مقدمة في العمليات التصادفية

Introduction To Stochastic Process

Probability vector متجه الاحتمالية

Arow VECTOR U= (U_1, U_2,....., U_n) is called a Probability vector if its components are non-negative and there sum is 1.

EX: consider the following vector

(1) U=(3/4,0,-1/4,1/2)
(2) V=(3/4,1/2,0,1/4)
(3) W=(1/4,1/4,0,1/2)

then

1- U is not Probability vector since it's third component is negative.
2- V is not Probability vector because its sum greater then 1.
3- W is Probability vector.

Definition: A square matrix المصفوفة المربعة

P=($P_{i,j}$) is called a stochastic matrix if each of its rows is Probability vector.

EX:

(1) $\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ is not Stochastic matrix since contain negative element

(2) $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ is not Stochastic matrix since the sum of the second row not equal 1

(3) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$ is Stochastic matrix

Theorem: If A and B are Stochastic matrices then the product AB is Stochastic matrix.

Definition: A Stochastic matrix P is said to be regular if all the entries of some power P^M are positive.

EX: the Stochastic matrix A=$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is regular since

A^2=$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \bullet \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

Consider the Stochastic matrix

A=$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Note: In fact every power A^M will have 1 and 0 in the first row ,hence A is not regular.

Definition: A non-zero vector U= (U_1, U_2,….., U_n) is called afixed point of a square matrix A if U is left fixed , i.e. not change where multiplied by A , i.e. UA=U

EX: consider the matrix A=$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ then the vector U= (2 -1) is a fixed point of A for UA= (2 -1) • $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ = (2 -1) =U

EX: suppose U is affixed point of a matrix A, UA=U , we claim that every scalar multiple of U say, ʎU is also affixed point of A i.e.(ʎU)A = ʎ(UA) = ʎU

Thus in particular the vector 2U=(4 -2) is afixed point of the matrix A of the example above

(4 -2)• $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$= = (4 -2)

We state the result of preceding example as theorem .

Theorem: if U is affixed vector of matrix A then for any real number ʎ‡0 the scalar multiple ʎU is also fixed vector of A.

النقاط الثابتة و المصفوفات التصادفية المنتضمة

Fixed points and regular Stochastic matrices

Let p be aregular Stochastic matrix then

1- P has a unique probability vector k, and the components of k are all positive.

2- The sequence p, p^2, p^3, p^4 .... Of power p approaches to the matrix T whose rows are each the fixed point t.

3- If p is any probability vector the sequence of vectors pp,p p^2, pp^3, pp^4 .... Approaches the fixed point t.

Note: p^n approaches T means that each entry of p^n approaches the corresponding entry of T and pp^n approaches t means that each components pp^n approaches the corresponding components of t .

EX: consider the regular Stochastic matrix p=$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ we seek a probability vector t=(x,1-x) such that tp=t

→ (x,1-x) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ =(x,1-x)

→ 1/2 -1/2x =x

→ 1/2 +1/2x=1-x

} X=1/3

:. T= (1/3,2/3)

Thus T= (1/3,2/3) is the unique fixed probability vector of P, the the sequence p, p^2, p^3, p^4 .... approaches the matrix T whose rows are each the vector t. T = $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ = $\begin{pmatrix} 0.33 & 0.67 \\ 0.33 & 0.67 \end{pmatrix}$ we exhibit some of the powers of p to indicate the above result.

$$p^2=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}=\begin{pmatrix}0.50 & 0.50\\ 0.25 & 0.75\end{pmatrix}$$

$$p^3=\begin{pmatrix}\frac{1}{4} & \frac{3}{4}\\ \frac{3}{8} & \frac{5}{8}\end{pmatrix}=\begin{pmatrix}0.25 & 0.75\\ 0.37 & 0.63\end{pmatrix}$$

$$p^4=\begin{pmatrix}\frac{3}{8} & \frac{5}{8}\\ \frac{5}{16} & \frac{11}{16}\end{pmatrix}=\begin{pmatrix}0.37 & 0.63\\ 0.31 & 0.69\end{pmatrix}$$

$$p^5=\begin{pmatrix}\frac{5}{16} & \frac{11}{16}\\ \frac{11}{32} & \frac{21}{32}\end{pmatrix}=\begin{pmatrix}0.31 & 0.69\\ 0.34 & 0.66\end{pmatrix}$$

EX: find the unique fixed probability vector of the regular Stochastic matrix

$$P=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$$

Sol : t=(x,y,1-x-y) such that tp=t

$$(x,y,1-x-y)\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}=(x,y,1-x-y)$$

1/2 -1/2x-1/2y=x → 3x+y=1 1

X+1/2 -1/2x-1/2y=y → x-3y=-1........2

Y=1-x-y → x+2y=1 3

X= 1/3 y= 2/3 → t= (1/5,2/5,2/5)

Markov chains

سلسلة ماكروف

Consider a sequence of trails where out comes say $x_1, x_2, x_3, \ldots.$ Satisfy the following two probability

(1) Each out comes belongs to a finite set of outcomes $(a_1, a_2, a_3, \ldots.., a_n)$ called the state space of the system if the outcomes on then^{th} trial is a_i, then we say that the system is in state a_i at time n or at the n^{th} step.

(2) The outcomes of any trial depend at most upon the outcomes of the immediately preceding trial and not upon any other previous outcomes with each pair of states (a_i, a_j) there is given the probability $\boldsymbol{p_{ij}}$ that a_j accurs immediately after a_i accurs. Such Stochastic process is called a (finite) Markov chains. The number $\boldsymbol{p_{ij}}$ called transition probabilities can be arranged in amatrix p.

P= $\begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{pmatrix}$ called transition matrix

Theorem :- the transition matrix p of markov chain is Stochastic matrix .

EX:- a man either drives his car or takes a train to work each day. Soppuse he never takes the train two days in arow, but if he drives to work then the next day he is just as likely to drive again as he is take the train.

The state space of the system is {(train)t,(drive)d}.
This Stochastic process is a markov's chain since the bout comes on any day depends only on what happened the preceding. The transition matrix of the Markov chains is:-

$$p=\begin{matrix} & t & d \\ t & 0 & 1 \\ d & \frac{1}{2} & \frac{1}{2} \end{matrix}$$

EX:- (Random walk with reflecting barriers) a man is at an integral point on the x-axis between the origin O and say the point 5.
He take a unite step to the right with probability p or to the left with probability q=1-p
Unless he is at the origin. He takes a step to the right or at the point 5 where he take a step to the right to 4.
Find the transition matrix of the markov chains ???
p=

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Higher transition probability

احتماليات الانتقال العليا

The entry $\boldsymbol{p_{ij}}$ in the transition matrix p of a markov chain is the probability that the system changes from the state $\boldsymbol{a_i}$ to the state $\boldsymbol{a_j}$ in one step: $\boldsymbol{a_i} \rightarrow \boldsymbol{a_j}$

Qua :- what is the probability denoted $\boldsymbol{p_{ij}^{(n)}}$ that the system changes from the state $\boldsymbol{a_i}$ to the state $\boldsymbol{a_j}$ in exactly n steps:

$$\boldsymbol{a_i} \longrightarrow \boldsymbol{a_{k1}} \longrightarrow \boldsymbol{a_{k2}} \longrightarrow \cdots \longrightarrow \boldsymbol{a_{k_{n-1}}} \longrightarrow \boldsymbol{a_j}$$

The $\boldsymbol{p_{ij}^{(n)}}$ are arranged in matrix $\boldsymbol{p^{(n)}}$ called the n-steps transition matrix..

Note :- now suppose that at some arbitrary time the probability that the system is in state a_i is p_i we denote these probability by the probability vector p=(p_1, p_2,….., p_n) which is called the probability distribution of the system at the time

In particular, we shall let $p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots\ldots, p_n^{(0)})$

Denoted the initial probability distribution

i.e. the distribution where the process begins and we shall let $p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots\ldots, p_n^{(n)})$ denote the $\boldsymbol{n}^{\,\boldsymbol{th}}$ step probability distribution i.e. the distribution after the first n steps.

Theorem :- let p be transition matrix of markov's chain process then the n-step transition matrix is equal to the $\boldsymbol{n}^{\,\boldsymbol{th}}$ power of p.

Proof:- suppose the system is in state $\boldsymbol{a_i}$ at say time k
we seek probability $\boldsymbol{p_{ij}^{(n)}}$ that the system is in state $\boldsymbol{a_j}$ at time k+n.

Now, the probability distribution of the system at time k, since the system is in state $\boldsymbol{a_i}$ is the vector $\boldsymbol{e_i}$ = (0,…,0,1,0,…,0) which has a 1 at the $\boldsymbol{i^{th}}$ position and zero every where else the probability distribution at time k+n is the product $\boldsymbol{e_i}\,\boldsymbol{p^n}$.

But $\boldsymbol{e_i}\,\boldsymbol{p^n}$ is the $\boldsymbol{i^{th}}$ row of the matrix $\boldsymbol{p^n}$ then $\boldsymbol{p_{ij}^{(n)}}$ is the $\boldsymbol{j^{th}}$ component at the $\boldsymbol{i^{th}}$ row of $\boldsymbol{p^n}$ and so $\boldsymbol{p^{(n)}} = \boldsymbol{p^n}$.

Theorem :- if p is transition matrix of the markov chain and $\boldsymbol{p^{(0)}}$ is the initial distribution of the system then $\boldsymbol{p^{(n)}}$=$\boldsymbol{p^n}\boldsymbol{p^{(0)}}$.

Proof:- let p=($\boldsymbol{p_{ij}}$) be the transition matrix of markov chain if p=($\boldsymbol{p_i}$) is the probability distribution of the system of some arbitrary time k. then pP is the probability distribution of the system one step later i.e. at time k+1 hence **P**$\boldsymbol{p^n}$ is the probability distribution of the system n-step later i.e. at time k+n .

In particular $p^{(1)}$ = $p^{(0)}$ p , $p^{(2)}$ = $p^{(1)}p$ …. And also $p^{(n)}$ =$p^{(0)}p^n$

Suppose the state space is {a_1, a_2,....., a_n} the probability that the system is in state $\boldsymbol{a_j}$ at time k and then in state $\boldsymbol{a_i}$ at time k+1 is the product $\boldsymbol{p_j\, p_{j\,1}}$.

thus the probability that the system is in state $\boldsymbol{a_i}$ at time k+1 is the sum $\boldsymbol{p_1\, p_{1j}} + \boldsymbol{p_2\, p_{2j}} + \ldots\ldots + \boldsymbol{p_m\, p_{mi}} = \sum_{j=1}^{m} \boldsymbol{p_j\, p_{ji}}$

Thus the probability distribution at time k+1 is

$$\boldsymbol{P^*} = \left(\sum_{j=1}^{m} \boldsymbol{p_j\, p_{j1}}\ , \ldots\ , \sum_{j=1}^{m} \boldsymbol{p_j\, p_{jm}}\right)$$

However the vector is precisely the product of the vector **p=(** $\boldsymbol{p_i}$ **)** by the matrix **(p=**$\boldsymbol{p_{ij}}$**) :** $\boldsymbol{P^* = pP}$

EX:- consider the markov chain (train and drive) whose transition matrix is

$$p = \begin{matrix} t \\ d \end{matrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\text{So } P^4 = \begin{matrix} t \\ d \end{matrix} \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix}$$

Thus the probability that the system changes from say state t to state d in exactly 4 steps is 5/8

i.e. P^4 td=5/8 , P^4 dt=5/16 , P^4 tt=3/8, P^4 dd=11/16 .

Now suppose that on the first day of work , the main tossed a fair die ande drove to work if $\boldsymbol{a_6}$ appeared what is the probability distribution on after 4-days ???

EX:- three boys A,B,and C are throwing a ball to each other A always throws the ball to C , but C is just likely to throw the ball to B as to A let x_n denote the n^{th} person to be thrown the ball the state space of the system is {A,B,C}

(1) Is this a markovchain ?
(2) Find the transition matrix ?
(3) Suppose C was the first person with the ball find the probability distribution after 1 step , 2 steps , 3 steps.

Sol:- 1-

This is markov chain since the person throwing the ball is not influenced by those who previously had the ball .

Sol:- 2-

The transition matrix of markov chain is

$$P= \begin{matrix} & A & B & C \\ A \\ B \\ C \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Sol:- 3-

$P^{(0)}$ = (0,0,1) ⟵ الابتدائي

$P^{(1)} = P^{(0)}\ P$

$$= (0,0,1) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (1/2\ ,\ 1/2\ ,\ 0)$$

$P^{(2)} = P^{(1)}\ P = P^{(0)}\ P^2$

$= = (1/2 , 1/2 , 0)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

= (0,1/2,1/2) after 2-steps

$P^{(3)} = P^{(0)} P^3 = P^{(2)} P$

$= (0,1/2,1/2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (1/4,1/4,1/2)$

After three steps (throws) the probability that A has the ball is 1/4 , that B has the ball is 1/4 and that C has the ball is 1/2 .

i.e. $P_A^{(3)}$= 1/4 , $P_B^{(3)}$= 1/4 , $P_C^{(3)}$= 1/2

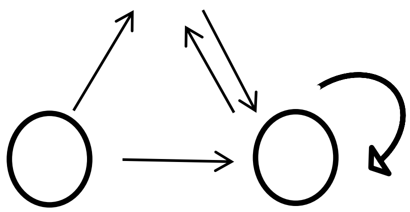
transition diagram

المخطط البياني للاحتمالات

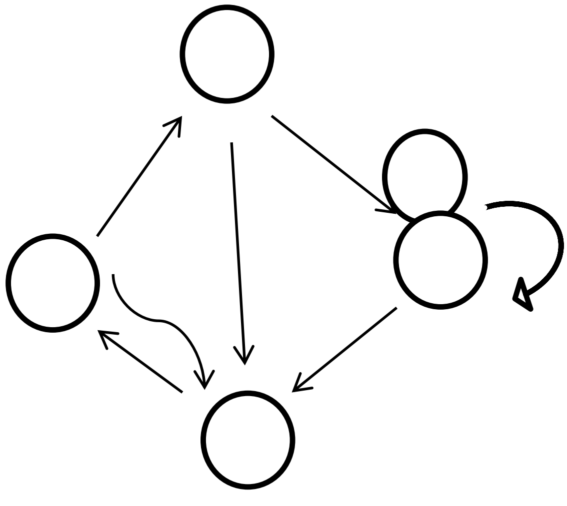
the transition probabilities of a markov chain can be represented by a diagram called a transition diagram where a positive probability $\boldsymbol{p_{ij}}$ is denoted by an arrow from the state $\boldsymbol{a_i}$ to the state $\boldsymbol{a_j}$.

EX:- find the transition diagram of the following transition matrix

$$P=\begin{matrix} & a_1 & a_2 & a_3 \\ a_1 & 0 & 0 & 1 \\ a_2 & \frac{1}{2} & 0 & \frac{1}{2} \\ a_3 & \frac{1}{2} & 0 & \frac{1}{2} \end{matrix}$$



EX:- find the transition matrix of the following transition diagram:



$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Classification of state
تصنيف الاوضاع

1. Absorbing state:- حالة الامتصاص (الاشباع)

A state a_i of a markov chain is called an Absorbing if the system remains in the state a_i once it enters there. Thus a state a_i is Absorbing if and only if the i^{th} row of the transition matrix **P** has a one on the main diagonal and zero every where else..

EX:- suppose the following matrix is a transition matrix of markov chain.

$$P= \begin{pmatrix} 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The state a_2 and a_5 are each Absorbing since each of the second and fifth rows has 1 on the main diagonal.

2. Accessibility :- حالة امكانية الوصول

The state **j** is accessible from **i** if from some integer **n ≥ 0** , then $P_{ij}^{(n)} \geq 0$ and we write i ⟶ j

3. Communication :- حالة المبادلة

The state i and j are Communicate if i ⟶ j and j ⟶ i , and we write i ⟷ j

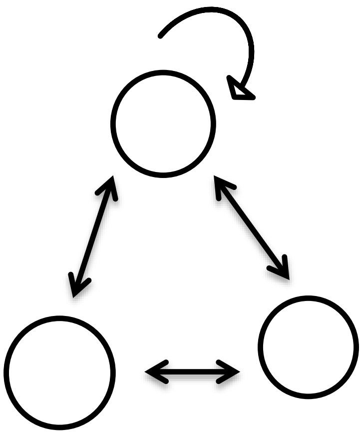
Note :- the relation ⟷ is an equivalence relation

- If i ⟷ i reflexive
- If i ⟷ j then j ⟷ i symmetric
- If i ⟷ j and j ⟷ k then i ⟷ k transitive

EX:- let x={x_n: n ≥0} is a markov chain with state space I={1,2,3} and transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.7 & 0 & 0.3 \\ 0.5 & 0.5 & 0 \end{pmatrix} \end{array}$$

Sol :-



From the diagram we can get

$1 \longleftrightarrow 2,\ 1 \longleftrightarrow 3,\ 2 \longleftrightarrow 3$

4. Recurrent and transient state :- حالة العودة والزوال

The state **j** is Recurrent if P_{jj}**= 1** , but if P_{jj}**< 1** , then state **j** is called transient state.

5. Periodic state :-

The Recurrent state j is called periodic if there exist integer S ≥ 2 such that the state j can occur only S,2S,3S…

i.e. S is high common factor of the values of **n ≥ 1** for which $P_{jj}^{(n)}$ **≥ 0**

6. Closed set :-

Let B set of state . then B is said to be closed set if $P_{ij}^{(n)}$ **= 0** ,

n ≥ 0 , i B and **j** B

7. Irreducible closed set :- المجموعة المغلقة عديمة الاختزال

The closed set B is called Irreducible if the all states in the set is Communicate

Note :- the markov chain is called Irreducible if the set of all state is unique closed set for this state

8. Null state :- الحالة الصفرية

A recurrent state j is called null state if

E j (T_i)=∞

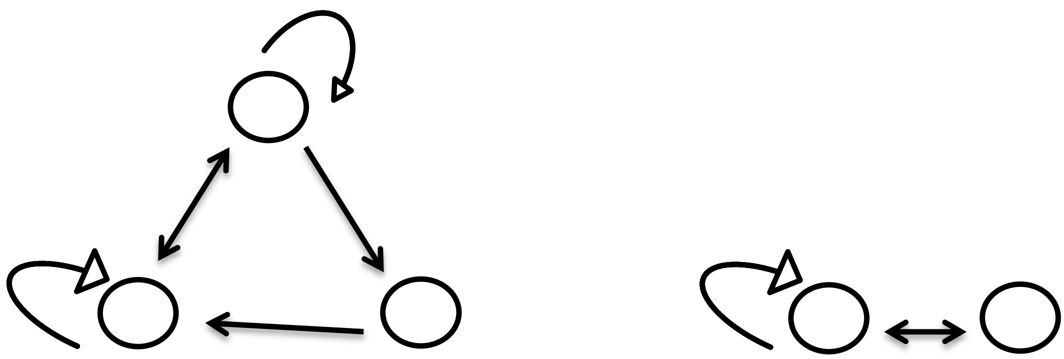
= M_i = mean recurrence of time

EX:- let the following transition matrix

$$P=\begin{bmatrix} 0.1 & 0.6 & 0.3 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

find the closed set and Irreducible closed set

sol :-



from the above diagram there is three closed set as following :

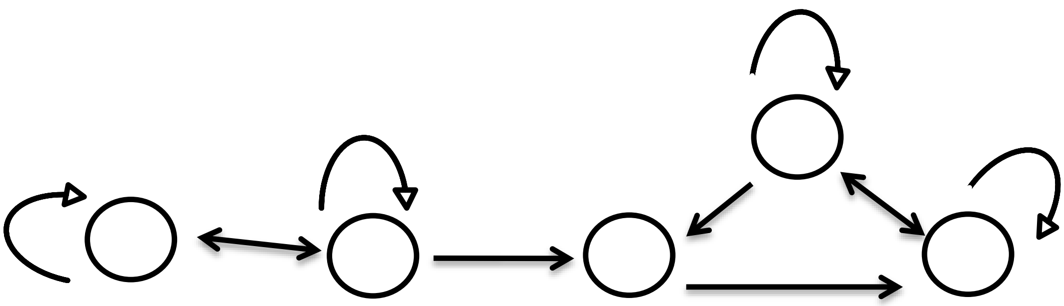
B1={1,2,3} B2={4,5} B3={1,2,3,4,5} and the Irreducible sets are :-

B1={1,2,3} B2={4,5}

EX :- let the markov chain with state space {a,b,c,d,e} and transition matrix

$$\begin{bmatrix} 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{bmatrix}$$

find the closed set



From the above diagram , get $\langle B1 = \{a, c, e\}\ \ B2 = \{a, b, c, d, e\}\rangle$ are closed sets.

The markov chain is not Irreducible because we can remove the second and the fourth rows and columns, we get

Q= $\begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

where Q is the transition matrix of markov process restricted by closed set {a,c,e}

Theorem 1 :- If j is recurrent and j $\longrightarrow$ k then k $\longrightarrow$ j and f(k,j)= 1

Proof :- let ɣ > 0 is probability that j $\longrightarrow$ k with out return to state j and

1-f(k,j) is the probability that the state j impossible occurrence if the chain start in state k , 1- f(j,j) ≥ ɣ(1-f(k,j)) ≥ 0

.: j is recurrent state by assumption

:. ɣ(1-f(k,j)) = 0 ,, → f(k,j) = 1 :. K $\longrightarrow$ j

Theorem :- let j is recurrent state and j → k then k is also recurrent state proof :- j → k by assumption

:. ∃ integer L p^L(j,k) > 0

By using theorem 1 get k→j

:. ∃ integer S p^s(j,k) > 0

{X_s=j , X_{s+n}=j , X_{s+n+L}= k } {X_{s+n+L}= k}

Then

p^{s+n+L} (k,k)= p_k{X_{s+n+L}= k} ≥ p_k{X_s=j , X_{s+n}=j , X_{s+n+L}= k }

= p^s(k,j) p^n(j,j) p^L(j,k)

:. R(k,k) = $E_k(N_k)$=$\sum_{m=0}^{\infty} p^m(\mathrm{k,k})$ ≥ $\sum_{m=L+s}^{\infty} p^m(\mathrm{k,k})$

=$\sum_{n=0}^{\infty} p^{k+s+L}(\mathrm{k,k})$ ≥ $p^s(\mathrm{k,j})\ p^L(\mathrm{j,k})\ \sum_{n=0}^{\infty} p^K(\mathrm{j,j})$

= $p^s(\mathrm{k,j})\ p^L(\mathrm{j,K})$ R(j,j)

:. j is recurrent state, then R(j,j)=∞

:. $p^s(\mathrm{k,j})\ p^L(\mathrm{j,k})$ > 0

:.R(k,k)=∞

:. K is recurrent state.

Theorem :- let x is irreducible markov chain then all the state of the markov chain are transient states or are null states or not null states as well as it is periodic with length S or not periodic

Proof:-

:. M.C is irreducible then for two states j and k are j → k and k → j that is mean ∃ integer L and S $p^{s}(\mathrm{k,j}) > 0$ and $p^{L}(\mathrm{j,k}) > 0$

Let B= $p^{s}(\mathrm{k,j})\, p^{L}(\mathrm{k,j})$

1- If the state j is recurrent state then k is is recurrent (by theorem) and if j is transient state then k is transient state and if k is recurrent then j is recurrent
2- Let j be null recurrent then p^{m}(j,j) → 0 when m → 0 because p^{n+s+L}(j,j) ≥ B p^{n}(k,k) then p^{n}(k,k) → 0 when n → ∞

:. K is recurrent then k must be null let j is periodic state of length S

:. p^{s+L}(j,j) ≥ $p^{s}(\mathrm{k,j})\, p^{L}(\mathrm{j,k})$= B > 0

Then from def. of periodic state (L+s) must be multiple of integer S.

And from this if n not multiple of S then n+L+s is also not multiple on integer S and p^{n+s+L}(j,j) = 0 then p^{n}(k,k) = 0

:. K must be periodic of length $\acute{S}$ ≥ S and if k is periodic with length $\acute{S}$ then the state j must be periodic with ($\acute{S}$ ≥ S) , then S = $\acute{S}$ and the state k and j have same length.

probability distribution of T_1:-

let f_n (j,j)= $p_i(T_1 = n)$, I I n=1,2,… where

f_1 (i,j)=p(I,j)= p_i (x1=j)…….(1

→ f_n (i,j)= p_i {x1ǂ j, x2ǂj , …,x_{n-1} ǂj , xn=j}

$=\sum_{c\neq j}^{i} p_i \{x1 = c, x2\neq j, \dots, x_{n-1} \neq j, xn = j\}$

$=\sum_{c\neq j}^{i} p_i \{x1 = c\} p_i \{x2\neq j, \dots, x_{n-1} \neq j, xn = j \mid x1 = c\}$

$=\sum_{c\neq j}^{i} p_i \{x1 = c\} p_c \{x1\neq j, \dots, x_{n-1} = j\}$

$=\sum_{c\neq j}^{i} p(i,c).f_{n-1}$(c,j)…….(2

Now let f(I,j)=p_1 (T_1<∞)=$\sum_{n=1}^{\infty} f_n$(i,j)

By using (1) and (2) get

f(i,j)=p(i,j) + $\sum_{n=1}^{\infty} \sum_{c\neq j} p(i,c)\ f_n$ (c,j)

=p(i,j)+ $\sum p(i,c) f(c,j), i\varepsilon I \dots . (3$

And f(i,j) can be written as

f(i,j)= $p_i \{xn = j, \text{for some n} \geq 1\}$

by using matrices if $f_k(f_k(i,j))$ is arow vector for element f_k(i,j) I I and

f=(f(i,j)) is row vector for elements f(i,j) i I then

f_1=(p(i,j)) is a column represent state j in transition matrix p and

f_k=f_{k-1} **G** when k ≥ 2

where **G** is the transpose of matrix p after change the elements of column of state j to zero .

EX:- let X={X_n , n ≥ 0} is markov chain with state space I={1,2,3}

and transition matrix p= $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$ and j=3. Find

1- Find The probability of the first recurrent to state 3

2- Find The probability not recurrent of state 3

The number of appearance time:- عدد مرات الضهور

Let N_i (w) represent the number of appearance times sor state j in chain

x_0 (w), x_1 (w),….. and if $I_j(X_n)$ $\begin{cases} 1 & , \; Xn = j \\ 0 & , \; Xn \neq j \end{cases}$

Then N_j(w)= $\sum_{n=0}^{\infty} \text{Ij(Xn)}$

:. E_j {I_j(w)} = p_i $\{\text{xn} = \text{j}\}$= p^n (i,j)

→ E_j {N_j(w)}=$\sum_{n=0}^{\infty} p^n$ (i, j)……(*

If we assume R(i,j) = E_j {N_j(w)} = $\sum_{n=0}^{\infty} p^n$ (i, j) …..(**

Then the matrix R which its elements in row i and column j

Called basic matrix and using (**) get R=I+P+P^2+…

Note that RP=PR=R-I

:. R(I-P) =(I-P)R =I

EX:- let S_n represent the number of successes in n of independent Bernoulli trials where the probability of success is P

1- Is S_n markov chain ?
2- If yes find transition matrix

Sol :- (1)

:. $P\{S_{n+1}=k \mid s_0,s_1,s_2,s_2,....\}$

$= P\{S_{n+1}=k \mid S_n\}$

:. $\{S_n , n \geq 0\}$ is markov chain

(2)

$p_0^0 = \{S_0 = 0\} = 1$, $p_0^0 = 0$, $j \geq 1$

$P(i,j)=p\{ S_{n+1}=j \mid S_n = i\}$

$= p\{ i+S_{n+1} - S_n = j \mid S_n = i\}$

$= p\{ S_{n+1} - S_n = j-i \mid S_n = i\}$

$= p\{ S_{n+1} - S_n = j-i \}$

$=p\{S_1 = j-i\}=p$, if $j-i=1$

i.e. $p\{S_1 = 1\}=p$

And $p=\{S_1 = j-i\}=q$ if $j=i$

i.e. $p\{S_1 = 0\}=q$, note $p+q=1$

$$p=\begin{bmatrix} q & p & 0 & \dots & \dots & 0 \\ 0 & q & p & 0 & .. & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

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