Some Special Distribution:

Discrete Distribution:

1) Discrete uniform distribution:

The r.v. X is said to have an uniform distribution if P.m.f. is give by

$$f(X,N) = \begin{cases} \frac{1}{N} & ;\ x = 1,2,3,\dots,N \\ 0 & ; o.w. \end{cases} \quad ; Where\ the\ parameter\ N \geq 1\ Natural\ numbere$$

And denoted by $X \sim D_u(N)$

Clear that

$$f(x) \geq 0 \quad and \quad \sum_{x=1}^{N} f(x) = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x.

$$Note:\ the\ mean\ \mu = \frac{N+1}{2} \quad and\ the\ variance\ \sigma^2 = \frac{N^2-1}{12} \quad (proof)$$

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx} \frac{1}{N} = \frac{1}{N} \sum_{\forall x} e^{tx}$$

$$= \frac{1}{N} [e^t + e^{2t} + e^{3t} + \dots + e^{Nt}]$$

$$= \frac{e^t}{N} \left[1 + e^t + e^{2t} + \dots + e^{(N-1)t} \right]$$

$$= \frac{e^t}{N} \left[\frac{1 - e^{Nt}}{1 - e^t} \right]$$

Example : If $X\sim D_u(6)$; find ❶ P(x > 1) ❷ P(x ≥ 3) ❸ mean and variance

Solution :

$$X\sim D_u(6) \implies f(x) = \begin{cases} \frac{1}{6} & ; x = 1,2,3,\dots,6 \\ 0 & ; o.w. \end{cases}$$

❶ P(x > 1) =1 – P(x ≤1)= 1- P(x=1)= 1 - 1/6 =5/6

❷ P(x ≥ 3) = P(x=3) + P(x=4)+ P(x=5)+ P(x=6) =1/6 + 1/6 +1/6 +1/6 =4/6

Or = 1- P(x<3) = 1- [P(x=1)+p(x=2)] = 1- [1/6 + 1/6]=1-2/6 =4/6

❸ Mean $\mu = \frac{N+1}{2} = \frac{6+1}{2} = \frac{7}{2}$

Variance $\sigma^2 = \frac{N^2-1}{12} = \frac{36-1}{12} = \frac{35}{12}$

2) Bernoulli distribution:

The r.v. X is said to have a Bernoulli distribution with parameter p if P.m.f. is give by

$$f(X,p) = \begin{cases} p^x\, q^{1-x} & ; x = 0,1 \\ 0 & ; o.w. \end{cases} \quad ; Where\ the\ parameter\ 0 \le p \le 1\ , q = 1-p$$

And denoted by $X\sim Ber(p)$

Clear that

$$f(x) \ge 0 \quad and \quad \sum_{x=0}^{1} f(x) = p + q = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

$Note: \ the\ mean\ \mu = p \quad and \quad the\ variance\ \sigma^2 = p\,q \quad (proof)$

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x)$$

$$= \sum_{\forall x} e^{tx}\ p^x\, q^{1-x} = \sum_{x=0}^{1} e^{tx}\ p^x\, q^{1-x}$$

$$= e^{0\,t}\ p^0\, q^{1-0} + e^{1\,x}\, p^1\, q^{1-1} = q + p\, e^t$$

3) Binomial distribution:

The r.v. X is said to have a Binomial distribution with two parameter n and p if

P.m.f. is give by

$$f(X,n,p) = \begin{cases} C_x^n\, p^x\, q^{n-x} & ;\ x = 0,1,2,\dots,n \\ 0 & ; o.w. \end{cases}$$

And denoted by $X \sim b(n,p)$

Where the two parameter n and p satisfy the following conditions:

1- n is a positive integer

2- $0 \le p \le 1$

3- $q = 1 - p$ or $p + q = 1$

Under the conditions, it is clear that

$$f(x) \ge 0\ \ \forall x\ \ and \quad \sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} C_x^n\, p^x\, q^{n-x} = (p+q)^n = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

Useful relationship: Binomial Formula is $\sum_{x=0}^{n} C_x^n\, a^x b^{n-x} = (a+b)^n$.

$Note:\ the\ mean\ \mu = n\,p \quad and \quad the\ variance\ \sigma^2 = n\,p\,q$

$Proof:$

$$\mu = E(x) = \sum_{\forall x} x\, f(x) = \sum_{x=0}^{n} x.\, C_x^n\ p^x\, q^{n-x}$$

$$\because\ C_x^n = \frac{n!}{x!\,(n-x)!}$$

$$\therefore \mu = \sum_{x=0}^{n} x.\, C_x^n\ p^x\, q^{n-x} = \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!}\ p^x\, q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!}\ p^x\, q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!}\ p^x\, q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!}\ p^{x-1}\, q^{n-x}$$

$let\ \ y = x-1 \quad and \quad m = n-1$

$$\Longmapsto \mu = np \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!}\ p^y\, q^{m-y}$$

$$\therefore \mu = np \sum_{y=0}^{m} C_y^m\ p^y\, q^{m-y} = np(p+q)^m = np\ 1^m = np$$

$$\therefore\ \ \mu = np$$

$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\, f(x) = \sum_{x=0}^{n} x(x-1) C_x^n\; p^x\, q^{n-x}$$

$$\therefore E[x(x-1)] = \sum_{x=2}^{n} x(x-1) \frac{n!}{x!\,(n-x)!}\; p^x\, q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!}\; p^x\, q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\; p^{x-2}\, q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} C_{x-2}^{n-2}\; p^{x-2}\, q^{n-x}$$

$let \quad y = x-2 \quad and \quad m = n-2$

$$= n(n-1)p^2 \sum_{y=0}^{m} C_y^m\; p^y\, q^{m-y} = 1$$

$$E[x(x-1)] = E(x^2) - E(x) = n(n-1)p^2$$

$$\Longrightarrow E(x^2) = np + n^2p^2 - np^2$$

$$\therefore \quad \sigma^2 = E(x^2) - E^2(x) = np + n^2p^2 - np^2 - n^2p^2$$

$$= np - np^2 = np(1-p) = npq$$

$$\therefore \quad \sigma^2 = npq$$

The moment generating function of Binomial distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx}\, f(x) = \sum_{\forall x} e^{tx}\; C_x^n\; p^x\, q^{n-x} = \sum_{x=0}^{n} C_x^n\, e^{tx}\; p^x\, q^{n-x}$$

$$= \sum_{x=0}^{n} C_x^n\, (e^t p)^x\, q^{n-x} = (q + p\, e^t)^n$$

The properties of binomial distribution is:

1- two possible outcome
2- n trials
3- independent trials

Example: Suppose X is binomially distributed with parameter n and p, further suppose

E(X) = 5 and var(X) = 4, find n and p.

Solution: $X \sim b(n,p) \Longmapsto E(x) = np = 5 \quad and \quad var(x) = npq = 4$

$$\Longmapsto q = \frac{4}{5} \Longmapsto p = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\because \quad np = 5 \quad \Longmapsto n = \frac{5}{\frac{1}{5}} = 25$$

$$\therefore \quad X \sim b\left(25, \frac{1}{5}\right)$$

4) Poisson distribution:

The r.v. X is said to have a Poisson distribution with parameter λ if P.m.f. is give by

$$f(X,\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & ; x = 0,1,2,\dots,\infty \\ 0 & ; o.w. \end{cases} \quad , where\ \lambda > 0$$

And denoted by $X \sim Po(\lambda)$

$$Since \ >0 \Longmapsto f(x) \geq 0 \ ; \forall x$$

$$Since \quad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \quad \Longmapsto \sum_{\forall x} f(x) = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

$Note:\ the\ Mean\ \mu = \lambda \quad and \quad the\ Variance\ \sigma^2 = \lambda$

$Proof:$

$$Mean = E(X) = \sum_{x=0}^{\infty} x\, f(x;\lambda) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda \lambda^{x-1}}{x(x-1)!}$$

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \ldots\ldots\ldots + \infty \right]$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda$$

$\therefore$ Mean of the Poisson distribution is λ

And

Variance = Var(X) = $E(X^2) - [E(X)]^2$

$\because$ Mean of the Poisson distribution is λ From (1)

Var(X) = $E(X^2) - [\lambda]^2 \ldots\ldots\ldots (1)$

$$E(X^2) = \sum_{x=0}^{\infty} x^2\, f(x;\lambda) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

= E(x)=λ

$$= e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^2 \lambda^{x-2}}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\dots\dots + \infty \right] + \lambda$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\Rightarrow \quad E(X^2) = \lambda^2 + \lambda \cdots\cdots\cdots\cdots (2)$$

Putting (2) in (1) we get

$$\text{Var(X)} = E(X^2) - [\lambda]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$\therefore$ Variance of the Poisson distribution is λ

Note: In Poisson distribution the mean and variance are equal i.e. λ

The properties of Poisson distribution is:

1- The event occurring randomly in time.
2- The number of event in non-overlapping time period are independent.
3- $\lambda > 0$, rate of Poisson process is average number of events per unite time.

Example :Suppose the number of flaws in a 100–foot roll of paper is a Poisson random variable with λ = 10. Then the probability that there are eight flaws in a 100-foot roll is:

Solution:

$$P(X=8 \mid \lambda = 10) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-10}(10)^8}{8!} = \frac{(2.71828)^{-10}(100{,}000{,}000)}{40{,}320} = .1126$$

The probability of seven flaws in a 50-foot roll is:

$$P(X=7 \mid \lambda = 5) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}(5)^7}{7!} = \frac{(2.71828)^{-5}(78{,}125)}{5{,}040} = .1044$$

Note:

The Poisson distribution can be limiting case of a binomial distribution under certain conditions.

1. Number of trails i.e. n is indefinitely large i.e. $n \to \infty$
2. p, the probability of success in each trail is indefinitely small i.e $p \to 0$
3. $np = \lambda$ is finite.

Proof:(للاطلاع)

If X is a binomial distribution then the probability mass function is given by

$$P(X=x) = C_x^n \, p^x q^{n-x}, \; x = 0,1,2,\ldots\ldots n$$

Under the above conditions

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty} C_x^n p^x q^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \qquad [\because np = \lambda]$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots\ldots(n-x+1)(n-x)!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty} \frac{n^x\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\left(1-\frac{x-1}{n}\right)\right]}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty} 1 \,.\, \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x} = \frac{\lambda^x}{x!}\frac{e^{-\lambda}}{1^x} = \frac{e^{-\lambda}\lambda^x}{x!}$$

Example : If n=5000 , p=0.001 and $X\sim b(\mathrm{n},\mathrm{p})$ find *P(x=7)=f(7)*

Solution : $X\sim b(\mathrm{n},\mathrm{p}) = X\sim Po(\lambda)$ λ= np=5000* 0.001=5

$$P(X=7|\lambda=5)=\frac{e^{-\lambda}\lambda^{x}}{x!}=\frac{e^{-5}(5)^{7}}{7!}=\frac{(2.71828)^{-5}(78,125)}{5,040}=.1044$$

$\therefore$ *f(7)=0.1044*

The moment generating function of Binomial distribution is given by

$$M_x(t)=e^{\lambda(e^t-1)}$$

Proof :

$$M_x(t)=E(e^{tx})=\sum_{\forall x}e^{tx}f(x)=\sum_{x=0}^{\infty}e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}=e^{-\lambda}e^{\lambda e^t}=e^{\lambda(e^t-1)}$$

Example :Let X be r.v. whose M.g.f is give by e^{2e^t-2} $find\ P(X\geq 1)$

Solution : $X\sim Po(2)$

$$\therefore f(X;\lambda)=\begin{cases}\frac{e^{-2}\,2^x}{x!} & ;\ x=0,1,2,\dots,\infty\\ 0 & ;o.w.\end{cases}$$

$$\therefore P(X\geq 1)=1-P(X<1)$$

$$=1-P(X=0)$$

$$=1-\frac{e^{-2}\,2^0}{0!}=1-e^{-2}=0.864$$

Question: If x has Poisson distribution and P(X=0)=1/2 find E(x) ?

Solution :

$$\because\ X\sim Po(\lambda)$$

$$\therefore f(X,\lambda)=\begin{cases}\frac{e^{-\lambda}\,\lambda^x}{x!} & ;\ x=0,1,2,\dots,\infty\\ 0 & ;o.w.\end{cases}$$

[H. W.]

5) Negative Binomial distribution

Perform independent Bernoulli trials repeatedly until a given number of success are observed i.e. let x the total number of failure before the r^{th} success, that means the r.v. X represent the number of failure Prior to the r^{th} success the P.m.f of X is:

$$f(X;r,p) = \begin{cases} C_x^{x+r-1}\, p^r\, q^x & ;\ x = 0,1,2,\ldots\ldots\ldots \\ 0 & ;o.w. \end{cases}$$

And denoted by $X \sim Nb(r,p)$

Where the parameter r and p satisfy r=1, 2, 3, ……… and $0 \le p \le 1$ and $q = 1-p$

Since $0 \le p \le 1$

$$\therefore f(x) \ge 0\ \forall x\ \ and\ \sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} C_x^{x+r-1}\, p^r\, q^x = p^r (1-q)^{-r} = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

Useful relationship: $\boxed{\sum_{x=0}^{\infty} C_x^{n+x-1}\, a^x = (1-a)^{-n}}$.

$$Note:\ the\ mean\ \mu = \frac{rq}{p} \quad and \quad the\ variance\ \sigma^2 = \frac{rq}{p^2}$$

$Proof:$

$$\mu = E(x) = \sum_{\forall x} x\, f(x) = \sum_{x=0}^{\infty} x.\, C_x^{x+r-1}\, p^r\, q^x$$

$$\because\ C_x^n = \frac{n!}{x!\,(n-x)!}$$

$$\therefore \mu = \sum_{x=1}^{\infty} x.\, C_x^{x+r-1}\, p^r\, q^x = \sum_{x=1}^{\infty} x\frac{(x+r-1)!}{x!\,(r-1)!}\, p^r\, q^x$$

$$= p^r \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(x-1)!\,(r-1)!}\, q^x$$

$$= r\, q\, p^r \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(x-1)!\, r!}\, q^{x-1}$$

$let \quad y = x-1 \quad and \quad s = r+1$

$$= r\, q\, p^r \sum_{y=0}^{\infty} \frac{(y+s-1)!}{y!\,(s-1)!}\, q^y$$

$$\Longrightarrow \mu = \frac{r\, q\, p^r}{(1-q)^s} = \frac{r\, q\, p^r}{p^{r+1}} = \frac{rq}{p}$$

$$\therefore \quad \mu = E(x) = \frac{rq}{p}$$

$$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\, f(x) = \sum_{x=0}^{\infty} x(x-1)\, C_x^{x+r-1}\, p^r\, q^x$$

$$\therefore E[x(x-1)] = \sum_{x=2}^{\infty} x(x-1) \frac{(x+r-1)!}{x!\,(r-1)!}\; p^r\, q^x$$

$$= q^2\, p^r\, r(r+1) \sum_{x=2}^{\infty} \frac{(x+r-1)!}{(x-2)!\,(r+1)!}\, q^{x-2}$$

$let \quad y = x-2 \quad and \quad s = r+2$

$$= q^2\, p^r\, r(r+1) \sum_{y=0}^{\infty} \frac{(y+s-1)!}{y!\,(s-1)!}\, q^y$$

$$\Longrightarrow E(x^2) - E(x) = \frac{q^2\, p^r\, r(r+1)}{(1-q)^s} = \frac{q^2\, p^r\, r(r+1)}{p^{r+2}} = \frac{r(r+1)q^2}{p^2}$$

$$\Longrightarrow E(x^2) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p}$$

$$\therefore \quad \sigma^2 = E(x^2) - E^2(x) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{rq^2}{p^2} + \frac{rq}{p} = \frac{rq(q+p)}{p^2} = \frac{rq}{p^2}$$

$$\therefore \quad \sigma^2 = \frac{rq}{p^2}$$

The moment generating function of **Negative Binomial** distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx} \; C_x^{x+r-1} \, p^r \; q^x$$

$$= p^r \sum_{x=0}^{\infty} C_x^{x+r-1} \, (e^t q)^x \quad = (\frac{p}{1-q\,e^t})^r$$

Other formula of Negative Binomial distribution given by

$$f(X;r,p) = \begin{cases} C_{r-1}^{x+r-1} \, p^r \, q^x & ; \; x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

Where X denote the number of failures before we get the first r successes

Remark: If X denote the number of trials required to get a total of r successes then

$$f(X;r,p) = \begin{cases} C_{r-1}^{x-1} \, p^r \, q^{x-r} & ; \; x = r, r+1, r+2, \dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

6) Geometric Distribution

In case of the binomial distribution, the number of trials was predetermined. Sometimes, however, we wish to know the number of trials needed before a certain outcome occurs.

For example, we wish to play until we win; you roll dice until you get an 11; a mechanic waits for the first plane to arrive at the airport that needs repair; a basketball player shoots until he makes it. These situations fall under the ***Geometric distribution.*** **(Special case of Negative Binomial at r=1)**

The r.v. X is said to have a ***Geometric distribution*** with parameter p if P.m.f. is given by

$$f(X;p) = \begin{cases} p\ q^x \ ; \ x = 0,1,2,\dots\dots\dots \\ \\ 0 \quad ; o.w. \end{cases} \qquad where\ 0 \le p \le 1; \ \ q = 1 - p$$

And denoted by $X \sim G(p)$

Since $0 \le p \le 1$

$$\therefore f(x) \ge 0 \ \ \forall x \ \ and \ \sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} p\ q^x = \frac{p}{(1-q)} = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

The moment generating function of Geometric distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx}\ p\ \ q^x$$

$$= p \sum_{x=0}^{\infty} (e^t q)^x = \frac{p}{1 - q\, e^t}$$

$$Note: \ the\ mean\ \mu = \frac{q}{p} \quad and \quad the\ variance\ \sigma^2 = \frac{q}{p^2}$$

$Proof:$

$$M_x(t) = \frac{p}{1 - q\, e^t} \Longmapsto M'_x(t) = \frac{p\, q\ e^t}{(1 - q\, e^t)^2} \Longmapsto E(x) = M'_x(0) = \frac{p\, q\ e^0}{(1 - q\, e^0)^2} = \frac{q}{p}$$

$$\therefore \quad \mu = E(x) = \frac{q}{p}$$

$$M''_x(t) = \frac{(1 - q\, e^t)^2 (p\, q\ e^t) + 2q((1 - q\, e^t)(p\, q\ e^t)}{(1 - q\, e^t)^4} \Longmapsto E(x^2) = M''_x(0)$$

$$\therefore\ E(x^2) = \frac{(1-q\,e^0)^2(p\,q\ \ e^0) + 2q((1-q\,e^0)(p\,q\ \ e^0)}{(1-q\,e^0)^4}$$

$$\therefore\ E(x^2) = \frac{(1-q\,)^2(p\,q\ \,) + 2q((1-q)(p\,q\,)}{(1-q\,)^4} = \frac{qp^3 + 2q^2p^2}{p^4} = \frac{qp\ + 2q^2}{p^2}$$

$$\sigma^2 = var(x) = E(x^2) - E^2(x) \Longrightarrow \sigma^2 = \frac{qp\ + 2q^2}{p^2} - \frac{q^2}{p^2} = \frac{q(p\ + q\,)}{p^2} = \frac{q}{p^2}$$

$$\therefore\quad \sigma^2 = \frac{q}{p^2}$$

The properties Geometric distribution

1. Each event falls into just one of two categories, which are generally referred to as a “success” or “failure.”
2. The probability of success, call it *p*, is the same for each observation.
3. The observations are all independent.
4. The variable of interest is the number of trials required to obtain the **first** success.

Remark: If X denote the number of trials required to get a first success then the P.m.f is:

$$f(X;p) = \begin{cases} p\ \ q^{x-1} & ;\ x = 1,2,3,\dots\dots\dots \\ 0 & ;o.w. \end{cases}$$

$\textbf{Note}:$ *By this case the mean* $\mu = \dfrac{1}{p}$ *and the variance* $\sigma^2 = \dfrac{1}{p^2}$

Proof (H.W.)

Example: On island of Oahu in the small village of Nanakuli, about 80% of the residents are of Hawaiian ancestry . Suppose you fly to Hawaii and visit Nanakuli.
What is the probability that the first villager you meet is Hawaiian?
What is the probability that you do not meet a Hawaiian until the third villager?

(على جزيرةِ أوهايو في القريةِ الصغيرةِ في نانا كولي، حوالي 80 % من السكّانِ مِنْ اصل هاواي. افترض انك سافرت إلى هاواي وزُرتُ نانا كولي. ما هو احتمال ان القرويِّ الأولِ الذي تلتقيه هاوايُ؟ ما هو احتمال بأنّك لا تلتقي باي هاواي حتى القرويِّ الثالثِ؟)

Solution :

$$f(X;p) = \begin{cases} p\ \ q^{x-1} & ;\ x = 1,2,3,\dots\dots\dots \\ 0 & ;o.w. \end{cases}$$

$p = 0.80 = .8 \quad ; \quad q = 1 - p = 1 - .8 = .2$

$f(1) = P(X = 1) = (1 - .8)^{1-1}(.80) = (.2)^0(.80) = .80$.

$f(3) = P(X = 3) = (.2)^2(.80) = .032$

What is the probability that you will meet at most three people from Hawaiian ancestry **(اصل)**?

$P(X \le 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$= (.2)^0(.8) + (.2)^1(.8) + (.2)^2(.8)$

$= .8 + .16 + .032 = .992$

How many people should you expect to meet before you meet the first Hawaiian?

$\mu = \dfrac{1}{.80} = 1.25$ ⟾ First integer number $\ge \mu$ ⟾ **Answer= 2**

Note : If X is a geometric random variable with probability *p* of success on each trial, the expected number of trials necessary to reach the first success is $\mu = E(x) = \dfrac{1}{P}$.

What is the probability that it takes more than three people before you meet a Hawaiian?

$P(X > 3) = (1 - .8)^4 = .0016$

Note, The probability that it takes more than *n* trials before we see the first success is $P(X > n) = (1 - p)^{n+1}$ and $P(X \ge n) = (1 - p)^n$. (Proof that H.W.)

Example: Suppose we flip a fair coin until we get a head. Let X be the number of trail to get a head. Find the probability mass function of X.

Solution:

$X \sim G(p) \quad p = 0.5 \quad and \quad q = 1 - p = 0.5$

$$P.m.f. \;\; is \quad f(X) = \begin{cases} (0.5)\ (0.5)^{x-1} = (0.5)^x & ;\ x = 1, 2, \ldots \ldots \\ 0 & ; o.w. \end{cases}$$

Question: A fair die cast on successive independent trail until the second six is observed.

Find the probability of observing exactly ten non-sixes before the second six is appear

Answer:

$$r = 2 \quad , \; p = \frac{1}{6} \qquad , \quad q = \frac{5}{6}$$

$$\therefore X \sim Nb(r,p) \;\Rrightarrow\; f(X;r,p) = \begin{cases} C_{r-1}^{x+r-1}\, p^r\, q^x & ;\; x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow\; f(x) = f\left(X; 2, \frac{1}{6}\right) = \begin{cases} C_1^{x+1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^x & ;\; x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow f(10) = P(x = 10) = C_1^{11} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10}$$

$$= 10 \times (0.0278) \times (0.1615) = 0.0449$$

Or

$$f(X;r,p) = \begin{cases} C_x^{x+r-1}\, p^r\, q^x & ;\; x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow\; f(x) = f\left(X; 2, \frac{1}{6}\right) = \begin{cases} C_x^{x+1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^x & ;\; x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow f(10) = P(x = 10) = C_{10}^{11} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10}$$

$$= 10 \times (0.0278) \times (0.1615) = 0.0449$$

Theorem: (Memory Loss Property)

If X has the geometric distribution with parameter *p* then

$$P(X \geq i + j \mid X \geq i) = P(X \geq j) \quad ; \quad i\,, j = 1, 2, 3, \dots\dots\dots$$

Proof:

$$P(X \geq i + j \mid X \geq i) = \frac{P(X \geq i + j)}{P(X \geq i)} = \frac{(1-p)^{i+j}}{(1-p)^i} = (1-p)^j = P(X \geq j)$$

7) The hyper-geometric distribution :

Suppose that we have population of N objects; D of one type and N-D a second type. A random sample of size n is drawn from the population without replacement. Then if we let x denote the number of objects of the first type selected, we get first of all $\binom{D}{x}$ ways of choosing x object of the first type, $\binom{N-D}{n-x}$ ways of choosing $n-x$ object of the second type, $\binom{N}{n}$ ways of choosing a sample of size n from the population of N objects then the r.v. X has P.m.f is given by:

$$f(x)=f(x;N,D,n)=\begin{cases}\dfrac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} & ;x=a,a+1,a+2,\dots,b\\ 0 & ;otherwise\end{cases}$$

where ❶ $a=\max\{0\,,n+D-N\}$ *and* $b=\min\{n,D\}$

❷ $0\le x\le D$ ❸ $0\le n-x\le N-D$

And denoted by $X\sim hyp(N,D,n)$

clear that $f(x)\ge 0\ \ \forall x$

$$and\quad \sum_{x=0}^{n} f(x)=\sum_{x=0}^{n}\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}=\frac{1}{\binom{N}{n}}\sum_{x=0}^{n}\binom{D}{x}\binom{N-D}{n-x}=\frac{\binom{N}{n}}{\binom{N}{n}}=1$$

Useful relationship: $\sum_{k=0}^{m}\binom{A}{k}\binom{B}{m-k}=\binom{A+B}{m}$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

Theorem: $X\sim hyp(N,D,n)$ *then* $\mu=n\frac{D}{N}$ *and* $\sigma^2=n\frac{D}{N}\,\frac{N-n}{N-1}\,\frac{N-D}{N}$

Proof :

$$\mu=E(x)=\sum_{\forall x}x\,f(x)=\sum_{x=0}^{n}x.\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$\because \ C_x^m = \frac{m!}{x!\,(m-x)!}$$

$$\therefore \mu = \sum_{x=1}^{n} x.\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}}\sum_{x=1}^{n} x\frac{D!}{x!\,(D-x)!}\binom{N-D}{n-x}$$

$$= \frac{D}{\binom{N}{n}}\sum_{x=1}^{n}\frac{(D-1)!}{(x-1)!\,(D-x)!}\binom{N-D}{n-x}$$

$let \quad y = x-1 \ , \ \ m = n-1 \quad and \quad A = D-1$

$$= \frac{D}{\binom{N}{n}}\sum_{y=0}^{m}\frac{A!}{y!\,(A-y)!}\binom{N-A-1}{m-y}$$

$$= \frac{D}{\binom{N}{n}}\sum_{y=0}^{m}\binom{A}{y}\binom{N-A-1}{m-y}$$

$$= \frac{D}{\binom{N}{n}}\binom{N-1}{m} = \frac{D}{\binom{N}{n}}\binom{N-1}{n-1} = \frac{D}{\binom{N}{n}}\,\frac{n}{N}\binom{N}{n}$$

$$\Longmapsto \mu = E(x) = n\frac{D}{N}$$

$$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\,f(x) = \sum_{x=0}^{n} x(x-1)\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$\therefore E[x(x-1)] = \frac{1}{\binom{N}{n}}\sum_{x=2}^{n}\frac{D!}{(x-2)!\,(D-x)!}\binom{N-D}{n-x}$$

$$= \frac{D(D-1)}{\binom{N}{n}}\sum_{x=2}^{n}\frac{(D-2)!}{(x-2)!\,(D-x)!}\binom{N-D}{n-x}$$

$$let \quad y = x - 2 \ , \ m = n - 2 \quad and \quad A = D - 2$$

$$= \frac{D(D-1)}{\binom{N}{n}} \sum_{y=0}^{m} \frac{A!}{y!\,(A-y)!} \binom{N-A-2}{m-y}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \sum_{y=0}^{m} \binom{A}{y} \binom{N-A-2}{m-y}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \binom{N-2}{m} = \frac{D}{\binom{N}{n}} \binom{N-2}{n-2}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \frac{n(n-1)}{N(N-1)} \binom{N}{n} = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)}$$

$$\Longmapsto E(x^2) = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)} + \frac{nD}{N}$$

$$\therefore \ \sigma^2 = E(x^2) - E^2(x) = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)} + \frac{nD}{N} - \left(\frac{nD}{N}\right)^2$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{(D-1)(n-1)}{(N-1)} + 1 - \frac{nD}{\mathrm{N}}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N(n-1)(D-1) + N(N-1) - nD(N-1)}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{nND - nN - ND + N + N^2 - N - nND + nD}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N^2 - nN - ND + nD}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N(N-n) - D(N-n)}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{(N-n)(N-D)}{N(N-1)}\right]$$

$$\therefore \quad \sigma^2 = n \frac{D}{N} \ \frac{N-n}{N-1} \ \frac{N-D}{N}$$

The moment generating function of **Negative Binomial** distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx} \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$= 1 + \sum_{k=1}^{\infty} [F(k+1) - 1] \frac{p}{k!}$$

Where: $F(k)$ is the sum of the first k-coefficient of the hyper-geometric series.

Note : If we set $p = \frac{D}{N}$ then the mean of the hyper-geometric distribution coincides with the mean of the binomial distribution, and the variance is $\frac{N-n}{N-1}$ times of the variance of binomial distribution.

Some continues Distribution:

1) Continues uniform distribution (Rectangular distribution) :

The r.v. X is said to have an uniform distribution if P.d.f. is give by

$$f(x,a,b) = \begin{cases} \frac{1}{b-a} & ; a \le x \le b \\ 0 & ; o.w. \end{cases} \quad ; Where\ a\ and\ b\ it\ satisfy -\infty < a < b < \infty$$

And denoted by $X \sim C_u(a,b)$

Clear that satisfy

$$f(x) \ge 0\ \forall x \quad and \quad \int_a^b f(x)\,dx = 1$$

Then f(x) satisfies the condition of begin a P.d.f. of continues type of random variable x.

$$Note:\ the\ mean\ \mu = \frac{a+b}{2} \quad and\ the\ variance\ \sigma^2 = \frac{(b-a)^2}{12} \quad (proof)$$

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)\, dx = \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx = \frac{e^{bt} - e^{at}}{t(b-a)}$$

Example : If x is uniform distribution over (0 , 10) , calculate the probability that :

❶ $P(x< 3)$ ❷ $P(3<x<8)$ ❸ $P(x>6)$ ❹The distribution function

Solution : $X \sim C_u(a,b)$ $\therefore$ $X \sim C_u(0,10)$

$$f(x,a,b) = \begin{cases} \dfrac{1}{b-a} & ;\ a \le x \le b \\ 0 & ; o.w. \end{cases} \Rightarrow f(x,0,10) = \begin{cases} \dfrac{1}{10} & ;\ 0 \le x \le 10 \\ 0 & ; o.w. \end{cases}$$

❶ $$P(x < 3) = \int_0^3 \frac{1}{10}\, dx = \left[\frac{x}{10}\right]_0^3 = \frac{3}{10}$$

❷ $$P(x > 6) = \int_6^{10} \frac{1}{10}\, dx = \left[\frac{x}{10}\right]_6^{10} = \frac{10}{10} - \frac{6}{10} = \frac{4}{10}$$

❸ $$P(3 < x < 8) = \int_3^8 \frac{1}{10}\, dx = \left[\frac{x}{10}\right]_3^8 = \frac{8}{10} - \frac{3}{10} = \frac{5}{10}$$

❹ $$F(x) = P(X < x) = \int_{-\infty}^{x} f(u)\, du = \int_0^x \frac{1}{10}\, du = \left[\frac{u}{10}\right]_0^x = \frac{x}{10}$$

$$\Longmapsto F(x) = \begin{cases} 0 & ;\ x \le 0 \\ \dfrac{x}{10} & ; 0 < x < 10 \\ 1 & ; x \ge 10 \end{cases}$$

2) Normal distribution :

The normal distribution is one of the widely used in application of statistical methods the P.d.f. of X is give by:

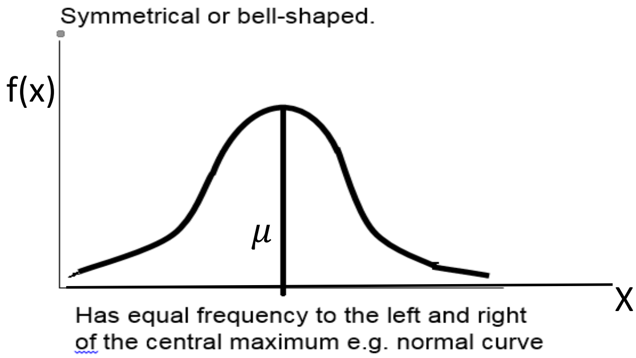
$$f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad ; -\infty<x<\infty$$

Where the parameters μ and σ^2 satisfy $-\infty<\mu<\infty$ and $\sigma^2>0$,

And denoted by $X\sim N(\mu,\sigma^2)$

The properties of this p.d.f. are

1) It is a bell shaped curve is symmetric about μ



2) The two parameter that appear in the density μ and σ^2 represent the mean and the variance of the random variable X.

Now

Consider the integral

$$I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$let\ y=\frac{x-\mu}{\sigma}\ \Rightarrow dy=\frac{dx}{\sigma}\ \Rightarrow\ dx=\sigma dy$$

$$\because\ -\infty<x<\infty\quad \therefore\ -\infty<y<\infty\quad \Rightarrow I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y^2}\,dy$$

$$\therefore\ I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx \quad and \quad I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

$$\therefore\ I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx\ .\ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\ .\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dx\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\ .\ e^{-\frac{1}{2}y^2}\, dx\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

We change the variables to polar coordination by using

$x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore\ I^2 = \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi} r e^{-\frac{1}{2}r^2}\, d\theta\, dr = \frac{1}{2\pi}\int_{0}^{\infty} r e^{-\frac{1}{2}r^2}[\theta]_0^{2\pi}\, dr$$

$$= \int_{0}^{\infty} r e^{-\frac{1}{2}r^2} dr = \left[-e^{-\frac{1}{2}r^2}\right]_0^{\infty} = 1 \implies I = 1$$

Then f(x) satisfies the condition of begin a P.d.f. of continues type to random variable x.

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{tx} . e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[2x\sigma^2 t + x^2 - 2\mu x + \mu^2]} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + \mu^2]} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + \mu^2 + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2]} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2]} e^{\frac{-\mu^2 + (\mu+\sigma^2 t)^2}{2\sigma^2}} dx$$

$$= \frac{e^{\frac{-\mu^2 + (\mu+\sigma^2 t)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2]} dx$$

$$= \frac{e^{\frac{-\mu^2 + (\mu+\sigma^2 t)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x - (\mu+\sigma^2 t)]^2} dx$$

$$If\ y = x - (\mu + \sigma^2 t) \Longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x - (\mu+\sigma^2 t)]^2} dx = I = 1$$

$$\therefore \quad M_x(t) = e^{\frac{-\mu^2+(\mu+\sigma^2 t)^2}{2\sigma^2}} = e^{\frac{-\mu^2+\mu^2+2\mu\sigma^2 t+(\sigma^2 t)^2}{2\sigma^2}}$$

$$\Longrightarrow \quad M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Note : The mean $\boldsymbol{\mu}$ and the variance σ^2 of normal distribution will be calculated from $M_x(t)$ as following :

$Since \quad M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $E(x) = M_x'(0)$ and $E(x^2) = M_x''(0)$

$$M_x'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t) \Longrightarrow M_x'(0) = \mu$$

$$M_x''(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.\, \sigma^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^2 \Longrightarrow M_x''(0) = \sigma^2 + \mu^2$$

$$\therefore \quad var(x) = E(x^2) - E^2(x) \Longrightarrow var(x) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

3) Standard normal distribution

If the normal random variable Z has mean zero and variance one instead of μ and σ^2 it is called standard normal distribution with P.d.f that:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad ; -\infty < z < \infty$$

And denoted by $Z \sim N(0,1)$

Theorem : If the r.v $X \sim N(\mu, \sigma^2), \sigma^2 > 0$; **then the r.v** $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

Note : ❶ If the r.v $X \sim N(\mu, \sigma^2), \sigma^2 > 0$; then $P(a < x < b) = Z\left(\frac{b-\mu}{\sigma}\right) - Z\left(\frac{a-\mu}{\sigma}\right)$

❷ *We can write* $Z(x)$ *as the following forms* $N(x), \emptyset(x)$ *or* $F(x)$.

$Remark$: $N(-x) = 1 - N(x)$ *or* $Z(-x) = 1 - Z(x)$.

Note: $M_z(t) = e^{\frac{1}{2}t^2}$ and $Z(x) = N(x) = F(x) = Pr(Z \le x) = Pr(\frac{X-\mu}{\sigma} \le x)$

Example: Let $X \sim N(3,4)$. *Find* $P(X \le 4)$

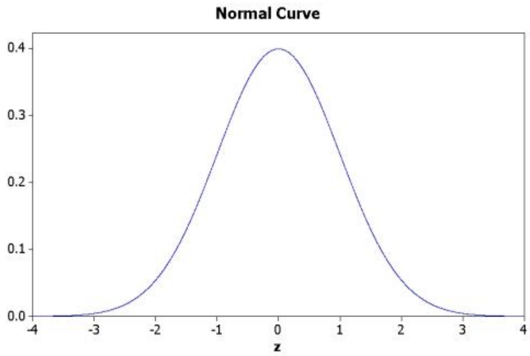
Solution :

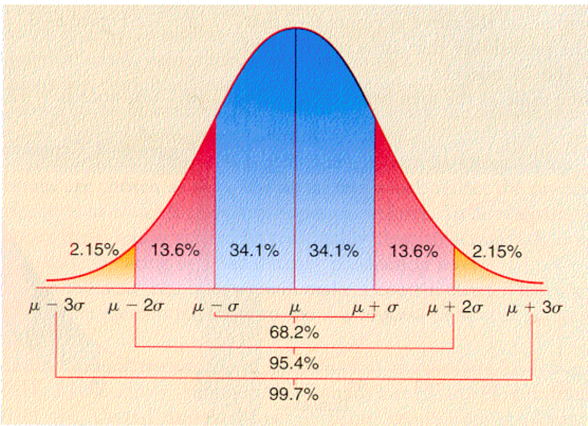
$$since\ X \sim N(3,4) \Longrightarrow \mu = 3 \ and \ \sigma^2 = 4 \Longrightarrow \sigma = 2$$

$$P(X \leq 4) = P\left(\frac{X-\mu}{\sigma} \leq \frac{4-\mu}{\sigma}\right) = P\left(Z \leq \frac{4-3}{2}\right) = Z\left(\frac{1}{2}\right) \; or \; \emptyset\left(\frac{1}{2}\right)$$

From Z table (standard normal table) we get $\emptyset\left(\frac{1}{2}\right) = Z\left(\frac{1}{2}\right) = 0.6915$

$\therefore \quad P(X \leq 4) = 0.6915$





Areas Under the Normal Curve