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Chapter one

Functions

***Real Numbers*:**

$\mathbb{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$ Natural Numbers.

$\mathbb{Z} = \{\dots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$ Integer Numbers.

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ Rational Numbers.

$\mathbb{IQ} = \pi, e, \sqrt{2}, \sqrt{3}, \sqrt{5}, \dots$ Irrational Numbers.

$\mathbb{R} = \mathbb{Q} \cup \mathbb{IQ}$

Definition **(1.1):** If a and b are real numbers, we define the Intervals as follows:

(1) Open Intervals $(a,b) = \{x \in R : a < x < b\}$

(2) Closed Intervals $[a,b] = \{x \in R : a \le x \le b\}$

(3) Half open, half closed Intervals $[a,b) = \{x \in R : a \le x < b\}$ and $(a,b] = \{x \in R : a < x \le b\}$.

(4) $[a,\infty) = \{x \in R : x \ge a\}$, $(-\infty,a] = \{x \in R : x \le a\}$, $(-\infty,a) = \{x \in R : x < a\}$ and $(-\infty,\infty) = \{x \in R : x \text{ is a real number}\} = R$.

Remark: If A and B are intervals, then

1. The union of A and B is denoted by $A \cup B$ and defined as the interval whose members belong to A or B (or both).
2. The intersection of A and B is denoted by $A \cap B$ and defined as the interval whose members belong to both A and B.

Example: Let $A = [0,5]$ *and* $B = [1,7]$ then $A \cup B = [0,7]$ and $A \cap B = [1,5]$.

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Functions

<u>Definition 1</u>: A relation $f : X \to Y$ is called a function if and only if for each element $x \in X$ there exist a unique element $y \in Y$ such that $y = f(x)$.

<u>Note:</u>

(1) x is the independent variable (input value of f) and y is dependent variable (output value of f at x)

(2) The set X of all possible input values is called the domain of f and it's denoted by D_f

(3) The set Y is called the co-domain of the function.

(4) The set of all possible output values $f(x)$ as x varies throughout D_x is called the range of f and it's denoted by R_f. (Note that $R_f \subseteq Y$.)

<u>Examples</u>: Find the Domain and the Range of the following:

(1) $y = x + 5$

$D_f = \{x \in R : -\infty < x < \infty\} = R$

To find the Range, we represent x in terms of y. $x = y - 5$

$R_f = \{y \in R : -\infty < y < \infty\} = R$

(2) $y = x^2$ $\quad D_f = \{x \in R : -\infty < x < \infty\} = R$

$x = \sqrt{y} \quad \Rightarrow \quad R_f = \{y \in R : y \geq 0\} = [0, \infty)$

(3) $y = \dfrac{1}{x+2}$ (set the denominator = ***0)***

$x + 2 = 0, \quad x = -2$

$\Rightarrow D_f = \{x \in \Re : x \neq -2\} = \Re / \{-2\}$

$x = \dfrac{1-2y}{y}$

$\Rightarrow R_f = \{y \in \Re : y \neq 0\} = \Re / \{0\}$

(4) $y = \sqrt{x+9}$ $\quad x \in [0,7] \quad \Rightarrow D_f = [0,7]$

Put $x = 0$ in the function we get $y = 3$

Put $x = 7$ in the function we get $y = 4$ then $R_f = [3,4]$

(5) $f(x) = \dfrac{1}{\sqrt{2-x}} + 5$ $\qquad$ **(6)** $y = \dfrac{3x}{x^2 - 5x + 6}$

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Definition: Let $f(x)$ be a function with domain D_f. The set of all points (x, y) in the plane with x in D_f and $y = f(x)$ is called the graph of $f(x)$.

$\{(x, f(x)): x \in D_f\}$

x				
$f(x)$				
$(x, f(x))$				

Algebraic Combination of Function

If f and g are two functions with domains D_f *and* D_g respectively, then

(1) $(f+g)(x) = f(x) + g(x)$ ***with*** $D_{f+g} = D_f \cap D_g$

(2) $(f-g)(x) = f(x) - g(x)$ ***with*** $D_{f-g} = D_f \cap D_g$

(3) $(f.g)(x) = f(x) . g(x)$ ***with*** $D_{f.g} = D_f \cap D_g$

(4) $((\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$ ***with*** $D_{f/g} = D_f \cap D_g$ and $g(x) \neq 0$

Examples: *1)* If $f(x) = x^2 - 5x - 6$ and $g(x) = 3x^2 + 4$ then find $(f+g)(x)$, $(f-g)(x), (g-f)(x), (f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains.

Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

Function	Formula	Domain
$f+g$	$(f+g)(x) = 4x^2 - 5x - 2$	$\mathbb{R}$
$f-g$	$(f-g)(x) = -2x^2 - 5x - 10$	$\mathbb{R}$
$g-f$	$(g-f)(x) = 2x^2 + 5x + 10$	$\mathbb{R}$
$f.g$	$(f.g)(x) = 3x^4 - 15x^3 - 14x^2 - 20x - 24$	$\mathbb{R}$
f/g	$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 5x - 6}{3x^2 + 4}$	$\mathbb{R}$
g/f	$\left(\frac{g}{f}\right)(x) = \frac{3x^2 + 4}{x^2 - 5x - 6}$	$(-\infty, -1) \cup (-1,6) \cup (6, \infty)$

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2) If $f(x) = \sqrt{x}$ *and* $gx) = \sqrt{1-x}$ then find $(f+g)(x), (f-g)(x), (g-f)(x),$ $(f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains.

Solution: $D_f = \{x \in \mathbb{R}: x \geq 0\} = \mathbb{R}^+ \cup \{0\} = [0,\infty)$, $D_g = \{x \in \mathbb{R}: x \leq 1\} = (-\infty, 1]$

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	[0,1]
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	[0,1]
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	[0,1]
$f.g$	$(f.g)(x) = \sqrt{x(1-x)}$	[0,1]
f/g	$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{1-x}} = \sqrt{\frac{x}{1-x}}$	[0,1)
g/f	$\left(\frac{g}{f}\right)(x) = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}}$	(0,1]

3) If $f(x) = 2$ *and* $gx) = x^2$ then find $(f+g)(x), (f-g)(x), (g-f)(x),$ $(f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains and ranges.

Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

Function	Formula	Domain	Rang
$f+g$	$(f+g)(x) = 2 + x^2$	$\mathbb{R}$	$[2,\infty)$
$f-g$	$(f-g)(x) = 2 - x^2$	$\mathbb{R}$	$[-\infty, 2)$
$g-f$	$(g-f)(x) = x^2 - 2$	$\mathbb{R}$	$[-2,\infty)$
$f.g$	$(f.g)(x) = 2x^2$	$\mathbb{R}$	$[0,\infty)$
f/g	$\left(\frac{f}{g}\right)(x) = \frac{2}{x^2}$	$(-\infty,0) \cup (0,\infty)$	$(0,\infty)$
g/f	$\left(\frac{g}{f}\right)(x) = \frac{x^2}{2}$	$\mathbb{R}$	$(0,\infty)$

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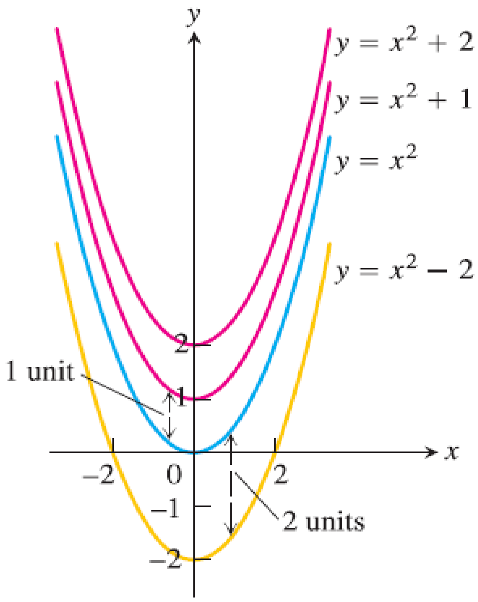
Shifting a Graph of a function:

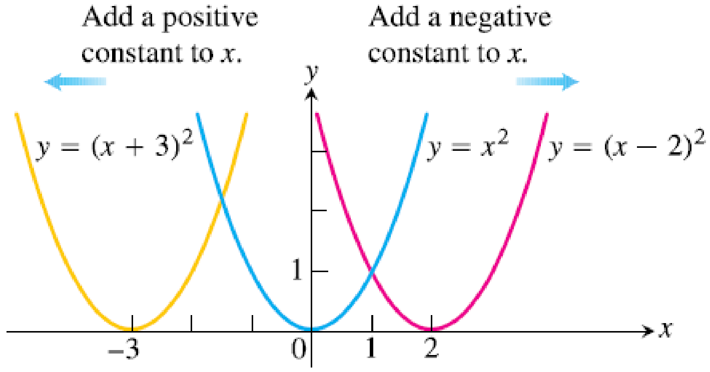
1. Vertical Shifts:

$$y = f(x) + k \Longrightarrow \text{Shifts } \textit{the graph of } f \begin{cases} up\ k\ units\ if\ k > 0 \\ down\ |k|\ units\ if\ k < 0 \end{cases}$$

2. Horizontal Shifts:

$$y = f(x + h) \Longrightarrow \text{Shifts } \textit{the graph of } f \begin{cases} left\ h\ units\ if\ h > 0 \\ right\ |h|\ units\ if\ h < 0 \end{cases}$$





Scaling and Reflecting a Graph of a Function:

For $c > 1$, the graph is scaled as:

$y = cf(x)$ Stretches the graph of f vertically by a factor of c.

$y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c.

$y = f(cx)$ Compresses the graph of f horizontally by a factor of c.

$y = f(\frac{1}{c}x)$ Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected as:

$y = -f(x)$ Reflects the graph f across the $x - axis$

$y = f(-x)$ Reflects the graph f across the $y - axis$

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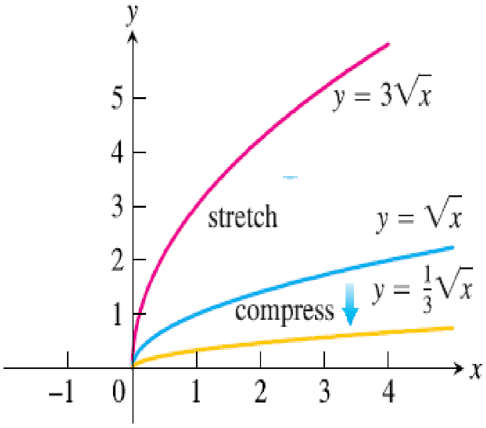
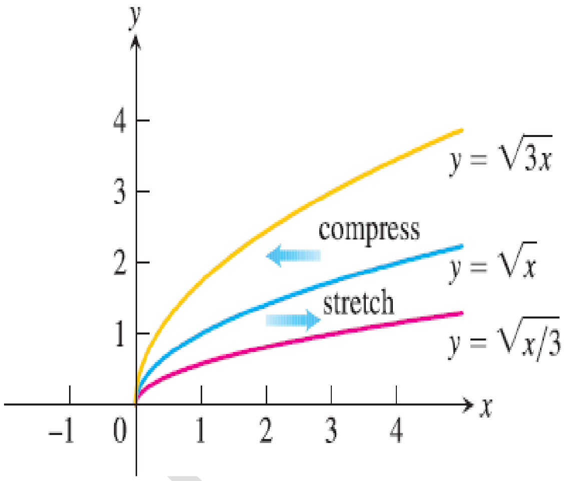
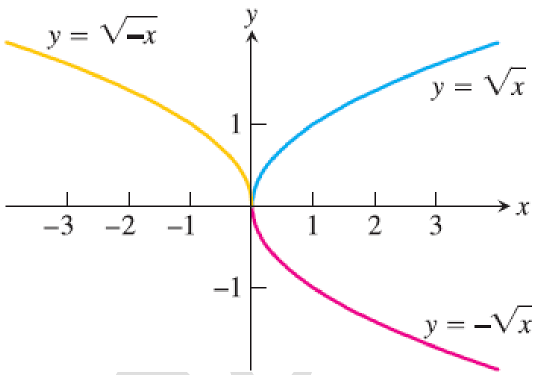
Example: Consider the function $y = \sqrt{x}$



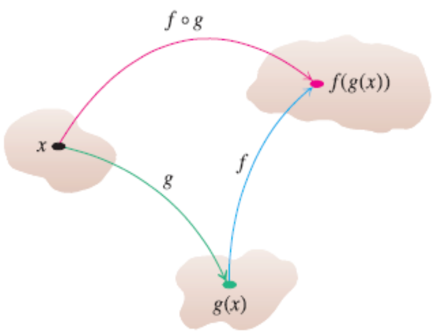




Composite Function

If the range of the function $g(x)$ is contained in the domain of the function $f(x)$, then the composition $f \circ g$ is the function defined by $(f \circ g)(x) = f\big(g(x)\big)$.
The domain of $f \circ g$ consists of the number x in the domain of g for which $g(x)$ lies in the domain of f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$



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Examples:

1) If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, then find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

Solution: $D_f = [0, \infty)$, $D_g = \mathbb{R}$

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[\mathbf{-1}, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[\mathbf{0}, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[\mathbf{0}, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$\mathbb{R}$

2) If $f(x) = x^2 - 3$ and $g(x) = x + 1$, then find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

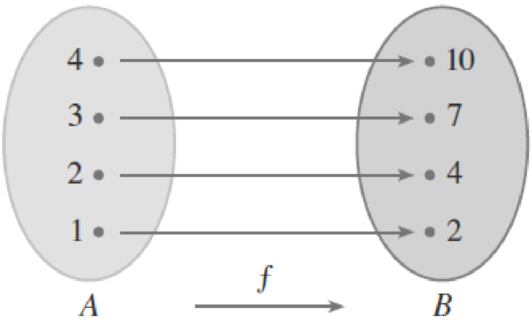
Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

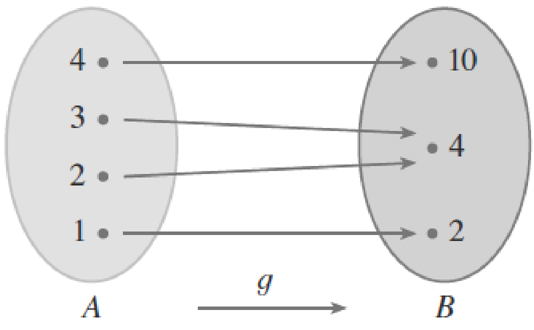
Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = (g(x))^2 - 3 = x^2 + 2x - 2$	$\mathbb{R}$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = x^2 - 2$	$\mathbb{R}$
(c) $(f \circ f)(x) = f(f(x)) = (f(x))^2 - 3 = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$	$\mathbb{R}$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$\mathbb{R}$

The Inverse of the functions:

Definition : A function is said to be one – to - one function if and only if there is no two elements of the domain have the same image in the range.

i.e. ***if*** $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ***or*** ***if*** $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$





Horizontal Line Test: A function is one-to-one if and only if there is no horizontal line can intersect its graph more than once.

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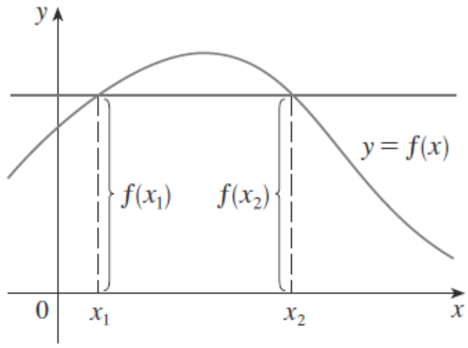

Definition: Let f be a one-to-one function with domain A and B (biective function). Then its inverse function f^{-1} has domain B and range A and is defined by
$f^{-1}(y) = x \Leftrightarrow f(x) = y, \forall y \in B$ or $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and $D_{f^{-1}} = R_f$, $D_f = R_{f^{-1}}$.

How to find the inverse function of a one-to-one function *f*
Step 1: Write $y = f(x)$.
Step 2: Solve this equation for x in terms of y (if possible).
Step 3: To express f^{-1} as a function of x, interchange x *and* y.
The resulting equation is $y = f^{-1}(x)$.

CAUTION
The -1 in f^{-1} is not a power number. Thus $f^{-1}(x)$ dose not mean $\frac{1}{f(x)}$.

Example: Find the inverse of f for $f(x) = x^3 + 2$
Solution: According to the above algorithm, we first write
$y = x^3 + 2$, then we solve this equation for x:
$x^3 = y - 2 \Rightarrow x = \sqrt[3]{y-2}$
Finally, we iterchange x *and* y:
$y = \sqrt[3]{x-2}$
$\therefore f^{-1}(x) = \sqrt[3]{x-2}$

❖ The graph of f^{-1} is obtained by reflecting the graph of f around the line $y = x$.