD or vise versa at P.

Example: Given $f(x) = x^2 \ln x$, find the intervals of concavity and the inflection points.

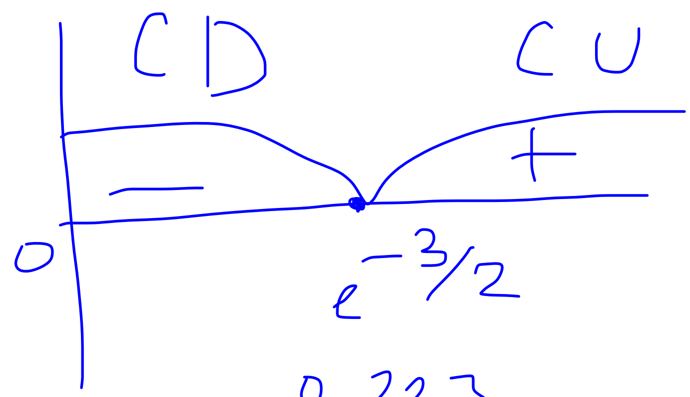
$$f'(x) = x^2 \cdot \frac{1}{x} + 2x \ln x = x + 2x \ln x$$

$$f''(x) = 1 + 2x \cdot \frac{1}{x} + 2 \cdot \ln x$$

$$= 3 + 2 \ln x = 0$$

$$\ln x = -\frac{3}{2}$$

$$x = e^{-3/2}$$



0.223

$f(x)$ is CD on $(0, e^{-3/2})$

CU on $(e^{-3/2}, \infty)$

$(0.223, f(0.223))$ is IP

Second Derivative Test: Let f'' be continuous near c.

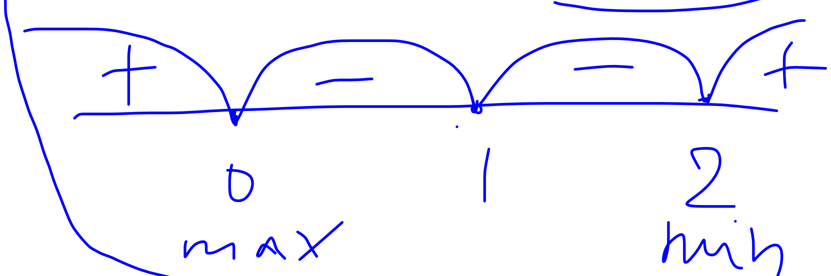
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c. CU
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c. CD

Example: $f(x) = \frac{x^2}{x-1}$, find the local minimum and maximum values.

$$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \boxed{\frac{x^2 - 2x}{(x-1)^2}} = \frac{x(x-2)}{(x-1)^2}$$

$x = 0, 2, 1$

1st Der. Test



$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x)2(x-1)}{(x-1)^4}$$

$$= \frac{2((x-1)^2 - (x^2-2x))}{(x-1)^3}$$

$$= \frac{2}{(x-1)^3}$$

2nd Der. Test

$f''(0) = -2 < 0 \rightarrow$ max

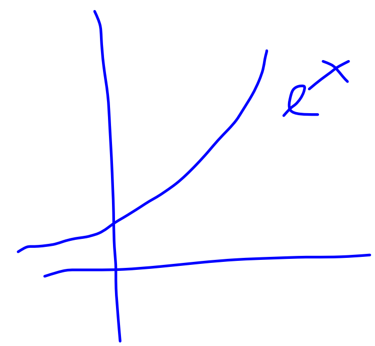
$f''(2) = 2 > 0 \rightarrow$ min

Example: Given $f(x) = \frac{e^x}{1-e^x}$. Use the first and second derivatives, together with asymptotes, to sketch the graph of f.

1. Domain $= \{x : 1-e^x \neq 0\} = \{x : x \neq 0\}$

2. $\lim_{x\to\infty} \frac{e^x}{1-e^x} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{1}{\underbrace{\frac{1}{e^x}}_{\to 0} - 1} = -1$

$\boxed{y = -1 \text{ is HA}}$



$\lim_{x\to-\infty} \frac{\overbrace{e^x}^{\to 0}}{\underbrace{1-e^x}_{\to 0}} = 0$

$\boxed{y = 0 \text{ is HA}}$

$\lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty$ $\qquad$ $\lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty$

$e^x \to 1^+$ $\qquad$ $\boxed{x = 0 \text{ is VA}}$ $\qquad$ $e^x \to 1^-$

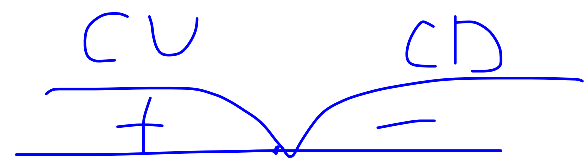
3. $f'(x) = \left(\dfrac{e^x}{1-e^x}\right)' = \dfrac{e^x(1-e^x) - e^x(-e^x)}{(1-e^x)^2}$

$= \dfrac{e^x}{(1-e^x)^2} > 0 \Rightarrow f(x) \nearrow$ on D

4. $f''(x) = \dfrac{e^x(1-e^x)^2 - e^x \cdot 2(1-e^x)(-e^x)}{(1-e^x)^4}$

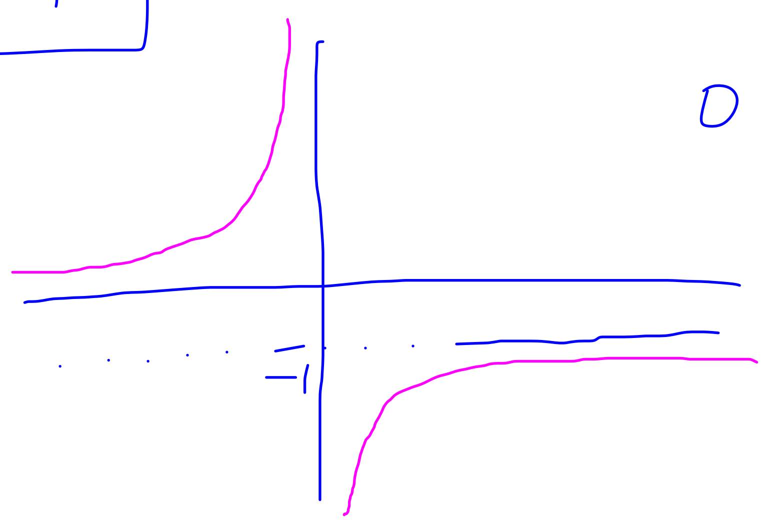
$= \dfrac{e^x(1-e^x) + 2e^{2x}}{(1-e^x)^3} = \dfrac{e^x + e^{2x}}{(1-e^x)^3}$

$= \dfrac{e^x(1+e^x)}{\underbrace{(1-e^x)^3}} > 0$



No IP

$0 \notin D$



Indeterminate Forms

What if we want to find $\lim_{x\to 1} \frac{\ln x}{x-1}$?

There are several types of indeterminate forms:

$$\frac{0}{0} \quad \text{and} \quad \frac{\pm\infty}{\pm\infty}$$

Indeterminate difference: $\infty - \infty$

Indeterminate product: $0 \cdot \infty$

Indeterminate power: 0^0 , ∞^0 , and 1^∞

The first two types are solvable with the help of *l'Hospital's Rule*.

<u>L'Hospital's Rule</u>: Let f and g be differentiable on (a, b) and $g'(x) \neq 0$ on (a, b). Let $c \in (a, b)$. If

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

<u>Example</u>:

$$\lim_{x\to 1} \frac{\ln x}{x-1} \quad \frac{0}{0} \quad \overset{H}{=} \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x}$$

$$= 1$$

Example:

$$\lim_{x\to\infty}\frac{x^2}{e^x}\overset{H}{=}\lim_{x\to\infty}\frac{2x}{e^x}\overset{H}{=}\lim_{x\to\infty}\frac{2}{e^x}=0$$

Example:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$0 \cdot (-\infty)$

$$\lim_{x\to 0^+} \frac{\sin x}{x} \frac{\ln x}{1/x} = \underbrace{\lim \frac{\sin x}{x}}_{=1} \cdot \lim_{x\to 0^+} \frac{\overset{-\infty}{\ln x}}{\underset{\infty}{1/x}}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} =$$

$$= \lim_{x\to 0^+} (-x) = 0$$

Example:

$$\lim_{x\to\infty}(\overset{\infty - \infty}{x - \ln x}) = \lim_{x\to\infty} x\left(1 - \frac{\overset{\to\infty}{\ln x}}{\underset{\to\infty}{x}}\right) \,\, \boxed{=}$$

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

$$\boxed{=} \lim_{x\to\infty} x \cdot \lim\left(1 - \underset{\to 0}{\frac{\ln x}{x}}\right)$$

$$= \infty$$

Example:

∞^0

$$\lim_{x\to\infty}(e^x+x)^{1/x}$$

$$y=(e^x+x)^{1/x}$$

$$\ln y=\frac{1}{x}\ln(e^x+x)$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{1}{e^x+x}(e^x+1)}{1}$$

$$\frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{e^x+1}\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{e^x}$$

$$=1$$

$$e^{\lim_{x\to\infty}\ln y} \qquad e$$

$$e^{\lim\ln y}=\lim e^{\ln y}=\lim y=e$$

Example:

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} (2x+1)\cdot \ln\left(\frac{2x-3}{2x+5}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{2x-3}{2x+5}\right)}{1/(2x+1)} = \lim_{x\to\infty} \frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}$$

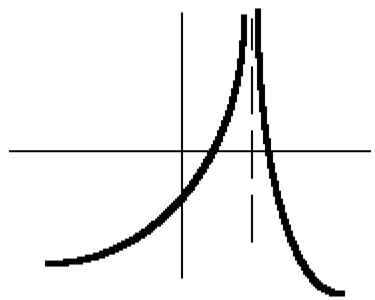
$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{\cancel{2}}{2x-3} - \frac{\cancel{2}}{2x+5}}{-\frac{\cancel{2}}{(2x+1)^2}} = \lim_{x\to\infty} \frac{-[2x+5-(2x-3)](2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{-8(2x+1)^2}{(2x-3)(2x+5)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-8\left(2+\frac{1}{x}\right)^2}{\left(2-\frac{3}{x}\right)\left(2+\frac{5}{x}\right)}$$

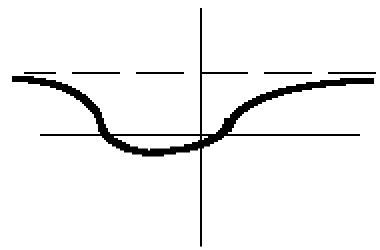
$$= -8\cdot\frac{2^2}{2\cdot 2} = -8 \qquad \boxed{\lim y = e^{-8}}$$

Slant Asymptotes

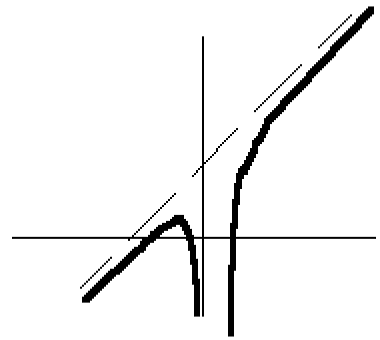
There are asymptotes that are neither horizontal nor vertical, but *oblique*.



V.A. (vertical asymptote)



H.A. (horizontal asymptote)



S.A. (slant asymptote)

Definition: The line $y = mx + b$ is called a **slant asymptote** if

$$\lim_{x \to \infty} [f(x) - (mx - b)] = 0$$

$\pm \infty$

Note: For rational functions, the slant asymptotes are present if the degree of the numerator is one more than the degree of the denominator.

Example: Find all the asymptotes of $r(x) = \frac{x^2-4x-5}{x-3}$ $= \underbrace{-\frac{8}{x-3}} + \underbrace{x-1}$

$$\begin{array}{r|l} x^2-4x-5 & x-3 \\ \hline -(x^2-3x) & x-1 \\ \hline -x-5 & \\ -(-x+3) & \\ \hline -8 & \end{array}$$

$$\lim_{x\to\infty}\left[r(x)-(x-1)\right]$$

$$= \lim_{x\to\infty} \frac{-8}{x-3} = 0$$

$\boxed{y = x-1 \text{ is a SA}}$

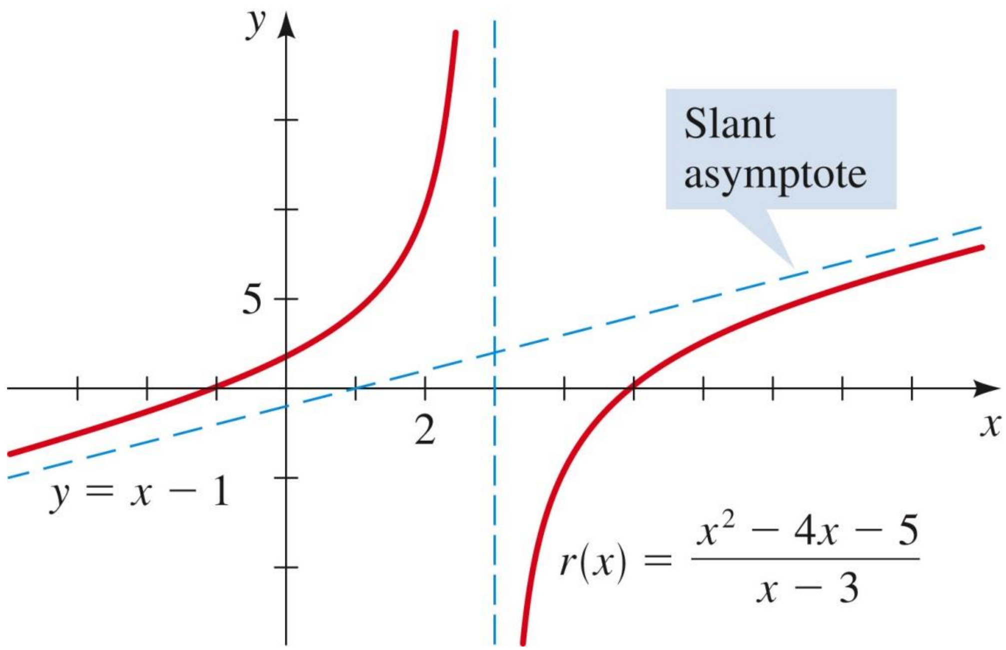
$$\lim_{x\to\infty} \frac{x^2-4x-5}{x-3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x-4-\frac{5}{x}}{1-\frac{3}{x}} = \infty$$

no HA.

$$\lim_{x \to 3^+} \frac{x^2 - 4x - 5}{x - 3} = \lim_{x \to 3^+} \frac{(x-5)(x+1)}{x-3} = -\infty$$

$$\lim_{x \to 3^-} \frac{x^2 - 4x - 5}{x - 3} = \infty$$

$\boxed{x = 3 \text{ is VA}}$



<u>Example</u>: Sketch the graph of $y = \frac{x^2+1}{x} = x + \frac{1}{x}$

Domain $= \{x \neq 0\}$

Asymptotes:

$\lim_{x \to \pm\infty} \frac{x^2+1}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \pm\infty} \frac{x + \frac{1}{x}}{1} = \pm\infty$

$\Rightarrow$ no HA

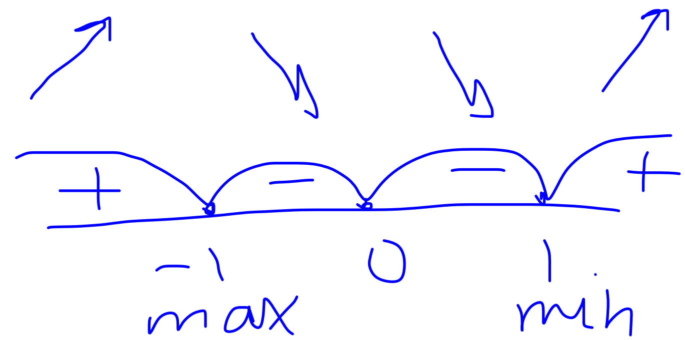
$\lim_{x \to 0^+} \frac{x^2+1}{x} = \infty$; $\lim_{x \to 0^-} \frac{x^2+1}{x} = -\infty$

$\Rightarrow x = 0$ is VA

$\lim_{x \to \infty} \left[\frac{x^2+1}{x} - x \right] = \lim_{x \to \infty} \frac{1}{x} = 0 \Rightarrow y = x$ is SA

$$y' = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

$x = \pm 1, 0$ are critical points



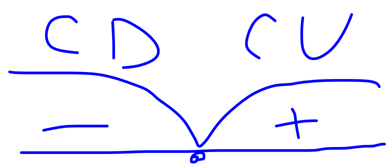
$f(x) \nearrow$ on $(-\infty,-1)\cup(1,\infty)$

$\searrow$ on $(-1,0)\cup(0,1)$

$(-1,-2)$ is local max

$(1,2)$ is local min

$$y'' = \frac{2}{x^3}$$

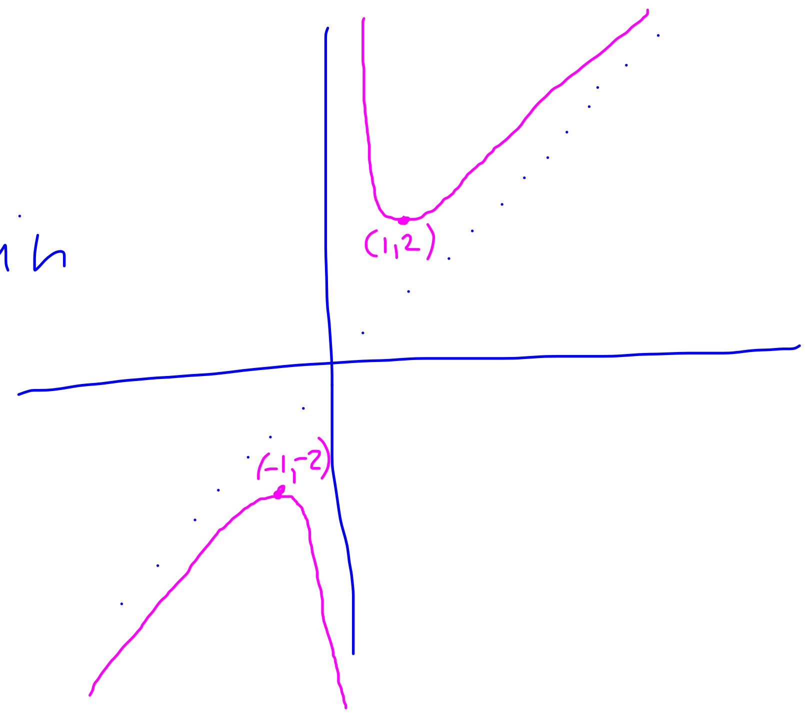


$0 \notin$ domain

no IP

$f(x)$ is CD on $(-\infty, 0)$

CU on $(0,\infty)$



Example: Sketch $f(x) = \frac{e^x}{x^2}$

Domain $= \{x \neq 0\}$

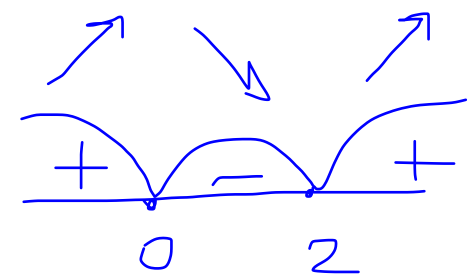
Asymptotes:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to-\infty} \frac{e^x}{x^2} = \lim_{x\to-\infty} e^x \cdot \frac{1}{x^2} = 0 \cdot 0 = 0 \Rightarrow y = 0 \text{ is HA}$$

$$\lim_{x\to 0} \frac{e^x}{x^2} = \infty \Rightarrow x = 0 \text{ is VA}$$

$$y' = \frac{e^x x^2 - 2xe^x}{x^4} = \frac{e^x(x-2)}{x^3}$$



$x = 0, 2$ are critical points $\quad 0 \notin$ domain $\quad$ min

$f(x) \nearrow$ on $(-\infty, 0) \cup (2, \infty)$, $\searrow$ on $(0, 2)$, $(2, e^2/4)$ − local min

$$y'' = \frac{[e^x(x-2)+e^x]x^3 - 3x^2e^x(x-2)}{x^6} = \frac{e^x x^2[(x-1)x - 3(x-2)]}{x^6}$$

$$= \frac{e^x[x^2-4x+6]}{x^4} > 0$$

$x^2-4x+6>0$
$b^2-4ac = 16-4\cdot6\cdot1<0$
no x-intercepts

$\Rightarrow f(x)$ is CU on $(-\infty,0)\cup(0,\infty)$

no IP