Probability

Probability concepts :

Probability is the study of **random experiment**.

Basic probability

Probability or **chance** can be measured on a scale, which runs from **zero**, which represents **impossibility**, to **one**, which represents **certainty**.

- If a die tossed in the air it is certain that, the die will be come down, but it's not certain say that, what will be appear. [Die (زهر النرد(حجر الزار)]
- As this experiment, its call **random experiment**.

Random experiment:

The experiment in which it is impossible to know it's result (because its related to the probability laws).

Definition (1-1)

A **sample space Ω**, is the set of all possible outcomes of a random experiment**.**

Example

If we roll a standard 6-sided die, a describe the sample space

S=Ω ={1,2,3,4,5,6}

Two dice

One die

Examples:

①The sample space of tossing a **coin** is

Ω_1={ H,T} [Coin قطعة نقدية , H; Head صورة , T; Tail كتابة]

②The sample space of rolling a **die** is

Ω_2={ 1, 2, 3, 4, 5, 6} [Rolling die دحرجة حجر النرد]

③The sample space of drawing a **card** from an ordinary **deck** of playing cards is

Ω_3={ ω: ω is one of the 52 cards } [Deck cards ورق القمار]

④The sample space of choice of a number on the **interval zero to one** is

Ω_4={ ω: 0< ω <1 } [Open interval فترة مفتوحة]

⑤The sample space of tossing a coin on infinity number of time is

Ω_5={ω: ω all sequences of the from HTHHT… } [Infinity number of time عدد غير نهائي من المرات]

⑥The sample space of count the number of defective items produced in a production line 24 hours is

Ω_6={ 0,1, 2, 3, 4, 5, ... }

⑦The sample space of a die is rolling two time in the air is

$$\Omega_7 = \begin{bmatrix} (\mathbf{1},\mathbf{1}) & (\mathbf{1},\mathbf{2}) & (\mathbf{1},\mathbf{3}) & (\mathbf{1},\mathbf{4}) & (\mathbf{1},\mathbf{5}) & (\mathbf{1},\mathbf{6}) \\ (\mathbf{2},\mathbf{1}) & (\mathbf{2},\mathbf{2}) & (\mathbf{2},\mathbf{3}) & (\mathbf{2},\mathbf{4}) & (\mathbf{2},\mathbf{5}) & (\mathbf{2},\mathbf{6}) \\ (\mathbf{3},\mathbf{1}) & (\mathbf{3},\mathbf{2}) & (\mathbf{3},\mathbf{3}) & (\mathbf{3},\mathbf{4}) & (\mathbf{3},\mathbf{5}) & (\mathbf{3},\mathbf{6}) \\ (\mathbf{4},\mathbf{1}) & (\mathbf{4},\mathbf{2}) & (\mathbf{4},\mathbf{3}) & (\mathbf{4},\mathbf{4}) & (\mathbf{4},\mathbf{5}) & (\mathbf{4},\mathbf{6}) \\ (\mathbf{5},\mathbf{1}) & (\mathbf{5},\mathbf{2}) & (\mathbf{5},\mathbf{3}) & (\mathbf{5},\mathbf{4}) & (\mathbf{5},\mathbf{5}) & (\mathbf{5},\mathbf{6}) \\ (\mathbf{6},\mathbf{1}) & (\mathbf{6},\mathbf{2}) & (\mathbf{6},\mathbf{3}) & (\mathbf{6},\mathbf{4}) & (\mathbf{6},\mathbf{5}) & (\mathbf{6},\mathbf{6}) \end{bmatrix}$$

Definition (1-2):

An event E is a subset of sample space Ω, or collection of points of Ω.

Let $\mathcal{F}$ be the class of all events in Ω, $\mathcal{F}$={E: E is event in Ω}.[$E \subseteq \Omega$, $E \in \mathcal{F}$]

Example:
In experiment: roll a die twice. Possible events are E1 ={1st face is a 6};
E2 ={sum of faces = 3};E3 = {sum of faces is odd};E4 = {1st face - 2nd face =3};
E5={ω: ω the sum of the pair is equal to seven}.
Identify the sample space and the above events. Obtain their probabilities when the die is fair.

Answer:

Sample space as Ω_7

E1={ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) } , E2={(1,2) , (2,1)}

E3={(1,2) , (1,4) , (1,6) , (2,1) ,(2,3) , (2,5) , (3,2) , (3,4) , (3,6) , (4,1) , (4,3) , (4,5) , (5,2) , (5,4) , (5,6) , (6,1) , (6,3) , (6,5) }

E4={(6,3) , (5,2) , (4,1) } , E5={(1,6) , (6,1) , (2,5) , (5,2) , (3,4) , (4,3) }.

Some algebra of events:

Given events A and B, further events can be identified as follows.

① The **complement** of any event A, written $\overline{A}$ or A^c , means that A does **not** occur.

② The **union** of any two events A and B, written A ∪ B, means that A or B or both occur.

③ The **intersection** of A and B, written as A ∩ B, means that both A and B occur.

④ The **difference** of A and B, written as A-B, means that $A \cap B^c$.

Notes1:

For convenience define the empty event ∅ to be the event containing on outcomes. Naturally, **Ω** may also be consider as an event.

Notes2:

(∅) is called the impossible event and **Ω** is called the certain or sure event.

Definition (1-3):

Two events E1 and E2 are said to be "**disjoint**" or "**mutually exclusive**" is

$E1 \cap E2 = \emptyset$

Definition (1-4):

***n* -factorial** ,it can find factorial by the following formula :

$$\boldsymbol{n! = n(n-1)(n-2)\ldots 2\times 1} \quad , \boldsymbol{n}! = \mathbf{1}\times\mathbf{2}\times\mathbf{3}\times\ldots\times\boldsymbol{n}; \quad \mathbf{0!=1}.$$

Note:

$$\boldsymbol{n}! = \boldsymbol{n}(\boldsymbol{n}-\mathbf{1})! \ , \ \boldsymbol{n}! = \boldsymbol{n}(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2})! \ , \ \mathbf{5!=5X4X3!} \ .$$

<u>**Example:**</u>

5!=120 3!=6 6!=720 6!=6X5! = 6X120 =720

Definition (1-5):

An ordered arrangement of r distinct objects is called a permutation.

Note:

The number of ways of ordering n distinct objects taken r at a time will be designated by the symbol P_r^n .

$$\mathrm{P}_r^n = \frac{n!}{(n-r)!}$$

Note : The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects, respectively, where each object appears in exactly one group and $\sum_{i=1}^{k} n_i = n$,is

$$\binom{n}{n_1\ n_2\ \dots\ n_k} = \frac{n!}{n_1!\ n_2!\ \dots\ n_k!}$$

Definition (1-6):

The number of combinations of n objects taken r at a time is the number of subsets, each of size r , that can be formed from the n objects. This umber will be denoted by $\mathrm{C}_r^n \equiv \binom{n}{r}$.

Note:

The number of unordered subsets of size r chosen (without replacement) from n available objects is

$$\mathrm{C}_r^n = \binom{n}{r} = \binom{n}{r\quad n-r} = \frac{\mathrm{P}_r^n}{r!} = \frac{n!}{r!\,(n-r)!}$$

Example1: How many ways can a committee consisting of 3 men and 2 women be chosen from 7 men and 5 women?

Solution:

$$\mathrm{C}_3^7\ \mathrm{C}_2^5 = \frac{7!}{3!\,(7-3)!}\ \frac{5!}{2!\,(5-2)!} \Longmapsto \frac{\mathbf{7\,.\,6\,.\,5}}{\mathbf{3!}}\ \frac{\mathbf{5\,.\,4}}{\mathbf{2!}} = \mathbf{350}$$

Example2:

How many permutation of letter a,b and c taken 2 at time?

Solution:

$$P_r^n = \frac{n!}{(n-r)!} \Longmapsto P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 6$$

Example3: Consider the word **STATISTICS**, find number of permutations?

Solution:

n= 10(all letter) **S**=3(n_1=3) **T**=3(n_2=3) **A**=1(n_3=1)

I=2(n_4=2) **C**=1(n_5=1) k=5

$$\binom{n}{n_1\ n_2 \dots n_k} = \frac{n!}{n_1!\ n_2! \dots n_k!} \Longmapsto \frac{10!}{3!3!1!2!1!} = 50400$$

Example4: A student is to answer 8 out of 10 question in an exam

a) How many choices he?
b) How many choices has he if he must answer first 3 question?
c) How many choices has he must answer at least 4 of first 5 question?

Solution:

a) $$C_r^n = \frac{n!}{r!\,(n-r)!} \Longmapsto C_8^{10} = \frac{10!}{8!\,(10-8)!} = \frac{10\ .\ 9\ .\ 8!}{8!\ 2!} = 45$$

b) $$C_3^3\ C_5^7 = \frac{3!}{3!\,(3-3)!}\ \frac{7!}{5!\,(7-5)!} \Longmapsto \frac{3!}{3!}\ \frac{7\ .\ 6\ .\ 5!}{5!} = 21$$

c) $$C_4^5\ C_4^5 + C_5^5\ C_3^5 = \frac{5!}{4!\,(5-4)!}\ \frac{5!}{4!\,(5-4)!} + \frac{5!}{5!\,(5-5)!}\ \frac{5!}{3!\,(5-3)!} = 35$$

H.W. (1) : If a+b=n then $\binom{n}{a} = \binom{n}{b}$ for all a and b ∈ N^0 . proof that?

H.W. (2) : Theorem $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ for all n and r ∈ N . proof that?

Note that: If $a, b \in R\ ; n \in N\ then\ (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r\, b^{n-r}$

<u>Example5</u>:

In how many ways can a Party of 7 Persons arrange their selves?

(i) In a row of 7 chairs

(ii) In a circle table

<u>Solution</u>:

(i) $n!=7! = 7.6.5.4.3.2.1 = 5040$

(ii) $(n-1)! = (7-1)! =6! = 6.5.4.3.2.1 = 720$

Classical Probability:

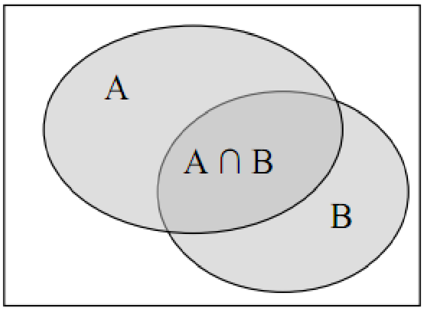
Let the sample space S contain (n) of elements and the event $A\subseteq S$ contain (m) of elements, then the probability of (A) denoted by P(A) and defined as:

$$P(A) = \frac{m}{n} = \frac{N(A)}{N(S)} = \frac{Number\ of\ elements\ of\ event\ A}{Number\ of\ elements\ of\ saple\ space\ S}$$

Probability :A measure of the likelihood of an event measured by a number between 0 and 1. Mathematically, probability is a function that to each event A assigns a number P(A) (called the probability of an event A)

Properties of probability:

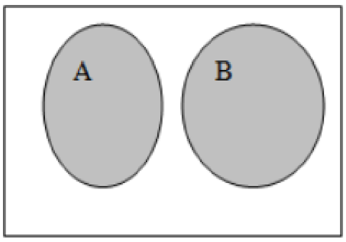
1) $0 \le P(A) \le 1$
2) $P(S) = 1$, $P(\Phi) = 0$
3) $P(A^c) + P(A) = 1$ [Rule of Complements]
   in particular $P(A) = 1- P(A^c)$ and $P(A^c) = 1- P(A)$
4) $P(A\cup B) = P(A) + P(B) - P(A \cap B)$



Events A and B are **mutually exclusive** if $A \cap B$ contains no sample points, that is, if A and B cannot occur at the same time.

For mutually exclusive events A and B

- $P(A \cup B) = P(A) + P(B)$



Example 1. A coin is tossed three times. Find the probabilities of the following events:

1. Exactly two "heads" (H) in three tosses
2. "Heads" in first toss
3. Exactly two "heads" in three tosses AND "heads" in the first toss
4. Exactly two "heads" in three tosses OR "heads" in the first toss

SOLUTION:

The sample space S =

HHH	HHT	HTH	THH
HTT	THT	TTH	TTT

Events: A = "exactly two H in three tosses" = {HHT, HTH, THH}

B = "H in first toss" = {HHH, HHT, HTH, HTT}

Probability: assume that the coin is fair and hence each sample point has the same probability

P(a single sample point) = 1/8 (*equally likely outcomes*)

1. P(exactly two H in three tosses) = P(A) = 3/8
2. P(H in the first toss) = P(B) = 4/8 = 1/2
3. P("exactly two H in three tosses" AND "H in the first toss") = $P(A \cap B)$ =
   = P({HHT, HTH}) = 2/8 = 1/4
4. P("exactly two H in three tosses" OR "H in the first toss") = $P(A \cup B)$ =
   = P{ HHH, HHT, HTH, HTT, THH}) = 5/8

Example 2 (cont). A coin is tossed three times. Find the probability of:

1. Exactly two H in three tosses OR H in the first toss using the *Additive Rule*
2. At least two H in three tosses
3. At most one H in three tosses using the *Rule of Complements*

SOLUTION

1. P("exactly two H in three tosses" OR "H in the first toss") =
   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 3/8 + 4/8 - 2/8 = 5/8$

HHH	HHT	HTH	THH
HTT	THT	TTH	TTT

Let D = "at least two H" = {HHH, HHT, HTH, THH}

2. P(D) = 4/8 = 1/2
3. P(most one H) = 1 – P(least two H) = 1 – P(D) = 1-1/2 = 1/2

HHH	HHT	HTH	THH
HTT	THT	TTH	TTT

Equally Likely Outcomes Model.

Assume that there are finitely many possible outcomes and all are equally likely. Then

$$P(A) = \frac{n(A)}{n(S)} = \frac{\#\,of\ outcomes\ in\ A}{\#\,of\ outcomes\ in\ S}$$

$$Or \qquad P(A)=\frac{n(A)}{n(S)}=\frac{number\ of\ ways\ to\ taken\ event\ A}{number\ of\ all\ \ ways\ possible\ taken}$$

$$P(A)=\frac{\text{عدد طرق الاختيار للحدث} A \text{(اوعدد عناصر } A)}{\text{عدد كل الطرق الممكنة للاختيار (اوعدد عناصر } S)}$$

Example1: S={HH, HT, TH, TT} and A={HH, HT, TH}

Solution: P(A)=3/4

Example2: A box contains on 8 red balls, 6 white balls and 7 blue balls, What is the probability that two balls drawn red and white?

Solution: N=number of all balls =21

① RW= drawn red and white

$$P(RW)=\frac{C_1^8\,C_1^6}{C_2^{21}}=\frac{8\ .\ 6}{21\ .10}=8/35$$

② RRB=drawn 2 red and blue

$$P(RRB)=\frac{C_2^6\,C_1^7}{C_3^{21}}=\frac{8\ .5\ .\ 7\ .\ 6}{21\ .20\ .19}=28/95$$

③ RWB=drawn red, white and blue

$$P(RWB)=\frac{C_1^8\,C_1^6\,C_1^7}{C_3^{21}}=\frac{8\ .\ 6\ .\ 7\ .\ 6}{21\ .20\ .19}=24/95$$

④ W^c =drawn 3ball all not white

$$P(W^c)=\frac{C_3^{15}}{C_3^{21}}=\frac{15\ .\ 14\ .\ 13}{21\ .20\ .19}=39/114$$

Theorem:

- *If $E_1, E_2, \dots, E_n$ are sequence of mutually exclusive (*M.e.*) events, then*
  $p(E_1 \cup E_2 \cup \dots \cup E_n) = p(E_1) + p(E_2) + \cdots + p(E_n)$

  And

- $If\ \ E_1 \cup E_2 \cup \dots \cup E_n = S\ (\equiv \Omega)\ \ then\ \ p(E_1) + p(E_2) + \cdots + p(E_n) = 1$

Theorem: If $\emptyset$ is the empty set (i.e. the impossible events) then $p(\emptyset)=0$.

Proof:

$A=A\cup\emptyset \Longmapsto p(A)=p(A\cup\emptyset)=p(A)+p(\emptyset)$ ($A,\emptyset$ disjoint; $A\cap\emptyset=\emptyset$)

$\Longmapsto p(A)=p(A)+p(\emptyset) \Rightarrow p(\emptyset)=0$

Theorem: If A^c is the complement of A then $p(A^c) = 1 - P(A)$

Proof:

$S=A\cup A^c \Longmapsto p(S)=p(A\cup A^c)=p(A)+p(A^c)$ (A, A^c disjoint(M.e); $A\cap A^c=\emptyset$)

$\Longmapsto p(S) = p(A)+p(A^c) \Rightarrow 1=p(A)+p(A^c) \Rightarrow p(A^c)=1-p(A)$

Example: If a die is tossed in the air and we deserve the number on the top and let:

A: is the event that even number appear.

B: is the event that odd number appear.

C: is the event that Prime number appear.

Find ① $P(A\cup B)$ ② $P(A\cap B)$ ③ $P(A\cup C)$ ④ $P(A\cap C)$ ⑤ $P(C^c)$

Solution:

$S=\{1, 2, 3, 4, 5, 6\}$ $A=\{2, 4, 6\}$ $B=\{1, 3, 5\}$ $C=\{2, 3, 5\}$

① $A\cup B=\{1,2,3,4,5,6\}=S \Longmapsto P(A\cup B)=1$ ② $A\cap B=\emptyset \Longmapsto P(\emptyset)=0$

③ $A\cup C=\{2,3,4,5,6\} \Longmapsto P(A\cup C)=5/6$ ④ $A\cap C=\{2\} \Longmapsto P(A\cap C)=1/6$

⑤ $C^c=\{1,4,6\} \Longmapsto P(C^c)=3/6=1/2$ $[P(C^c)=1-P(C) \Rightarrow P(C^c)=1-3/6=1/2]$

Example:

A coin is weighted so that heads is twice as likely to appear as tail find P(H) and P(T)

Solution:

$S=\{H, T\}$ $\therefore S=H\cup T \Rightarrow p(S)=p(H\cup T)=p(H)+p(T)$ [H and T are M.e.]

$\because p(H)=2p(T)$ let $p(T)=p \Rightarrow p(H)=2p$

$\therefore p(S)=p(H)+p(T) \Rightarrow 1=p+2p \Rightarrow 3p=1 \Rightarrow p=1/3$

$\therefore p(T)=1/3$ and $p(H)=2/3$.

Theorem : Let A and B subset of S then $p(A)=p(A\cap B)+p(A\cap B^c)$

Proof : $A= A\cap S \Rightarrow A=A\cap(B\cup B^c) \Rightarrow A=(A\cap B)\cup(A\cap B^c)$

$\because (A\cap B) \cap (A\cap B^c) =\emptyset \Rightarrow (A\cap B)$ and $(A\cap B^c)$ are mutually exclusive

$\therefore p(A)=p(A\cap B)+p(A\cap B^c)$

Example: If two dice are rolled and observed the number appear on the tops

1) Find the Probability of the event A that get sum of seven.
2) Find the probability of the event B that the Sum is more than 7.
3) Find Probability of the event C that the sum is less than seven.

Solution:

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

1) $A=\{(3,4),(4,3),(2,5),(5,2),(1,6),(6,1)\} \Rightarrow p(A)=6/36=1/6.$
2) $B=\{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),(5,3),(5,4),(5,5),(5,6),(6,2),(6,3),$
$(6,4),(6,5),(6,6)\} \Rightarrow p(B)=15/36=5/12$
3) $C=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(4,1),$
$(4,2),(5,1)\} \Rightarrow p(C)=15/36=5/12.$

<u>Independence</u> : [الاستقلالية]

Definition: the events A and B are independent if and only if

$$\mathbf{P(A \cap B) = P(A).\ P(B)}$$

Theorem : the two events A and B are independent if and only if A^c and B^c are independent

proof:(H.W)

Example: Toss coin two time let A the event is first head and B the event is second Head and C is two head respectively.

Test (A,B), (A,C), (B,C) are independent or no

Solution:

S={HH , HT , TH , TT} A={HH,HT} B={HH,TH} C={HH}

$A\cap B$={HH} $A\cap C$={HH} $B\cap C$={HH}

$\therefore P(A)=2/4, \quad P(B)=2/4, \quad P(C)=1/4, \quad P(A\cap B)=1/4,$

$P(A\cap C)=1/4, \quad P(B\cap C)=1/4$

$\because$ P(A∩B)=P(A).P(B) then A and B are independent

$\because$ P(A∩C)≠P(A).P(C) then A and C are not independent (dependent)

$\because$ P(B∩C)≠P(B).P(C) then B and C are not independent (dependent)

Field and Sigma Field:

A non-empty class of sets $\mathcal{F}$ is said to be a field if it is closed under complementation and finite union. Thus a field $\mathcal{F}$ is a non-empty class of subset of S if:

1- $S\in\mathcal{F}$
2- If A and B $\in\mathcal{F}$ then $A\cup B\in\mathcal{F}$.
3- If $A\in\mathcal{F}$ then $A^c\in\mathcal{F}$.

Definition: Any field $\mathcal{F}$ on S is called Sigma field or Borel field on S if it is closed under the numerable intersection and union. Thus, a Borel field $\mathcal{F}$ is afield on S such that:

1- *If* $A_i \in \mathcal{F}$ *,i=1,2,3, … ,then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

2- *If* $A_i \in \mathcal{F}$ *,i=1,2,3, … ,then* $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example: Let S ={H,T} check the sets on the following, are field yes or no if no why?

(1) $\mathcal{F}_1=\{\emptyset,S\}$ answer: yes is a field
(2) $\mathcal{F}_2=\{\emptyset,\{H\},\{T\},S\}$ answer: yes is a field
(3) $\mathcal{F}_3=\{\{H\},S\}$ answer: no is not field because $H^c=T\notin\mathcal{F}_3$
(4) $\mathcal{F}_4=\{\emptyset,\{H\}\}$ answer: no is not field because $H^c=T\notin\mathcal{F}_4$

Theorem : Let $\mathcal{F}$ is a sigma field then

① $\emptyset\in\mathcal{F}$ ② *If* $A_i^c\in\mathcal{F}$ *; i =1,2,3, … ,then* $\bigcap_{i=1}^{\infty} A_i^c\in\mathcal{F}$.

③ *If* $A_i^c\in\mathcal{F}$ *; i =1,2,3, … ,then* $\bigcup_{i=1}^{\infty} A_i^c\in\mathcal{F}$.

Proof:

① $\because$ $\mathcal{F}$ is a field ⇨ $S\in\mathcal{F}$ ⇨ $S^c\in\mathcal{F}$

$\because S^c=\emptyset$ then $\emptyset\in\mathcal{F}$

② $\because$ $A_i^c\in\mathcal{F}$ ⇨ $A_i\in\mathcal{F}$ $[(A^c)^c=A]$ for all A_i ; i =1,2,3,….

$\therefore \bigcup_{i=1}^{\infty} A_i\in\mathcal{F}$ ⇨ $(\bigcup_{i=1}^{\infty} A_i)^c\in\mathcal{F}$ ⇨ $\bigcap_{i=1}^{\infty} A_i^c\in\mathcal{F}$

③ H. W.

Theorem : the two events A and B are independent if and only if A and B^c are independent

proof :

$A= A\cap S \Longmapsto A=A\cap(B\cup B^c) \Longmapsto A=(A\cap B)\cup(A\cap B^c)$; $[\, S= B\cup B^c \,]$

$\because (A\cap B) \cap (A\cap B^c) =\emptyset \Rightarrow (A\cap B)$ and $(A\cap B^c)$ are mutually exclusive

$\therefore p(A)=p(A\cap B)+p(A\cap B^c) \Rightarrow p(A\cap B^c) = p(A)-p(A\cap B)$

$\Rightarrow p(A\cap B^c) = p(A)-p(A).p(B)$; [A and B are independent]

$\Rightarrow p(A\cap B^c) = p(A)(1-p(B))=p(A).p(B^c)$; $[p(B^c)=1- p(B)\,]$

$\therefore$ A and B^c are independent

Theorem : the two events A and B are independent if and only if A^c and B are independent **(H. W.)**

Definition: let $\mathcal{F}$ be a sigma field on S and $\mathcal{P}$ is the Probability function, then $(S, \mathcal{F}, \mathcal{P})$ is called the probability space.

<u>Conditional probability</u>:

Suppose that $P(E_2) \neq 0$.The conditional probability of the event E_1 given E_2 is defined as

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Note : The conditional probability is undefined if P(E2) = 0.

The conditional probability formula above yields the multiplication rule:

$$P(E_1 \cap E_2) = P(E_1)P(E_2|E_1)$$
$$= P(E_2)P(E_1|E_2)$$

Note that:

This implies that $P(E_1|E_2) = P(E_1)$ and $P(E_2|E_1) = P(E_2)$.Thus knowledge of the occurrence of one of the events does not affect the likelihood of occurrence of the other.

Events $E_1, E_2, \ldots, E_k$ are ***pairwise independent*** if $P(E_i\cap E_j) = P(E_i)P(E_j)$ for all $i \neq j$. They are ***mutually independent*** if for all subsets $P(\bigcap_j E_j) = \prod_j P(E_j)$.

Clearly, *mutual independence* $\Longrightarrow$ *pairwise independence*, but the converse is false

Law of total probability (partition law)

Suppose that $B_1, \ldots, B_k$ are **mutually exclusive** and **exhaustive** events (*i.e.* $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $\cup_i B_i = \Omega$).

Let A be any event. Then

$$P(A) = \sum_{j=1}^{k} P(A|B_j)P(B_j)$$

Bayes' Rule

Suppose that events $B_1, \ldots, B_k$ are mutually exclusive and exhaustive and let A be any event. Then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$$

Example: Two dice are rolled and let the event A that the number 2 appear at least in one die and the event B that get sum 6 find:

1. P(A|B) and P(B|A)
2. Are A and B independent?

solution:

Sample space S is as $\Omega 7$ ⇨ S={(1,1),(1,2),…(1,6),(2,1),…,(2,6),…,(6,6)}

1. A=((2,1),(1,2),(2,2),(2,3),(3,2),(2,4),(4,2),(2,5),(5,2),(2,6),(6,2)}

B={(1,5),(5,1),(4,2),(2,4),(3,3)}

A∩B={(4,2),(2,4)}

Then P(A)=11/36 , P(B)=5/36 and P(A∩B)=2/36

$\therefore$ P(A/B) = $\frac{2/36}{5/36} = \frac{2}{5}$ and P(B/A) = $\frac{2/36}{11/36} = \frac{2}{11}$

2. A and B are not independent because P(A/B) ≠P(A)

Note : *If A and B independent event then P(A/B)= P(A) and P(B/A)=P(B)*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

Theorem: *Let A, B two event such that P(B) > 0 then:*

① $P(A^c/B) = 1 - P(A/B)$ ② $P(A/S) = P(A)$ ③ $P(\emptyset/B) = 0$

④ $P(S/B) = 1$ ⑤ $P(S/S) = 1$

Proof:

$$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} \qquad \because\ P(B) = P(B \cap A) + P(B \cap A^c)$$

$$\therefore\ P(B \cap A^c) = P(B) - P(B \cap A) \Rightarrow\ P(A^c|B) = \frac{P(B) - P(B \cap A)}{P(B)}$$

$$\Rightarrow\ P(A^c|B) = 1 - \frac{P(B \cap A)}{P(B)} = 1 - P(A|B)$$

Theorem: *Let A, B and C three event such that P(C)>0 then*

$$P\big((A \cup B)\big|C\big) = P(A|C) + P(B|C) - P((B \cap A)|C)$$

Proof:

$$P\big((A \cup B)\big|C\big) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$= \frac{P\big((A \cap C) \cup (B \cap C)\big)}{P(C)}$$

$$= \frac{P(A \cap C) + P(B \cap C) - P\big((A \cap C) \cap (B \cap C)\big)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P\big((A \cap B) \cap C)\big)}{P(C)}$$

$$= P(A|C) + P(B|C) - P((B \cap A)|C)$$

Theorem: If *P(A)=a and P(B)=b then* $P(A/B) \geq \frac{a+b-1}{b}$

Proof:

$$\because\ P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1\ ; \quad [(A \cup B) \in \mathcal{F} \Rightarrow P(A \cup B) \leq 1]$$

$$\Longrightarrow P(A \cap B) \geq P(A) + P(B) - 1\ ; \ [\div P(B)]$$

$$\Longrightarrow \frac{P(A \cap B)}{P(B)} \geq \frac{P(A) + P(B) - 1}{P(B)}$$

$$\therefore\ P(A|B) \geq \frac{a + b - 1}{b}$$

Note : The **Deck** of Cards contain on 52 cards as that:

1- Four group is

Spade ♣ سنك - heart ♥ قلب - Club ♠ ماجة - Diamond ♦ دينار

2- All group contain 13 cards has nine numbered [2,3,…,10] and

four picture [Ace(1) أس - King(13) ملك - Queen(12) ملكة - Jack(11) ولد]

Example :

In a deck of cards draw a single card; if A is the event "drawing an ace" and B is the event "drawing a diamond " find :

1- P(A) and P(B)

2- P(A∪B) , P(A∩B) and P(B|A)

3-if drawing 2 card respectively find P(B|A)

Solution :

N(A)= $\binom{4}{1} = 4$ N(B)= $\binom{13}{1} = 13$ N(S)= $\binom{52}{1} = 52$

N(A)= $\binom{4}{2} = 6$ N(B)= $\binom{13}{2} = 78$ N(S)= $\binom{52}{2} = 1326$

1- P(A)=N(A)/N(S)= 4/52 =1/13.

P(B)= 13/52 =1/4

2- P(A∪B)= 16/52 = 4/13 and P(A∩B)=1/52

P(B|A)=(1/52)/(1/13)=1/4 then A and B are independent

3- P(A)=6/1326 P(B)=78/1326 P(A∩B)= 0

P(B|A)=0 then A and B are mutually exclusive (m.e.)

Theorem: ((**Decomposition theorem**))

If $\{A_n\}$ is a sequence of mutually exclusive (disjoint) events with $\cup_{i=1}^{n} A_i = S$ and $P(A_i) \geq 0$, $i=1,2,\ldots, n$ then for any event $B \in \mathcal{F}$

$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

Proof:

$$B = B\cap S \Longrightarrow B = B\cap(\bigcup_{i=1}^{n} A_i) \Longrightarrow B = \bigcup_{i=1}^{n}(B \cap A_i)$$

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n)$$

$$\therefore\ P(B) = P(\bigcup_{i=1}^{n}(B \cap A_i))$$

$$or \quad P(B) = P[(B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n)]$$

$$\because\ (B \cap A_1), (B \cap A_2), \ldots, (B \cap A_n)\ \ are\ \ m.e.(disjoint)$$

$$\therefore\ P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$$

$$or \quad P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$\because\ P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)} \Longrightarrow P(B \cap A_i) = P(A_i)\, P(B|A_i)$$

$$\therefore\ P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

Theorem: ((**Bays theorem**))

If $\{A_n\}$ is a sequence of mutually exclusive (disjoint) events with $\bigcup_{i=1}^{n} A_i = S$ and $P(A_i) > 0$, $i=1,2,\ldots, n$ then for any event $B \in \mathcal{F}$

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B|A_i)}$$

Proof:

$$\because\ P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \Longrightarrow P(A_i \cap B) = P(B)\, P(A_i|B)$$

$$and\ \ P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)} \Longrightarrow P(B \cap A_i) = P(A_i)\, P(B|A_i)$$

$$\because\ (A_i \cap B) = (B \cap A_i) \Longrightarrow P(A_i \cap B) = P(B \cap A_i)$$

$$\therefore\ P(A_i)\, P(B|A_i) = P(B)\, P(A_i|B)$$

$$\therefore\ P(A_i|B) = \frac{P(A_i)\, P(B|A_i)}{P(B)}$$

$$\because\ P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

$$\therefore\ P(A_i|B) = \frac{P(A_i)\, P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B|A_i)}$$

Example: Consider two urns. Urn1 contains 5 white and 7 red balls. Urn2 contains 6 white and 4 red balls. One of the urns is selected at random and ball is drawn from it:

1) Find the Probability that the ball drawn will be white
2) If the ball drawn was white what is the probability that if from urn2.

Solution:

Let A_1 ="urn1 is chosen" ; A_2 ="urn2 is chosen" and B=" white ball is drawn "

$\therefore\ P(A_1)=P(A_2)=1/2 \quad , \quad P(B|A_1)=5/12 \quad , \quad P(B|A_2)=6/10$

① $\because\ P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i) \quad , n=2$

$\therefore\ P(B)=P(A_1)P(B|A_1)+P(A_2)P(B|A_2)$

$= (1/2)(5/12)+(1/2)(6/10)$

$= 0.51$

② $$\because \quad P(A_2|B) = \frac{P(A_2)\, P(B|A_2)}{P(B)} = \frac{P(A_2)\, P(B|A_2)}{\sum_{i=1}^{2} P(A_i)\, P(B|A_i)}$$

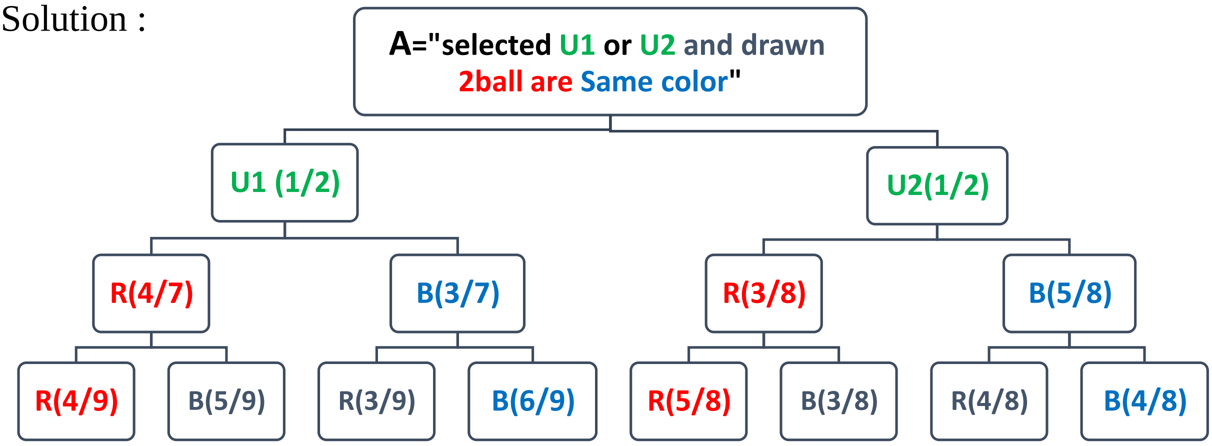
$$= \frac{\frac{1}{2}\,\frac{6}{10}}{0.51} = \frac{0.3}{0.51} = 0.59$$

<u>Tree of probability</u>: We can find ***P(E)*** by using ***tree of probability***

<u>Example</u>:

we are given two urns as following: Urn U1 contain 4 red and 3 blue balls; Urn U2 contain 3 red and 5 blue balls; an urn is selected at random a ball is drawn and put into the other urn, then a ball is drawn from it, find the probability that the drawn balls are of the Same color.

Solution :



$$P(A) = \left(\frac{1}{2}\right)\left(\frac{4}{7}\right)\left(\frac{4}{9}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{8}\right)\left(\frac{5}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{8}\right)\left(\frac{4}{8}\right)$$

$$= \frac{4381}{8064} = 0.5433$$

Example: We are given three urns as following :

Urn A: contain 3 red and **5 white** marble.

Urn B: contain 2 red and **1 white** marble.

Urn C: contain 2 red and **3 white** marble.

An urn is selected at random and a marble is drawn from the urns; If marble is red , what is the probability that it come from urn A.

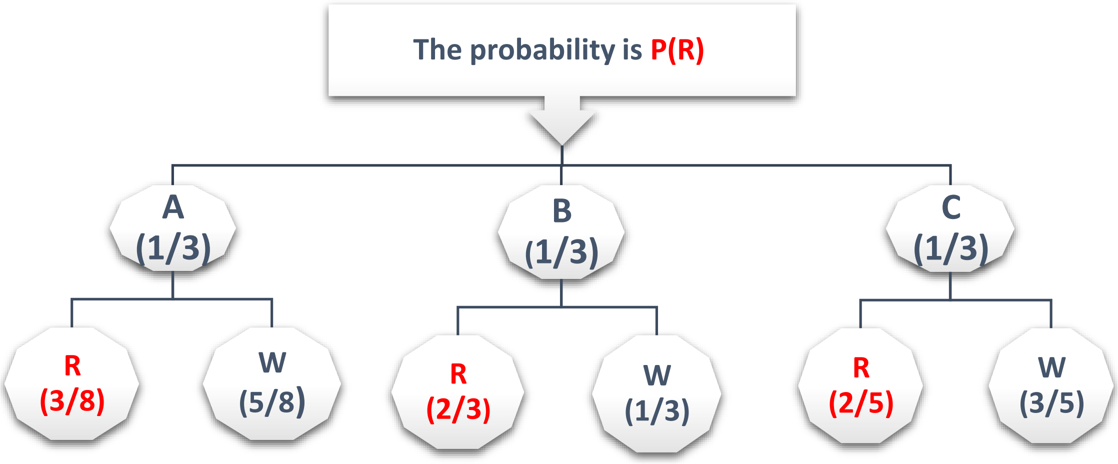
Solution:

We seek the probability that A was selected, given that marble is red, that is **P(A|R)** ; In order to find P(A|R) it is necessary first to compute P(A∩R) and P(R). The probability that urn A is selected and a red marble drawn is that

$$P(A\cap R) = \left(\frac{1}{3}\right)\left(\frac{3}{8}\right) = \frac{1}{8}$$

Since there are three paths leading to a red marble.

❶ Using tree to compute P(R)



$$P(R) = \left(\frac{1}{3}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) = \left(\frac{173}{360}\right)$$

$$\because\ P(A|R) = \frac{P(A\cap R)}{P(R)} = \frac{\left(\frac{1}{8}\right)}{\left(\frac{173}{360}\right)} \Rightarrow P(A|R) = \frac{45}{173} = \mathbf{0.2601}$$

❷ using Bayes' Theorem

$$\because \quad P(R) = P(A)\,P(R|A) + P(B)\,P(R|B) + P(C)\,P(R|C)$$

$$\because\ P(A|R)=\frac{P(A)P(R|A)}{P(R)}=\frac{P(A)P(R|A)}{P(A)\,P(R|A)+P(B)\,P(R|B)+P(C)\,P(R|C)}$$

$$\therefore\ P(A|R)=\frac{\left(\frac{1}{3}\right)\left(\frac{3}{8}\right)}{\left(\frac{1}{3}\right)\left(\frac{1}{8}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)}=\frac{\left(\frac{1}{8}\right)}{\left(\frac{173}{360}\right)}$$

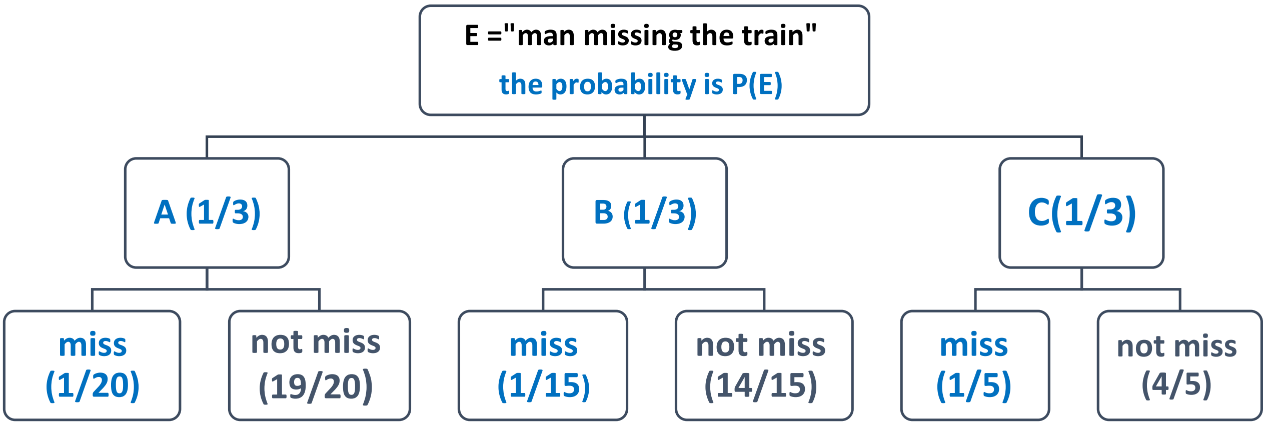
$$\Longmapsto\ P(A|R)=\frac{45}{173}=\mathbf{0.2601}$$

Example : A man is equally likely to choose any one of three routes A, B, and C from his house to the railway station and his choice of route is not influenced by the weather if the weather is dry, the probability of missing the train by routes A, B and C are 1/20 , 1/15 and 1/5 respectively. He sets out on a dry day and misses the train, what is the Probability that the route chosen was C?

Solution : Let E the event that " a man missing a train "

The required Probability is ***P(C/E)***⟵(المطلوب)

To find P(E) we will using tree



$$\therefore\ P(E)=\left(\frac{1}{3}\right)\left(\frac{1}{20}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{15}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)=\left(\frac{19}{180}\right)$$

$$\Rightarrow\ P(C|E)=\frac{P(E|C).P(C)}{P(E)}=\frac{\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)}{\left(\frac{19}{180}\right)}=\frac{12}{19}=0.6316$$

<u>**Another Solution**</u>**:** (Bayes' Theorem)

$$\because\quad P(E)=P(A)\,P(E|A)+P(B)\,P(E|B)+P(C)\,P(E|C)$$

$$\because\ P(C|E)=\frac{P(C)P(E|C)}{P(E)}=\frac{P(C)P(E|C)}{P(A)\,P(E|A)+P(B)\,P(E|B)+P(C)\,P(E|C)}$$

$$\therefore P(C|E) = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)}{\left(\frac{1}{3}\right)\left(\frac{1}{20}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{15}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)} = \left(\frac{12}{19}\right) = 0.6316$$

Example: A box contain black and white balls each ball is labeled either A or Z the composition of the box is show below:

Labeled \ Color	Black	White	Sum
A	5	3	8
Z	1	2	3
Sum	6	5	11

Let us now assume that a ball is to be selected at random from this box. What is the probability of getting a black ball if it was labeled A?

Solution:

A =" a ball is Labeled A" , B =" black ball"

∴ P(A)=8/11 P(B)=6/11 P(A∩B)=5/11

∴ P(B|A)= P(A∩B)/P(A)

= (5/11)/(8/11) =5/8

Question: Gavin the digits 1, 2, 3, 4 and 5, find haw many 4- digits numbers can be formed from them if:

1) A=" No digits may be repeated" 2) B="Repetitions of digits are allowed"

3) C="The number must be odd, without any repeat digit"

Answer :

N(A)= 5×4×3×2 =120 N(B)= 5×5×5×5 =5^4 = 625 N(C)= 5×4×3×1=60

Question: two men M1 , M2 and three women W1 , W2 , W3 are in chess tournament these of the same sex have equal probability of winning, but each men is twice as likely to win as women.

1) Find the probability a women wins the tournament. **(H.W.)**

2) if M1 and M2 are married find the Probability that one of them wins the tournament.

Question: Three horses A, B and C are in race; A is twice as likely to win as B and is twice as likely to win as C; find the probability of wining

Answer :

Let $P(C) = p$ then $P(B)=2p$ and $P(A)=2(2p)=4p$

$$\because P(A)+P(B)+P(C)=1$$

$$\therefore p+2p+4p=1 \Longrightarrow 7p=1 \Longrightarrow p=1/7$$

Then the probability of wining are

$$P(C)= 1/7 \quad P(B)= 2/7 \quad \text{and} \quad P(A)= 4/7$$

Example: Let three coin is tossed and number of heads observed.

1) Find the variable space of the observations.
2) Find P(0), P(2) and P(3)
3) Let A is the event that at least one head appears, find P(A).
4) Let B is the event that all head or tails appears, find P(B).
5) Let C is the event that one head appears at last, find P(C).

Solution :

S={ HHH , HHT , HTH , THH , THT, HTT , TTH , TTT }

❶ X= { 0 , 1 , 2 , 3 } [simply variable to sample space]

❷ $P(0)=1/8$ $P(2)=3/8$ $P(3)= 1/8$

❸ A={HHH , HHT , HTH , THH , TTH , THT,HTT }

⇨$P(A)=7/8$

❹ B={ HHH , TTT }

⇨ $P(B)=2/8 =1/4$

❺ C={HHH , HTH , THH , TTH }

⇨ $P(C)=4/8 =1/4$

Random Variable (r.v.):

A **random variable** is a variable whose value depends on the outcome of a probability experiment. As in algebra, random variables are represented by capital letters as X.

- *T* = the number of tails when a coin is flipped 3 times.
- *S* = the sum of the values showing when two dice are rolled.
- *Y* = the height of a woman chosen at random from a group.
- *V* = the liquid volume of soda in a can marked 12 ml.

Note:

A random variable (r.v) is a real number (value) X that associated with outcome of the random experiment (sample space)

Discrete Random Variable :

A numerical r.v. that takes on a countable number of values (there are gaps in the range of possible values).

Examples:

1. Number of phone calls received in a day by a company
2. Number of heads in 5 tosses of a coin

Continuous Random Variable :

A numerical r.v. that takes on an uncountable number of values (possible values lie in an unbroken interval).

Examples:

1. Length of nails produced at a factory
2. Time in 100-meter dash for runners

Other examples?

- The variables *T* and S from above are discrete random variables
- The variables Y and *V* from above are continuous random variables.

The **probability distribution** of a random variable is a graph, table, or formula which tells what values the r.v. can take and the probability that it takes each of those values.

Example:

Roll 1 die. The r.v. *X* = number of dots showing.

x	**1**	**2**	**3**	**4**	**5**	**6**
P(*x*)	**1/6**	**1/6**	**1/6**	**1/6**	**1/6**	**1/6**

Graph:

Example :

Toss 2 coins. The r.v. *X* = number of heads showing.

x	0	1	2
P(*x*)	¼	½	¼

Note that:

For any discrete probability distribution:

(1) P(*x*) is between 0 and 1 for any value of *x*.

(2) $\sum_{x} P(x)$ **=** 1. That is,

the sum of the probabilities for all possible *x* values is 1.

Example : P(*x*) = *x* / 10 for *x* = 1, 2, 3, 4.

P(1)=1/10 P(2)=2/10

P(3)=3/10 P(4)=4/10

Then P(x) $\geq$ 0

$$\sum_{x=1}^{4} P(x_i) = P(1) + P(2) + P(3) + P(4) = \frac{1+2+3+4}{10} = \frac{10}{10} = 1$$

Probability Distributions of Discrete Random Variables

A **probability distribution** for a discrete random variable *x* is a discrete is a list of each possible value for *x* together with the probability that when the experiment is run, *x* will have that value. This probability is denoted by $P(x)$.

Examples:

- As above, let *T* be the random variable that represents the number of tails obtained when a coin is flipped three times. Then *T* has 4 possible values: 0, 1, 2, and 3. The probability distribution for *T* is given in the following table:

T	0	1	2	3
$P(T)$	1/8	3/8	3/8	1/8

- A statistics class of 25 students is given a 5 point quiz. 3 students scored 0, 1 scored 1 , 4 scored 2 , 8 scored 3 , 6 scored 4 , and 3 students scored 5. If a student is chosen at random, and the random variable *s* is the student's quiz score then the probability distribution of *s* is:

S	0	1	2	3	4	5
$P(s)$	0.12	0.04	0.16	0.32	0.24	0.12

Note: For any discrete random variable *x*:

$$0 \le P(x) \le 1 \quad \text{and} \quad \sum P(x) = 1$$

Probability function

Is function which represent the probability of the random Variable (x) take real number from the sample space. There are two kinds of Probability function

1) *Probability mass function: (P.m. f.)*

If X is discrete random variable (d.r.v.) which distinct values $x_1, x_2, \dots, x_n$ then the probability mass function denoted by f(x) and defined by:

$$f(x) = P(x) = \begin{cases} P_r(X = x_i) \ , & X = x_i \ , i = 1, 2, \dots, n \\ 0 \ , & otherwise \end{cases}$$

With the following properties:

❶ $f(x) \ge 0 \quad \forall\, x \in X$

❷ $\sum_{all\ x} f(x) = 1$

❸ $P(x \in A) = \sum_{x \in A} P_r(x) = \sum_{x \in A} f(x)$

Example : Let X be a d.r.v (discrete random variable) with

$$f(x) = \begin{cases} \frac{1}{6} & , \quad X = 1,2,\dots,6 \\ 0 & , \quad otherwise \end{cases}$$ Is f(x) P.m.f ?

Solution

1) f(x) never negative (f(x)≥0 for all x)
2) $\sum_{all\ x} f(x) = \sum_{x=1}^{6} f(x) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$

∴ f(x) is P.m.f of x

Example : Let X be a d.r.v (discrete random variable) with

$$f(x) = \begin{cases} \frac{x}{15} & ; X = 1,2,3,4,5 \\ 0 & ; o.w.\ (other\ wise) \end{cases}$$

a) Is f(x) P.m.f of x ? b) If A={2,3} , find P(A)

Solution (a)

1) f(x) never negative (f(x)≥0 for all x)
2) $\sum_{all\ x} f(x) = \sum_{x=1}^{5} f(x) = \frac{1}{15}+\frac{2}{15}+\frac{3}{15}+\frac{4}{15}+\frac{5}{15}=\frac{15}{15}=\mathbf{1}$

∴ f(x) is P.m.f of x

Solution (b)

$$P(A)=P(x=2)+P(x=3)$$
$$=f(2)+f(3)$$
$$=\frac{2}{15}+\frac{3}{15}=\frac{5}{15}=\frac{1}{3}$$

2) Probability density function: (P.d. f.)

If the distribution of a continuous random variable(c.r.v) has a ***probability density function***, *f(x)*, then for any interval (a, b), we have $P(a \le X \le b) = \int_a^b f(x)\,dx$.

The probability density function (p.d.f.) has the following properties:

1) $f(x) \ge 0$ Everywhere;

2) $\int_{-\infty}^{+\infty} f(x)\,dx = 1 \quad [(-\infty,+\infty)\ mean\ that\ \forall\ x \in X]$.

Note: If X is a continuous r.v., then $P(X=x)=0$ for any x.

Note: As a result, we have

$P(a \le X \le b) = P(a < X < b) = P(a < X \le b) = P(a \le X < b)$.

Example : Let X be a c.r.v (continues random variable) with

$$f(x) = \begin{cases} e^{-x} & , \quad X > 0 \\ 0 & , \quad o.w. \end{cases}$$ (a) Is f(x) P.d.f of X ? (b) graph f(x)

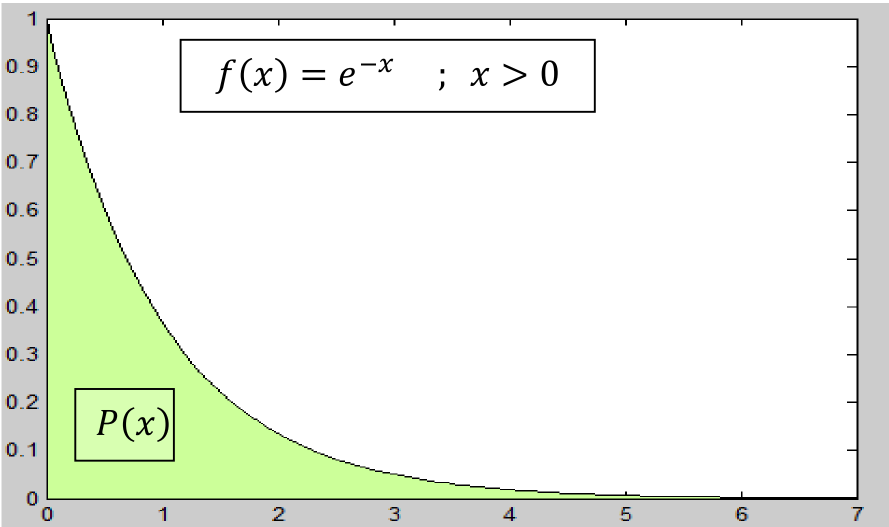
Solution (a)

1) f(x) never negative (f(x)≥0 for all x)

2) $\int_{\forall x} f(x)dx = \int_0^{\infty} e^{-x}dx = [-e^{-x}]_0^{\infty} = [e^{-\infty} - e^0] = 1$

$\therefore$ f(x) is P.d.f of x

Solution (b)



Example : Let $f(x) = \begin{cases} cx^2 & ,0 < X < 1 \\ 0 & , \quad o.w. \end{cases}$ be P.d.f of X

(a) Find value of *c* ? (b) compute $P\left(X < \frac{1}{2}\right)$ (c) compute $P\left(X > \frac{1}{2}\right)$

Solution (a)

$$\therefore \int_{\forall x} f(x)dx = 1 \Longrightarrow \int_0^1 cx^2dx = c\left[\frac{x^3}{3}\right]_0^1 = 1 \Longrightarrow c = 3$$

[(b), (c) H.W.]

Expected Value and Variance:

(I): Discrete Random Variable:

(a) Expected Value:

Examples: ***X :*** the random variable representing the point of throwing a fair die. then,

$$P(X=i)=f_x(i)=\frac{1}{6},\ i=1,2,3,4,5,6.$$

Intuitively, the average point of throwing a fair die is

$$\frac{1+2+3+4+5+6}{6}=3.5.$$

The expected value of the random variable ***X*** is just the average,

$$E(X)=\sum_{i=1}^{6} i f_x(i)=1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}=3.5=\text{average point}.$$

Formula for the expected value of a discrete random variable:

Let $a_1, a_2, \ldots, a_n, \ldots$ be all the possible values of the ***discrete random variable*** *X* and $f(x)$ is the probability distribution. Then, the expected value of the discrete random variable *X* is

$$E(X)=\mu=\sum_i a_i f(a_i)=a_1 f(a_1)+a_2 f(a_2)+\cdots+a_n f(a_n)+\cdots$$

Examples: (gambling X= -4 , 0 , 3 ; Y= -40 , 0 , 30)(i.e $Y=10.X$)

$$f(x)=\begin{cases}\frac{1}{6} & ,X\geq 0\\ \frac{2}{3} & ,X<0\end{cases}$$

Find the expected value of the random variable X is

$$E(X)=3\cdot f_x(3)+(-4)\cdot f_x(-4)+0\cdot f_x(0)=3\cdot\frac{1}{6}+(-4)\cdot\frac{2}{3}+0\cdot\frac{1}{6}=\frac{-13}{6}.$$

Therefore, on the average, the gambler will lose $\frac{-13}{6}$ for every bet.

Similarly, the expected value of the random variable Y is

$$E(Y) = 30 \cdot f_y(30) + (-40) \cdot f_y(-40) + 0 \cdot f_y(0) = 30 \cdot \frac{1}{6} + (-40) \cdot \frac{2}{3} + 0 \cdot \frac{1}{6} = \frac{-130}{6}.$$

(b) Variance:

Examples:

Suppose we want to measure the variation of the random variable ***X*** in the dice example. Then, the square distance between the values of ***X*** and its mean ***E(X)=3.5*** can be used, i.e., $(1-3.5)^2, (2-3.5)^2, (3-3.5)^2, (4-3.5)^2, (5-3.5)^2, (6\text{-}3.5)^2$ can be used. The average square distance is

$$\frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2}{6} = \frac{8.75}{3}.$$

Intuitively, large average square distance implies the values of X scatter widely.

The variance of the random variable ***X*** is just the average square distance (the expected value of the square distance). The variance for the dice example is

$$Var(X) = E\left[X - E(X)\right]^2 = E(X - 3.5)^2 = \sum_{x=1}^{6} (x-3.5)^2 f(x)$$

$$= (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5$$

$$= \frac{8.75}{3} = \text{the average square distance}$$

Formula for the variance of a discrete random variable:

Let $a_1, a_2, \ldots, a_n, \ldots$ be all the possible values of the ***discrete random variable*** *X* and $f(x)$ is the probability distribution. Let $\mu = E(X)$ be the expected value of *X*. Then, the variance of the discrete random variable *X* is

$$Var\ (X) = \sigma^2 = E\left[X - E(X)\right]^2 = \sum_i (a_i - \mu)^2 f(a_i)$$

$$= (a_1 - \mu)^2 f(a_1) + (a_2 - \mu)^2 f(a_2) + \cdots + (a_n - \mu)^2 f(a_n) + \cdots$$

Examples: (continues)

In the gambling example, the variance of the random variable ***X*** is

$$Var(X) = \left[3 - \left(\frac{-13}{6}\right)\right]^2 \cdot f_x(3) + \left[-4 - \left(\frac{-13}{6}\right)\right]^2 \cdot f_x(-4) + \left[0 - \left(\frac{-13}{6}\right)\right]^2 \cdot f_x(0)$$

$$= \left(\frac{31}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{-11}{6}\right)^2 \cdot \frac{2}{3} + \left(\frac{13}{6}\right)^2 \cdot \frac{1}{6} = 7.472$$

Similarly, the variance of the random variable ***Y*** is

$$Var(Y) = \left[30 - \left(\frac{-130}{6}\right)\right]^2 \cdot f_y(30) + \left[-40 - \left(\frac{-130}{6}\right)\right]^2 \cdot f_y(-40) + \left[0 - \left(\frac{-130}{6}\right)\right]^2 \cdot f_y(0)$$

$$= \left(\frac{310}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{-110}{6}\right)^2 \cdot \frac{2}{3} + \left(\frac{130}{6}\right)^2 \cdot \frac{1}{6} = 747.2$$

Examples:

The probability distribution function for a discrete random variable X is

$$f(x) = \begin{cases} 2k, & x = 1 \\ 3k, & x = 3 \\ 4k, & x = 5 \\ 0, & \text{otherwise} \end{cases}$$

where k is some constant. Please find

(a) value of k. (b) $P(X > 2)$ (c) $E(X)$ and $Var(X)$

Solution:

(a) $\sum_x f(x) = f(1) + f(3) + f(5) = 2k + 3k + 4k = 9k = 1$

$$\Leftrightarrow \quad k = \frac{1}{9}.$$

(b)

$$P(X > 2) = P(X = 3 \text{ or } X = 5) = P(X = 3) + P(X = 5)$$

$$= f(3) + f(5) = 3k + 4k = 7k = 7 \cdot \frac{1}{9} = \frac{7}{9}$$

(c)

$$u = E(X) = \sum_{x} xf(x) = 1\cdot f(1) + 3\cdot f(3) + 5\cdot f(5)$$
$$= 1\cdot\frac{2}{9} + 3\cdot\frac{3}{9} + 5\cdot\frac{4}{9} = \frac{31}{9}$$

and

$$Var(X) = \sum_{x} (x-u)^2 f(x)$$
$$= \left(1-\frac{31}{9}\right)^2 \cdot f(1) + \left(3-\frac{31}{9}\right)^2 \cdot f(3) + \left(5-\frac{31}{9}\right)^2 \cdot f(5)$$
$$= \frac{(-22)^2}{81}\cdot\frac{2}{9} + \frac{16}{81}\cdot\frac{3}{9} + \frac{14^2}{81}\cdot\frac{4}{9} = \frac{200}{81}$$

(II): Continuous Random Variable:

(a) Expected Value:

Examples:

Z: the random variable representing the delay flight time taking values in [0,1].

$$P(0 \le Z \le 0.5) = P(0.5 < Z \le 1) = \frac{1}{2}.$$

Then, the probability density function for Z is

$$f_2(x) = 1,\ \ 0 \le x \le 1.$$

Intuitively, since there is equal chance for any delay time in [0,1], 0.5 hour seems to be a sensible estimate of the average delay time.

The expected value of the random variable Z is just the average delay time.

$$E(Z) = \int_0^1 x f_2(x)dx = \int_0^1 x dx = \frac{x^2}{2}\Big|_0^1 = 0.5 = \text{average delay time}.$$

Formula for the expected value of a continuous random variable:

Let the ***continuous random variable*** *X* taking values in [a,b] and $f(x)$ is the probability density function. Then, the expected value of the continuous random variable *X* is

$$E(X) = \mu = \int_a^b xf(x)dx .$$

Examples: (continues)

In the flight time example, suppose the probability density function for Z is

$$f_1(x) = \frac{4}{3},\ 0 \le x \le 0.5;\quad f_1(x) = \frac{2}{3},\ 0.5 < x \le 1.$$

Then, the expected value of the random variable Z is

$$E(Z) = \int_0^1 xf_1(x)dx = \int_0^{0.5} x\cdot\frac{4}{3}dx + \int_{0.5}^1 x\cdot\frac{2}{3}dx = \frac{x^2}{2}\cdot\frac{4}{3}\Big|_0^{0.5} + \frac{x^2}{2}\cdot\frac{2}{3}\Big|_{0.5}^{1}$$

$$= \left(\frac{0.5^2}{2}\cdot\frac{4}{3} - \frac{0^2}{2}\cdot\frac{4}{3}\right) + \left(\frac{1^2}{2}\cdot\frac{2}{3} - \frac{0.5^2}{2}\cdot\frac{2}{3}\right) = \frac{5}{12}$$

Therefore, on the average, the flight time is $\frac{5}{12}$ hour.

(b) Variance:

Examples: (continues)

Suppose we want to measure the variation of the random variable Z in the flight time example. Suppose $f_2(x)$ is the probability density function for Z. Then, the square distance between the values of Z and its mean $E(Z) = \frac{1}{2}$ can be used, i.e., $\left(x - \frac{1}{2}\right)^2,\ 0 \le x \le 1$ can be used. The average square distance is

$$E\left(Z - \frac{1}{2}\right)^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 f_2(x)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) - \left(\frac{0}{3} - \frac{0}{2} + \frac{0}{4}\right) = \frac{1}{12}.$$

The variance of the random variable Z is just the average square distance (the expected value of the square distance). The variance for the flight time example is

$$Var\ (Z) = E[Z - E(Z)]^2 = E\left(Z - \frac{1}{2}\right)^2 = \frac{1}{12} = \text{the average square distance}.$$

Formula for the variance of a continuous random variable:

Let the ***continuous random variable*** *X* taking values in [a,b] and $f(X)$ is the probability distribution. Let $\mu = E(X)$ be the expected value of *X*. Then, the variance of the continuous random variable *X* is

$$Var(X) = \sigma^2 = E[X - E(X)]^2 = \int_a^b (x-u)^2 f(x)dx$$

Examples: (continues)

In the flight time example, suppose $f_1(x)$ is the probability density function for Z.

Then, the variance of the random variable ***Z*** is

$$Var(Z) = E[Z - E(Z)]^2 = \int_0^1 \left(x - \frac{5}{12}\right)^2 f_1(x)dx = \int_0^{0.5} \left(x - \frac{5}{12}\right)^2 \cdot \frac{4}{3}dx + \int_{0.5}^1 \left(x - \frac{5}{12}\right)^2 \cdot \frac{2}{3}dx$$

$$= \left(\frac{x^3}{3} - \frac{5x^2}{12} + \frac{25x}{144}\right) \cdot \frac{4}{3}\Big|_0^{0.5} + \left(\frac{x^3}{3} - \frac{5x^2}{12} + \frac{25x}{144}\right) \cdot \frac{2}{3}\Big|_{0.5}^1 = \frac{11}{144}$$

Examples:

The probability density function for a continuous random variable X is

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where a, b are some constants. Please find

(a) a, b if $E(X) = \frac{3}{5}$ (b) $Var(X)$.

Solution:

$$\text{(a)} \int_0^1 f(x)dx = 1 \Leftrightarrow \int_0^1 (a + bx^2)dx = 1 \Leftrightarrow ax + \frac{b}{3}x^3 \Big|_0^1 = 1$$

$$\Leftrightarrow a + \frac{b}{3} = 1$$

and

$$E(X)=\int_0^1 xf(x)dx=\int_0^1 x\left(a+bx^2\right)dx=\frac{a}{2}x^2+\frac{b}{4}x^4\Big|_0^1=\frac{a}{2}+\frac{b}{4}=\frac{3}{5}$$

Solve for the two equations, we have

$$a=\frac{3}{5},\ b=\frac{6}{5}.$$

(b) $$f(x)=\begin{cases}\frac{3}{5}+\frac{6}{5}x^2, & 0\le x\le 1\\ 0 & \text{, otherwise.}\end{cases}$$

Thus,

$$Var(X)=E[X-E(X)]^2=E(X^2)-[E(X)]^2=E(X^2)-\left(\frac{3}{5}\right)^2$$

$$=\int_0^1 x^2 f(x)dx-\frac{9}{25}=\int_0^1 x^2\left(\frac{3}{5}+\frac{6}{5}x^2\right)dx-\frac{9}{25}$$

$$=\frac{1}{5}x^3+\frac{6}{25}x^5\Big|_0^1-\frac{9}{25}=\frac{1}{5}+\frac{6}{25}-\frac{9}{25}=\frac{2}{25}$$

Examples:

The probability density function for a continuous random variable X is

$$f(x)=\begin{cases}\dfrac{x+2}{18} & ,-2<x<4\\ 0 & \text{, otherwise}\end{cases}$$

Find (a) $P(|X|<1)$ **(b)** $P(X^2<9)$ **(c)** $E(X)$ **and** $Var(X)$

Solution:

(a) $P(|X|<1)=P(-1<X<1)=\int_{-1}^{1}\frac{x+2}{18}dx=\left[\frac{x^2}{36}+\frac{x}{9}\right]_{-1}^{1}=\left(\frac{1}{36}+\frac{1}{9}\right)-\left(\frac{1}{36}-\frac{1}{9}\right)=\frac{2}{9}$

(b)

$$P\left(X^2 < 9\right) = P\left(-3 < X < 3\right) = \int_{-3}^{3} \frac{x+2}{18} dx = \int_{-3}^{-2} 0 dx + \int_{-2}^{3} \frac{x+2}{18} dx = \left[\frac{x^2}{36} + \frac{x}{9}\right]_{-2}^{3} = \frac{25}{36}$$

(c)

$$E(X) = \mu = \int_{-2}^{4} x \cdot \frac{(x+2)}{18} dx = \int_{-2}^{4} \left(\frac{x^2}{18} + \frac{x}{9}\right) dx = \left[\frac{x^3}{54} + \frac{x^2}{18}\right]_{-2}^{4} = 2.$$

Since

$$E\left(X^2\right) = \int_{-2}^{4} x^2 \cdot \frac{(x+2)}{18} dx = \int_{-2}^{4} \left(\frac{x^3}{18} + \frac{x^2}{9}\right) dx = \left[\frac{x^4}{72} + \frac{x^3}{27}\right]_{-2}^{4} = 6,$$

$$Var(X) = E(X-\mu)^2 = E\left(X^2\right) - \mu^2 = 6 - 2^2 = 2.$$

Examples:

The probability distribution functions (discrete random variable) or probability density functions (continuous random variable) **,Find** c for a random variable X are

(a) $$f(x) = \begin{cases} c\exp(-6x), & x > 0 \\ -cx, & -1 < x \le 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) $$f(x) = cx^2 \exp\left(-x^3\right), x > 0$$

(c) $$f(x) = c\left(\frac{1}{3}\right)^x, x = 0, 1, 2, \ldots$$

solution:

(a) $$\int_{-1}^{0} -cx dx + \int_{0}^{\infty} c\exp(-6x) dx = 1 \Leftrightarrow \left[\frac{-cx^2}{2}\right]_{-1}^{0} + \left[\frac{-c\exp(-6x)}{6}\right]_{0}^{\infty} = 1$$

$$\Leftrightarrow \frac{c}{2} + \frac{c}{6} = 1 \Leftrightarrow \frac{3c+c}{6} = 1 \Leftrightarrow c = \frac{3}{2}$$

(b) $$\int_{0}^{\infty} cx^2 \exp\left(-x^3\right) dx = 1 \Leftrightarrow c\int_{0}^{\infty} \frac{1}{3} \exp\left(-x^3\right) dx^3 = 1 \Leftrightarrow \left[\frac{-c\exp\left(-x^3\right)}{3}\right]_{0}^{\infty} = 1$$

$$\Leftrightarrow \frac{c}{3} = 1 \Leftrightarrow c = 3$$

(c)

$$\sum_{x=0}^{\infty} c\left(\frac{1}{3}\right)^{x} = 1 \Leftrightarrow c\left[1+\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\cdots\right] = 1 \Leftrightarrow c\left(\frac{1}{1-\frac{1}{3}}\right) = 1 \Leftrightarrow \frac{3c}{2} = 1$$

$$\Leftrightarrow c = \frac{2}{3}$$

Definition : The ***cumulative distribution function*** (or c.d.f.)

Discrete case: $F(x) = \sum_{u \le x} p(u)$

Continuous case: $F(x) = \int_{-\infty}^{x} f(u)du$ and $F'(x) = f(x)$.

for a continuous r.v. X is given by $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\ dx$, for all $x \in \Re$. If the distribution does not have a p.d.f., we may still define the c.d.f. for any x as the probability that X takes on a value no greater than x.

Note: The c.d.f. for the distribution of a r.v. is unique, and completely describes the distribution.

Mean and Variance

Definition: The mean, or expected value, or expectation, of a continuous r.v. X with p.d.f. f(x) is given by

$$\mu = E[X] = \int_{-\infty}^{+\infty} xf(x)dx.$$

Note:

We interpret the mean in terms of relative frequency. If we were to repeated take a measurement of the random variable X, recording all of our measurements, and calculating the average after each measurement, the value of the average would approach a limit as we continued to take measurements, and this limit is the expectation of X.

Definition: Let X be a continuous r.v. with p.d.f. f(x), and mean μ. The variance of X, or the variance of the distribution of X, is given by

$\sigma^2 = V(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx$. The standard deviation of X is just the square root of the variance.

Note: In practice, it is easier to use the computational formula for the variance, rather than the defining formula:

$$\sigma^2 = E\left[X^2\right] - \mu^2 = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2 .$$

Properties of Means and Variances

In the following, ***X*** and ***Y*** are random variables (continuous or discrete); *where a* and *b* are real constants.

① $\mu_X = E(X) = \begin{cases} \sum_x x p_X(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x f_X(x)dx & \text{continuous'} \end{cases}$

❷ $\sigma_X{}^2 = Var(X) = \begin{cases} \sum_x (x-\mu_X)^2 p_X(x) & \text{discrete} \\ \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx & \text{continuous} \end{cases}$

① E(*a*) = *a* ② E(*X* + *a*) = E(*X*) + *a* ③ E(*a.X*) = *a* .E(*X*)

❶ Var(*a*) = 0 ❷ Var(*X* +*a*) = Var(*X*) ❸ Var(*a.X*) = a^2.Var(*X*)

❹ SD(*a.X*) = |*a*| SD(*X*) (SD: means standard deviation)

① E(*X* + *Y*) = E(*X*) + E (*Y*) ② E(*a.X* + *b.Y*) = *a*.E(*X*) + *b*.E(*Y*)

If *X* and *Y* are independent:

❶Var(*X* + *Y*) = Var(*X*) + Var(*Y*) ❷Var(*a.X* + *b.Y*) = a^2.Var(*X*) + b^2.Var(*Y*)

Example 7: $X = 0, 1, 2$ with probabilities $1/4, 1/2, 1/4$.
Find $E(X)$, $E(X-1)$, $E(X^2)$ and $\text{Var}(X)$.

Answer: $E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$

$E[X-1] = E[X] - 1$, $E[X^2] = 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} = \frac{3}{2}$

$\text{Var}[X] = E[X^2] - E[X]^2 = \frac{1}{2}$.

Example 8: $f(x) = k(1+x)^{-4}$ on $(0, \infty)$. Find k and hence obtain $E(X)$, $E\{(1+X)^{-1}\}$, $E(X^2)$ and $\text{Var}(X)$.

Answer: $1 = k \int_0^\infty (1+x)^{-4} dx = k[-\frac{1}{3}(1+x)^{-3}]_0^\infty = \frac{k}{3} \Rightarrow k = 3$

$E[X] = 3\int_0^\infty x(1+x)^{-4} dx = 3\int_1^\infty (u-1)u^{-4} du = 3[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}]_1^\infty = 3(\frac{1}{2} - \frac{1}{3}) = \frac{1}{2}$

$E[(1+X)^{-1}] = 3\int_0^\infty (1+x)^{-5} dx = 3[-\frac{1}{4}(1+x)^{-4}]_0^\infty = \frac{3}{4}$

$E[X^2] = 3\int x^2(1+x)^{-4} dx = 3\int_1^\infty (u-1)^2 u^{-4} du = 3[-u^{-1} + u^{-2} - \frac{1}{3}u^{-3}]_1^\infty = 1$

$\text{Var}[X] = E[X^2] - E[X]^2 = \frac{3}{4}$.

Example

Verify that $f(x) = 2x \quad (0 \le x \le 1)$ is a legitimate probability density function ?

find $\text{P}\left[-\frac{1}{2} < X < \frac{1}{2}\right]$.

Note that, by default, $f(x) = 0$ for all values of x not mentioned in the definition.

On $0 \le x \le 1$, $f(x) = 2x \ge 0$. Elsewhere $f(x) = 0$. $\therefore f(x) \ge 0 \quad \forall x$.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 2x\, dx + \int_{1}^{\infty} 0\, dx = 0 + \left[x^2\right]_0^1 + 0 = 1$$

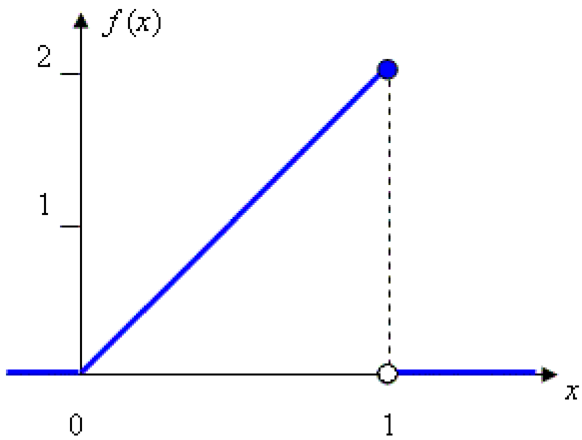
OR:

The total area under the graph of $f(x)$

= (area of the triangle, width 1, height 2)

= ½ (1)(2) = 1

Therefore $f(x)$ is a valid *p.d.f.*

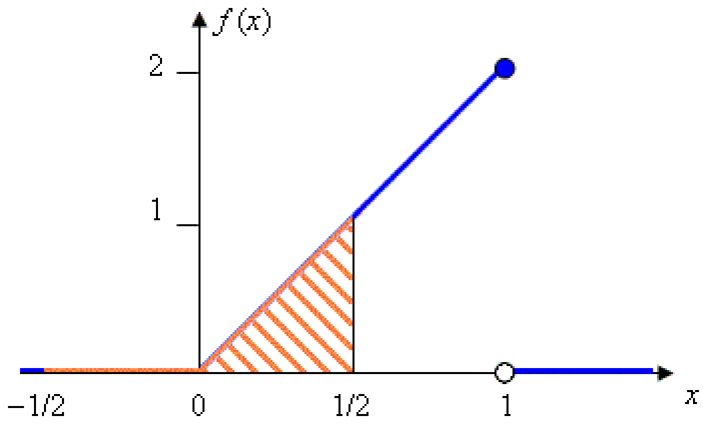


$P\left[-\frac{1}{2} < X < \frac{1}{2}\right]$ = area under $f(x)$

between $X = -½$ and $X = ½$

= (area of triangle, width ½, height 1)

= ½ (½) 1 = ¼ .



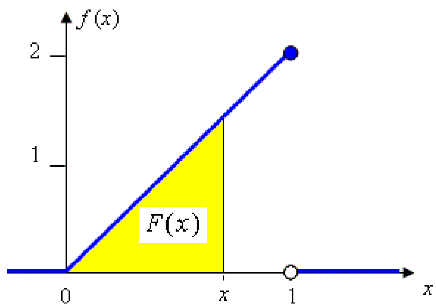
OR:

$$P\left[-\frac{1}{2} < X < \frac{1}{2}\right] = \int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{0} 0\,dx + \int_{0}^{1/2} 2x\,dx = 0 + \left[x^2\right]_0^{1/2} = \mathbf{\frac{1}{4}}$$

The cumulative distribution function (*c.d.f.*)

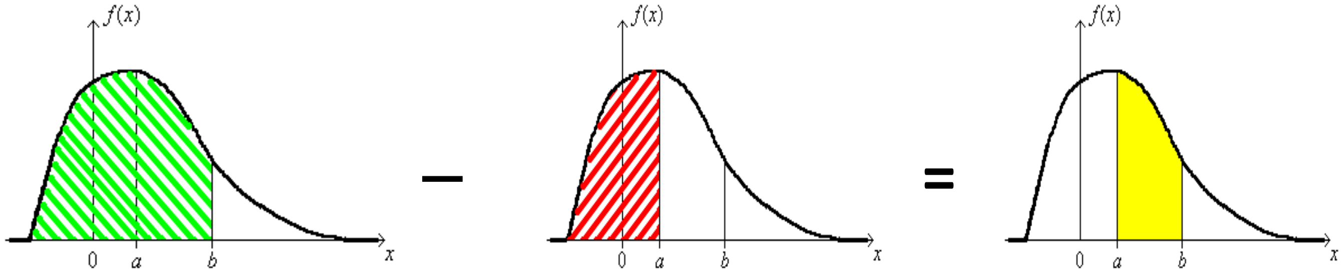
is defined by

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\,dt$$



Note :

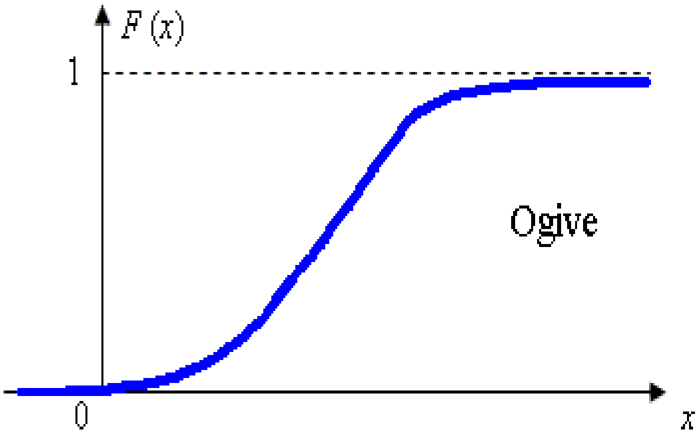
$$\boxed{P[a < X < b] \;=\; F(b) - F(a)}$$



$F(-\infty) = \mathbf{0}$ $F(+\infty) = \mathbf{1}$ $0 \le F(x) \le 1$ for all x.

The *c.d.f.* is a non-decreasing function of x and $\frac{d}{dx}(F(x)) = f(x) \ge 0 \quad \forall x$.

[Note : Many *c.d.f.s* like this:]



<u>Example</u> (continued)

Find the cumulative distribution function for $f(x) = 2x \quad (0 \le x \le 1)$.

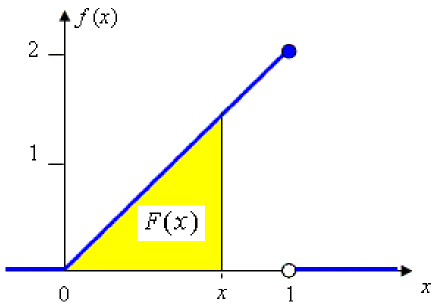
[Note that $f(x)$ is assumed to be zero for any x not mentioned in the definition]

<u>Graphical method</u>:

$x < 0 \quad \Rightarrow \quad F(x) = \mathbf{0}$

$0 \le x \le 1 \quad \Rightarrow \quad F(x) = \tfrac{1}{2}(x)(2x) = x^2$

$x > 1 \quad \Rightarrow \quad F(x) = \tfrac{1}{2}(1)(2) = 1$



<u>Calculus method</u>: $F(x) = \int_{-\infty}^{x} f(t)\, dt$

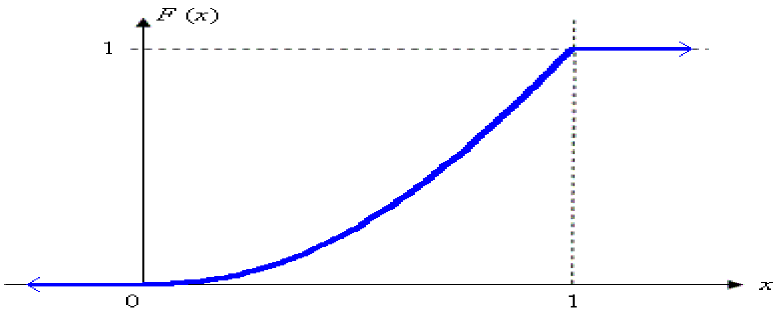
$x < 0 \quad \Rightarrow \quad F(x) = \int_{-\infty}^{x} 0\, dt = 0$

$0 \le x \le 1 \quad \Rightarrow \quad F(x) = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} 2t\, dt = F(0) + \left[t^2\right]_0^x$

$= 0 + \left(x^2 - 0\right) = x^2$

$x > 1 \quad \Rightarrow \quad F(x) = \int_{-\infty}^{1} f(t)\, dt + \int_{1}^{x} 0\, dt = F(1) + 0 = 1^2 = 1$

$$\therefore \quad F(x) = \begin{cases} 0 & (x < 0) \\ x^2 & (0 \le x \le 1) \\ 1 & (x > 1) \end{cases}$$



Note how the *c.d.f.* is a non-decreasing continuous function between $F = 0$ and $F = 1$.

Note :

For each integer n, the *nth moment of X*, μ_n, is $\mu_n = E(X^n)$.

And the ***nth central moment of*** *X* is $\sigma_n = E(X - \mu)^n$.

- Mean = 1st moment of X and 1st central moment of X is equal to 0
- Variance = 2nd central moment of X

The ***Moment generating function (mgf)*** *of X* is defined by $M_X(t) = E(e^{tX})$.

NOTE:

The nth moment is equal to the nth derivative of the ***mgf*** evaluated at t=0.

$$i.e.\ \ M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0} = E(x^n).$$

Useful relationship: $\boxed{\text{Binomial Formula is } \sum_{x=0}^{n} \binom{n}{x} u^x v^{n-x} = (u+v)^n}$.

Example :Find the p.d.f of Y where Y=X^2 if X a r.v. with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2} ; & -1 < X < 1 \\ 0 & ; o.w. \end{cases}$$

Solution :

Let G(y) is c.d.f. of Y where $0 \le Y < 1$

Then $G(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}dx = \frac{1}{2}\left[x\right]_{-\sqrt{y}}^{\sqrt{y}}$$

$$= \frac{1}{2}\left[\sqrt{y} + \sqrt{y}\right] = \sqrt{y} \quad \left\{ i.e.\ G(y) = y^{\frac{1}{2}} \right\}$$

$$\therefore \quad G(y) = \begin{cases} 0 & ; y \le 0 \\ \sqrt{y} & ; \ 0 < y < 1 \\ 1 & ; \ y \ge 1 \end{cases}$$

$$\because \quad g(y) = G'(y) \Longrightarrow g(y) = \frac{1}{2\sqrt{y}}$$

$$or \qquad g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & ; \ 0 < y < 1 \\ 0 & ; o.w. \end{cases}$$

Example : If X a r.v. is the number of heads obtained in four tosses of a coins,

Find the p.m.f (pdf) of y where $y = \frac{1}{1+x}$.

Solution :

S={ HHHH,HHHT,HHTH,HTHH,THHH,TTHH,THHT,THTH,HHTT,HTTH,HTHT,TTTH, TTHT, THTT, HTTT, TTTT } , N(S) $=2^4 = 16$

Then X= { 0, 1, 2, 3, 4 }

$$f(x) = \begin{cases} \dfrac{1}{16} & ; x = 0 \\ \dfrac{4}{16} = \dfrac{1}{4} & ; x = 1 \\ \dfrac{6}{16} = \dfrac{3}{8} & ; x = 2 \\ \dfrac{4}{16} = \dfrac{1}{4} & ; x = 3 \\ \dfrac{1}{16} & ; x = 4 \end{cases}$$

Or $$f(x) = \frac{\binom{4}{x}}{16} = \frac{3}{2\,X!\,(4-x)!} \qquad , x = 0, 1, 2, 3, 4$$

$$\because \quad y = \frac{1}{1+x} \quad \Longrightarrow \quad y = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

$$\because \quad y = \frac{1}{1+x} \Longrightarrow y(1+x) = 1 \Longrightarrow y + xy = 1 \Longrightarrow x = \frac{1-y}{y}$$

$$\therefore \quad g(y) = P(Y = y) = P\left(X = \frac{1-y}{y}\right) \;; \; y = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

Or

$$g(y)=\begin{cases}\frac{1}{16} & ;y=1\\ \frac{4}{16}=\frac{1}{4} & ;y=\frac{1}{2}\\ \frac{6}{16}=\frac{3}{8} & ;y=\frac{1}{3}\\ \frac{4}{16}=\frac{1}{4} & ;y=\frac{1}{4}\\ \frac{1}{16} & ;y=\frac{1}{5}\\ 0 & ;otherwise\end{cases}$$

The Mode and the median of the distribution:

A ***mode*** of distribution of one r.v. X of the continuous or discrete type is a value of x that maximizes the *f(x),* if there is only one such x if is called the mode of the distribution where $f'(x)=0$.

A ***median*** of distribution of one r.v. X of the continuous or discrete type is a value of $\boldsymbol{x}$ such that $P(X<x)\le\frac{1}{2}$ and $P(X\le x)\ge\frac{1}{2}$ if there is only one such $\boldsymbol{x}$ it called the median of the distribution [*i.e.* $\mathrm{F}(x)=\frac{1}{2}$]

Example : Find the mode of the distribution if X having the p.d.f.

$$f(x)=\begin{cases}\frac{1}{2}x^2e^{-x} & ;0< X<\infty\\ 0 & ;\ o.w.\end{cases}$$

Solution:

$f'(x)=\frac{1}{2}[-x^2e^{-x}+2xe^{-x}]$, set $f'(x)=0$.

$\Rightarrow \frac{1}{2}[-x^2e^{-x}+2xe^{-x}]=0 \Rightarrow x^2e^{-x}=2xe^{-x} \Rightarrow x=2$

$f''(x)=\frac{1}{2}[x^2e^{-x}-2xe^{-x}-2xe^{-x}+2e^{-x}] \Rightarrow f''(2)=\frac{1}{2}[-8e^{-2}+6e^{-2}]$

$\Rightarrow f''(2)=-2e^{-2}<0$

Then the mode of dist. is 2

Example :Find the mode of the distribution if X a r.v with p.d.f.

$$f(x) = \begin{cases} \frac{2^x e^{-2}}{x!} & ; X = 0, 1, 2, 3, \dots \dots \\ 0 & ; \ o.w. \end{cases}$$

Solution:

x	0	1	2	3	4	……
f(x)	e^{-2}	$2e^{-2}$	$2e^{-2}$	$\frac{4}{3}e^{-2}$	$\frac{16}{24}e^{-2}$	……

Then the mode is 1 and 2 because the f(x) are maximum at x= 1 and 2

Or f(0) < f(1) = f(2) > f(3) > f(4) > ……

Example :Find the median of the distribution if X having the following p.d.f.

$$f(x) = \begin{cases} 3x^2 & ; 0 < X < 1 \\ 0 & ; \ o.w. \end{cases}$$

Solution:

$$F(x) = \int_0^x f(u)du = \int_0^x 3u^2 du = [u^3]_0^x = x^3$$

Put $F(x) = \frac{1}{2}$

$$\Rightarrow x^3 = \frac{1}{2} \quad \Rightarrow \quad x = \sqrt[3]{\frac{1}{2}}$$

Then the median of dist. is $\sqrt[3]{\frac{1}{2}}$

Example : Find the **m.g.f.** and $\boldsymbol{\partial_x^2}$ for the r.v. X which having the p.d.f.

$$f(x)=\begin{cases} e^{-2x} & ; 0< X<\infty \\ 0 & ; o.w. \end{cases}$$

Solution:

$$M_x(t)=E(e^{tx}) = \int_0^\infty e^{tx}\, f(x)dx = \int_0^\infty e^{tx}\, .\, e^{-2x}\, dx$$

$$= \int_0^\infty e^{(t-2)x}\, dx = \frac{1}{(t-2)}\left[e^{(t-2)x}\right]_0^\infty = \frac{1}{(t-2)}(e^{-\infty}-e^0)$$

$$\therefore\ M_x(t)=\frac{1}{(t-2)}(0-1)=\frac{1}{(2-t)}$$

$$M_x'(t)=\frac{1}{(t-2)^2} \Longrightarrow M_x'(0)=\frac{1}{(0-2)^2}=\frac{1}{4}=E(x)$$

$$M_x''(t)=\frac{2}{(2-t)^3} \Longrightarrow M_x''(0)=\frac{2}{(2-0)^3}=\frac{1}{4}=E(x^2)$$

$$\therefore\ \partial_x^2=E(x^2)-[E(x)]^2 \Longrightarrow \partial_x^2=\frac{1}{4}-\frac{1}{16}=\frac{3}{16}$$

Theorem : Let X be a r.v. with P.d.f. f(x) and M.g.f. $M_x(t)$ then

① If y=ax+b then $M_y(t)=e^{bt}\, M_x(at)$ where a and b are constant

② If $Z=\frac{x-\mu}{\sigma_x}$ then $M_z(t)=e^{-\frac{\mu}{\sigma_x}t}\, M_x(\frac{t}{\sigma_x})$ where μ is mean and σ_x is S.D=$\sqrt{var(x)}$

Proof ① and ② are **H.W.**

Example: Let X be a r.v with $M_x(t)=(0.6+0.4e^t)^8$

① If y=3x+2 , find $M_y(t)$ ② If $Z=\frac{x-\mu}{\sigma}$, find $M_z(t)$

Solution :

① $\because M_y(t)=e^{bt}\, M_x(at)$, a=3 and b=2

$\therefore\ M_y(t) = e^{2t}\,(0.6+0.4e^{3t})^8$

② $Z = \dfrac{x-\mu}{\sigma}$ $\quad \mu = E(x) = M'_x(0) = 3.2$

Note that $M'_x(t) = 8(0.6+0.4e^t)^7 \times 0.4e^t = 3.2 \times e^t(0.6+0.4e^t)^7$

$$\sigma_x^2 = var(x) = E(x^2) - E(x)^2 = M''_x(0) - M'_x(0)^2$$

Note that $M''_x(t) = 8.96(0.6+0.4e^t)^6\, e^{2t} + 3.2(0.6+0.4e^t)^7\, e^t$

$\therefore\ M''_x(0) = 8.96 + 3.2 = 12.16 \Rightarrow \sigma = \sqrt{var(x)} = \sqrt{12.16 - 10.24} = \sqrt{1.92} = 1.386$

$\therefore\ Z = \dfrac{x-3.2}{1.386} \Rightarrow M_z(t) = e^{-\frac{3.2}{1.386}t}\, M_x\!\left(\frac{t}{1.386}\right) = e^{-\frac{3.2}{1.386}t}\left(0.6 + 0.4e^{t/1.386}\right)^8$

Characteristic function $[\emptyset_x(it)]$

Let X be a r.v. with P.d.f. f(x) , then the characteristic function denoted by $\emptyset_x(it)$ and defined as

$$\emptyset_x(it) = E\left(e^{ixt}\right) = \begin{cases} \sum_{\forall x} e^{ixt} f(x) & ; x \text{ is } d.r.v. \\ \int_{\forall x} e^{ixt} f(x)dx & ; x \text{ is } c.r.v. \end{cases} \qquad \text{such that } i = \sqrt{-1}$$

Note : ❶ $\emptyset_x(0) = 1$ ❷ $\emptyset'_x(0) = iE(x)$ ❸ $\emptyset''_x(0) = -E(x^2)$

❹ var(x)= $[i\emptyset'_x(0)]^2 - \emptyset''_x(0)$ ❺ we can write $\emptyset_x(it)$ *as* $\emptyset_x(t)$

{Proof H.W.}

Theorem : Let X be a r.v. with P.d.f. f(x) and $\emptyset_x(t)$ **is** characteristic function then

① If y=ax+b then $\emptyset_y(t) = e^{ibt}\,\emptyset_x(at)$ where a and b are constant

② If $Z=\frac{x-\mu}{\sigma_x}$ then $\emptyset_z(t)=e^{-\frac{\mu}{\sigma_x}it}\,\emptyset_x(\frac{t}{\sigma_x})$ where μ is mean and σ_x is S.D=$\sqrt{var(x)}$

Proof ① and ② are **H.W.**

Example : Let X be a r.v. with P.d.f.

$$f(x)=\begin{cases} e^{-x} & ;0<x<\infty \\ 0 & ;o.w. \end{cases} \quad , \text{ find } \emptyset_x(t)$$

Solution :

$$\emptyset_x(t)=\emptyset_x(it)=E(e^{ixt})=\int_0^\infty e^{ixt}e^{-x}dx$$

$$=\int_0^\infty e^{-(1-it)x}dx=\frac{-1}{1-it}\left[e^{-(1-it)x}\right]_0^\infty$$

$$=\frac{-1}{1-it}\left[e^{-\infty}-e^{0}\right]=\frac{-1}{1-it}(0-1)=\frac{1}{1-it}$$

Question: Let X be a r.v. with P.d.f.

$$f(x)=\begin{cases} a\left(\frac{2}{3}\right)^x & ;x=1,2,3,4,\dots\dots\dots \\ 0 & ;o.w. \end{cases} \quad , \text{ find the value of a}$$

Question: Let X be a r.v. with

$$f(x)=\begin{cases} \frac{\sin x}{2} & ;0<x<\pi \\ 0 & ;o.w. \end{cases} \quad , \text{Answer that:}$$

1) Is f(x) P.d.f. of x ?
2) $compute\ P(x>\frac{\pi}{3})$?
3) find cumulative distribution function ?
4) if $P(x\geq m)=\frac{1}{3}$ $find\ the\ value\ of\ m$?
5) find M.g.f and Ch.f ?

Joint Probability Distribution (Mass, Density) Function:

- The joint probability distribution function is

$$f(x_1, x_2) = P(X_1 = x_1, X_2 = x_2),$$

where x_1 and x_2 are possible values of X_1 and X_2, respectively.

- Let X_1 and X_2 be *discrete* random variables. The joint probability mass function satisfies the following conditions:

1. $f(x_1, x_2) \geq 0$ for all x_1, x_2;
2. $\sum_{\forall x_2} \sum_{\forall x_1} f(x_1, x_2) = 1$.

- Let X_1 and X_2 be *continuous* random variables. The joint probability density function satisfies the following conditions:

1. $f(x_1, x_2) \geq 0$ for $-\infty < x_1 < \infty, -\infty < x_2 < \infty$;
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 dx_2 = 1$.

Joint Cumulative Distribution Function:

- For any random variables X_1 and X_2, the joint (cumulative) distribution function is given by

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2).$$

- As X_1 and X_2 are *discrete,*

$$P(a \leq X_1 \leq b, c \leq X_2 \leq d) = \sum_{c \leq x_2 \leq d} \sum_{a \leq x_1 \leq b} f(x_1, x_2).$$

- As X_1 and X_2 are *continuous,*

$$P(a \leq X_1 \leq b, c \leq X_2 \leq d) = \int_c^d \int_a^b f(x_1, x_2) dx_1 dx_2.$$

- For any random variables X_1 and X_2 with joint cumulative distribution function $F(x_1, x_2)$,

1. $F(-\infty,-\infty)=F(-\infty,x_2)=F(x_1,-\infty)=0;$

2. $F(\infty,\infty)=1;$

3. if $b\geq a, d\geq c$, then

$$P(a\leq X_1\leq b, c\leq X_2\leq d)=F(b,d)-F(b,c)-F(a,d)+F(a,c).$$

Example 1:Form a collection of 3 white balls, 2 black balls, and 1 red ball, 2 balls is to be randomly selected. Let X_1 denote the number of white balls and X_2 the number of black balls.

(a) Find the joint probability distribution table of X_1 and X_2.
(b) $F(1,1)$ and $F(2,0)$.

solution:

(a) The joint probability distribution table is

x_2 \ x_1	0	1	2	Total
0	0	$3/15$	$3/15$	$6/15$
1	$2/15$	$6/15$	0	$8/15$
2	$1/15$	0	0	$1/15$
Total	$3/15$	$9/15$	$3/15$	1

Or

$$P(X_1=i, X_2=j)=f(i,j)=\frac{\binom{3}{i}\binom{2}{j}\binom{1}{2-i-j}}{\binom{6}{2}}, 0\leq i, j\leq 2; i+j=1 \text{ or } 2;$$

For example,

$$P(X_1=1, X_2=1)=f(1,1)=\frac{\binom{3}{1}\binom{2}{1}\binom{1}{0}}{\binom{6}{2}}=\frac{3\cdot 2}{15}=\frac{6}{15}.$$

(b) $F(1,1)=P(X_1 \le 1, X_2 \le 1)= f(0,0)+ f(0,1)+ f(1,0)+ f(1,1)=\dfrac{0+2+3+6}{15}=\dfrac{11}{15}$.

$$F(2,0)=P(X_1 \le 2, X_2 \le 0)= f(0,0)+ f(1,0)+ f(2,0)=\frac{0+3+3}{15}=\frac{6}{15}$$

Example 2:

Let $f(x_1,x_2)=2x_1 \quad ,0\le x_1 \le k; \quad 0\le x_2 \le 1;$

$=0, \text{otherwise.}$ be a j.p.d.f of x_1 and x_2

(a) Find k.

(b) Find $F(0.7,0.5)$, $F(2,0)$ and $F(0.2,3)$.

solution:

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)dx_1dx_2 = \int_0^1\int_0^k 2x_1dx_1dx_2 = \int_0^1 \left[x_1^2\right]_0^k dx_2 = \int_0^1 k^2 dx_2 = k^2 = 1 \Leftrightarrow k=\pm 1.$$

$\Rightarrow k=1$.

(b)

$$F(0.7,0.5)=P(X_1 \le 0.7, X_2 \le 0.5)=\int_{-\infty}^{0.5}\int_{-\infty}^{0.7} f(x_1,x_2)dx_1dx_2 = \int_0^{0.5}\int_0^{0.7} 2x_1dx_1dx_2$$

$$=\int_0^{0.5}\left[x_1^2\right]_0^{0.7} dx_2 = \int_0^{0.5} 0.49dx_2 = 0.49\cdot 0.5 = 0.245$$

$$F(2,0)=P(X_1 \le 2, X_2 \le 0)=\int_{-\infty}^{0}\int_{-\infty}^{2} f(x_1,x_2)dx_1dx_2 = \int_0^{0}\int_0^{1} 2x_1dx_1dx_2$$

$$=\int_0^0 \left[x_1^2\right]_0^1 dx_2 = \int_0^0 1dx_2 = 1\cdot 0 = 0$$

$$F(0.2,3)=P(X_1 \le 0.2, X_2 \le 3)=\int_{-\infty}^{3}\int_{-\infty}^{0.2} f(x_1,x_2)dx_1dx_2 = \int_0^{1}\int_0^{0.2} 2x_1dx_1dx_2$$

$$=\int_0^1 \left[x_1^2\right]_0^{0.2} dx_2 = \int_0^1 0.04dx_2 = 0.04$$