Definition of Laplace Transform

Definition

If $f(t)$ is defined for $t \geq 0$, then

$$\int_0^\infty K(s,t) f(t)\,dt = \lim_{b \to \infty} \int_0^b K(s,t) f(t)\,dt \qquad (1)$$

Definition: Laplace Transform

If $f(t)$ is defined for $t \geq 0$, then

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad (2)$$

is said to be the **Laplace Transform** of f.

Example 1:

Evaluate $L\{1\}$

Solution:

Here we keep that the bounds of integral are 0 and ∞ in mind.

From the definition

$$\mathsf{L}(1) = \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= \lim_{b\to\infty}\frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

Since $e^{-st} \to 0$ as $t \to \infty$, for $s > 0$.

Evaluate $L\{t\}$

Solution

$$L\{t\} = \frac{-te^{-st}}{s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt$$

$$= \frac{1}{s}L\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Evaluate $L\{e^{3t}\}$

Solution

$$L\{e^{-3t}\} = \int_0^\infty e^{-st} e^{-3t} dt = \int_0^\infty e^{-(s+3)t} dt$$

$$= \frac{-e^{(s+3)t}}{s+3} \Big|_0^\infty$$

$$= \frac{1}{s+3}, \; s > -3$$

Example 4:

Evaluate $L\{\sin 2t\}$

Solution

$$L\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t \; dt$$

$$= \frac{-e^{-st} \sin 2t}{s} \Big|_0^\infty + \frac{2}{s} \int_0^\infty e^{-st} \cos 2t \; dt$$

$$= \frac{2}{s} \int_0^\infty e^{-st} \cos 2t \; dt \;, \; s > 0$$

Example 4 "cont":

$$\lim_{t \to \infty} e^{-st} \cos 2t = 0 \, , \, s > 0$$

Laplace transform of $\sin 2t$

$$= \frac{2}{s} \left[\left. \frac{-e^{-st} \cos 2t}{s} \right|_0^\infty - \frac{2}{s} \int_0^\infty e^{-st} \sin 2t \ dt \right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2} \, \mathsf{L} \left\{ \sin 2t \right\}$$

$$\mathsf{L} \left\{ \sin 2t \right\} = \frac{2}{s^2 + 4} \, , \, s > 0$$

L.T. is Linear

❖ We can easily verify that

$$L\{\alpha f(t) + \beta g(t)\}$$
$$= \alpha L\{f(t)\} + \beta L\{g(t)\} \tag{3}$$
$$= \alpha F(s) + \beta G(s)$$

Transform of Some Basic Functions

$$L\{1\} = \frac{1}{s}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} , \quad n \in \mathbb{N}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{\sin k t\} = \frac{k}{s^2 + k^2}$$

$$L\{\cos k t\} = \frac{s}{s^2 + k^2}$$

$$L\{\sinh k t) = \frac{k}{s^2 - k^2}$$

$$L\{\cosh k t\} = \frac{s}{s^2 - k^2}$$

Exponential Order

Definition

A function $f(t)$ is said to be of ***exponential order***,

if there exists constants c, $M > 0$, and $T > 0$, such

that

$$|f(t)| \le Me^{ct} \quad \text{for all } t > T.$$
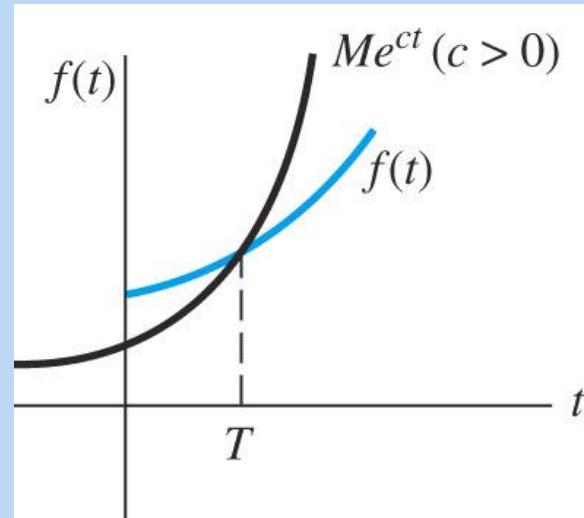
Functions with colored graphs are of exponential order

$$|t| \leq e^t$$

$$\left|e^{-t}\right| \leq e^t$$

$$|2\cos t| \leq 2e^t$$

(a)

(b)

(c)
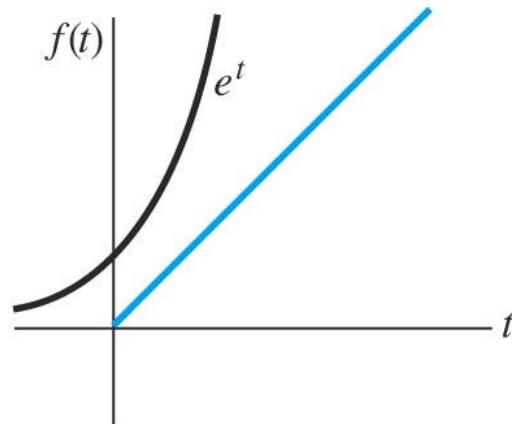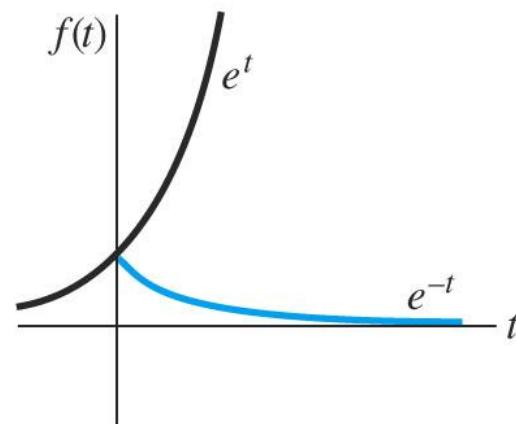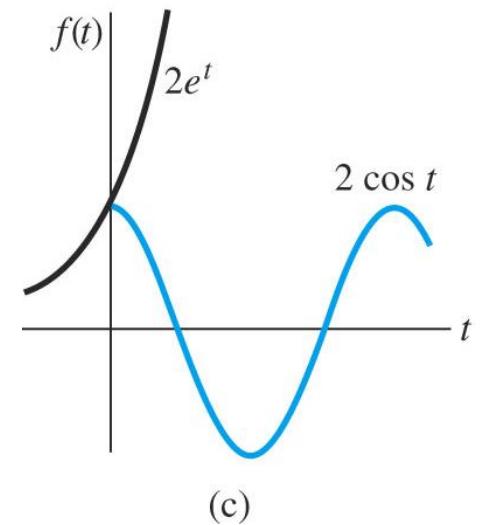
❖ A function such as e^{t^2} is not of exponential order,
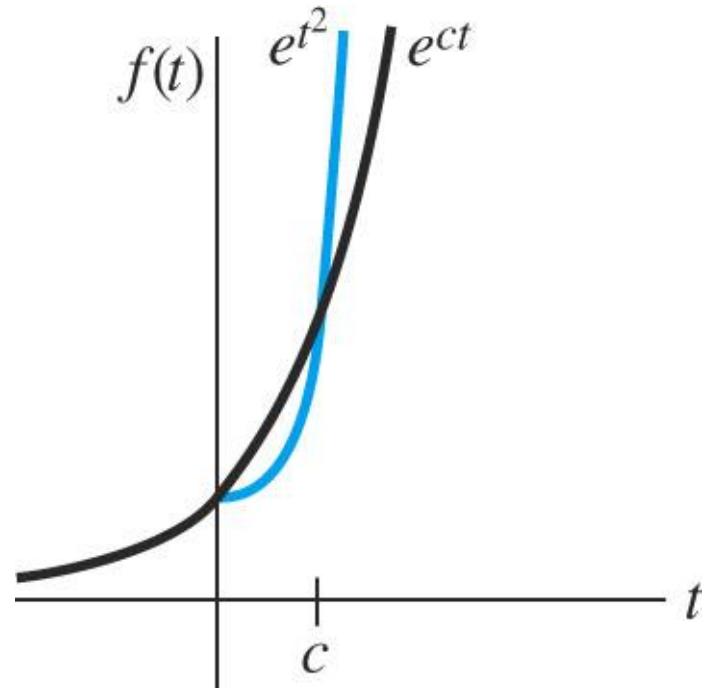
Sufficient Conditions for Existence

Theorem

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order c, then $L\{f(t)\}$ exists for $s > c$.

Example 5:

Find $L\{f(t)\}$ for
$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

Solution

$$L\{f(t)\} = \int_0^3 e^{-st} 0\,dt + \int_3^\infty e^{-st} 2\,dt$$

$$= \frac{-2e^{-st}}{s}\bigg|_3^\infty = \frac{2e^{-3s}}{s}, s > 0$$

The Inverse Transform and Transform of Derivatives

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^{n} \qquad n \in \mathbb{N}$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L^{-1}\left\{\frac{k}{s^2 + k^2}\right\} = \sin k\,t$$

$$L^{-1}\left\{\frac{s}{s^2 + k^2}\right\} = \cos k\,t$$

$$L^{-1}\left\{\frac{k}{s^2 - k^2}\right\} = \sinh k\,t$$

$$L^{-1}\left\{\frac{s}{s^2 - k^2}\right\} = \cosh k\,t$$

Find the inverse transform of

$$a)\ L^{-1}\left\{\frac{1}{s^5}\right\} \qquad\qquad b)\ L^{-1}\left\{\frac{1}{s^2+7}\right\}$$

Solution

$$a)\ L^{-1}\left\{\frac{1}{s^5}\right\}=\frac{1}{4!}L^{-1}\left\{\frac{4!}{s^5}\right\}=\frac{1}{24}t^4$$

$$b)\ L^{-1}\left\{\frac{1}{s^2+7}\right\}=\frac{1}{\sqrt{7}}L^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\}=\frac{1}{\sqrt{7}}\sin\sqrt{7}t$$

L⁻¹ is also linear

❖ We can easily verify that

$$L^{-1}\{\alpha F(s) + \beta G(s)\}$$

$$= \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\}$$

Example 2:

Find $\mathsf{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$

Solution

$$L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = L^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\}$$

$$= -2L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t$$

Example 3:

Find $L^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

Solution

Using partial fractions

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

Then

$$s^2+6s+9$$

$$=A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

If we set $s=1,\ 2,\ -4$, then

Example 3 "cont":

$$A = -16/5, \; B = 25/6, \; c = 1/30 \qquad (4)$$

Thus

$$L^{-1}\left\{ \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \right\}$$

$$= -\frac{16}{5} L^{-1}\left\{ \frac{1}{s-1} \right\} + \frac{25}{6} L^{-1}\left\{ \frac{1}{s-2} \right\} + \frac{1}{30} L^{-1}\left\{ \frac{1}{s+4} \right\}$$

$$= -\frac{16}{5} e^t + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t} \qquad (5)$$

Transform of Derivatives

❖ $L\{f'(t)\}$

$$= \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + sL\{f(t)\}$$

$$L\{f'(t)\} = sF(s) - f(0) \tag{6}$$

❖ $L\{f''(t)\}$

$$= \int_0^\infty e^{-st} f''(t)dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt$$

$$= -f'(0) + sL\{f'(t)\}$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0) \tag{7}$$

$$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0) \tag{8}$$

Transform of a Derivative

If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise-continuous on $[0, \infty)$, then

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where
$$F(s) = L\{f(t)\}.$$

Solving Linear ODEs

❖ $a_n \dfrac{d^n y}{dt^n} + a_{n-1} \dfrac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t)$

$y(0) = y_0 \, , \, y'(0) = y_1 \, , \, y^{(n-1)}(0) = y_{n-1}$

Then

$a_n L\left\{\dfrac{d^n y}{dt^n}\right\} + a_{n-1} L\left\{\dfrac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0 L\{y\} = L\{g(t)\}$ (9)

$a_n [s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)]$

$\quad + a_{n-1}[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0)]$

$\quad + \cdots + a_0 Y(s)$ (10)

$= G(s)$

Solving Linear ODEs

We have $P(s)Y(s) = Q(s) + G(s)$

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \qquad (11)$$

where $P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$

Strategy of Solving Linear ODEs

Find unknown $y(t)$ that satisfies a DE and initial conditions → **Apply Laplace transform L** → **Transformed DE becomes an algebraic equation in $Y(s)$**

Solution $y(t)$ of original IVP ← **Apply inverse transform L^{-1}** ← **Solve transformed equation for $Y(s)$**
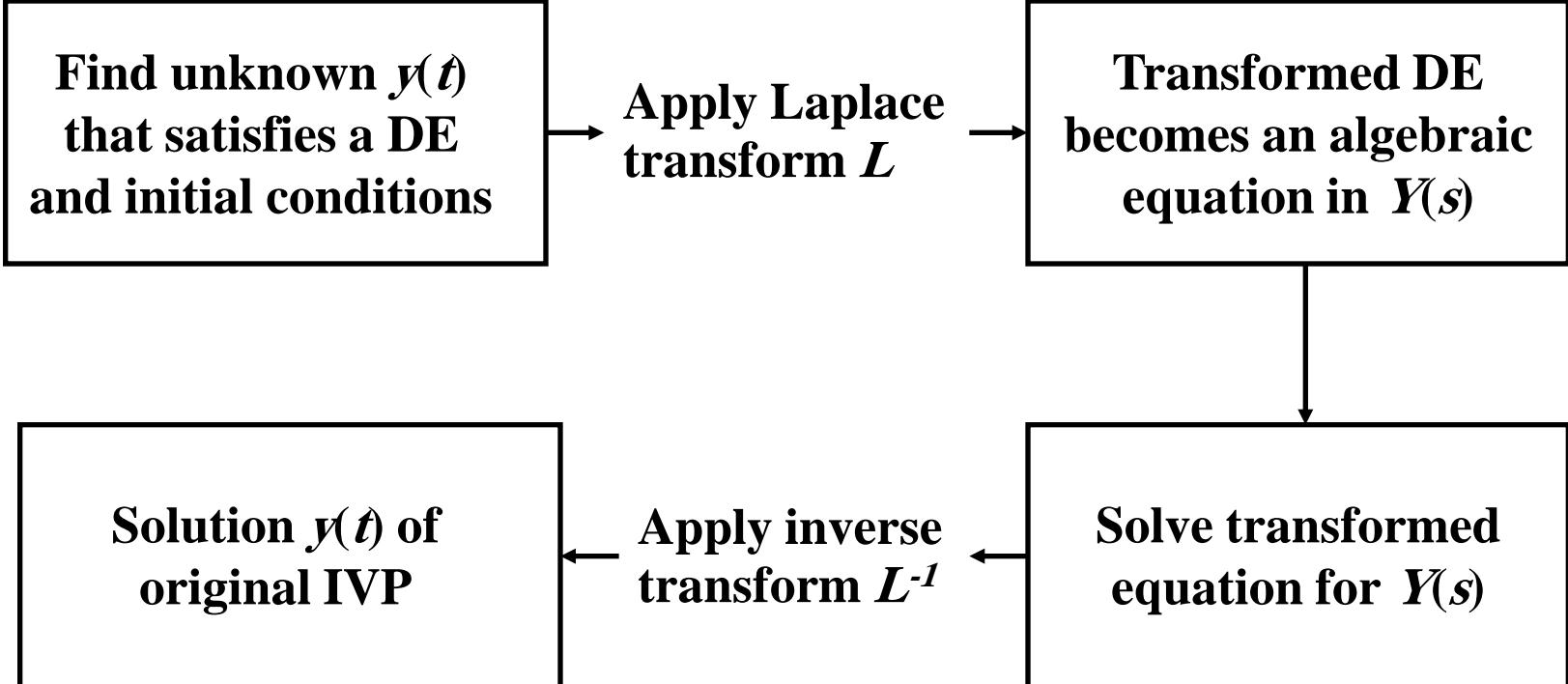
Example 4:

Solve $\dfrac{dy}{dt} + 3y = 13\sin 2t$, $y(0) = 6$

Solution

$$L\left\{\dfrac{dy}{dt}\right\} + 3L\{y\} = 13L\{\sin 2t\} \qquad (12)$$

$$sY(s) - 6 + 3Y(s) = \dfrac{26}{s^2 + 4}$$

$$(s + 3)Y(s) = 6 + \dfrac{26}{s^2 + 4}$$

$$Y(s) = \dfrac{6}{s + 3} + \dfrac{26}{(s + 3)(s^2 + 4)} = \dfrac{6s^2 + 50}{(s + 3)(s^2 + 4)} \qquad (13)$$

Example 4 "cont":

$$\frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$6s^2 + 50 = A(s^2+4) + (Bs+C)$$

We can find $A = 8, \; B = -2, \; C = 6$

Thus

$$Y(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = 8L^{-1}\left\{\frac{1}{s+3}\right\} - 2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

Example 5:

Solve $\quad y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

Solution

$$L\left\{\frac{d^2 y}{dt^2}\right\} - 3L\left\{\frac{dy}{dt}\right\} + 2L\{y\} = L\{e^{-4t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \quad (14)$$

Thus

$$y(t) = L^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

If $L\{f\} = F(s)$ and a is any real number, then

$$L\{e^{at}f(t)\} = F(s-a).$$

Proof

$$L\{e^{at}f(t)\} = \int e^{-st}e^{at}f(t)\,dt$$

$$= \int e^{-(s-a)t}f(t)\,dt = F(s-a)$$

$$L\{e^{at}f(t)\} = L\{f(t)\}_{s \to s-a}$$
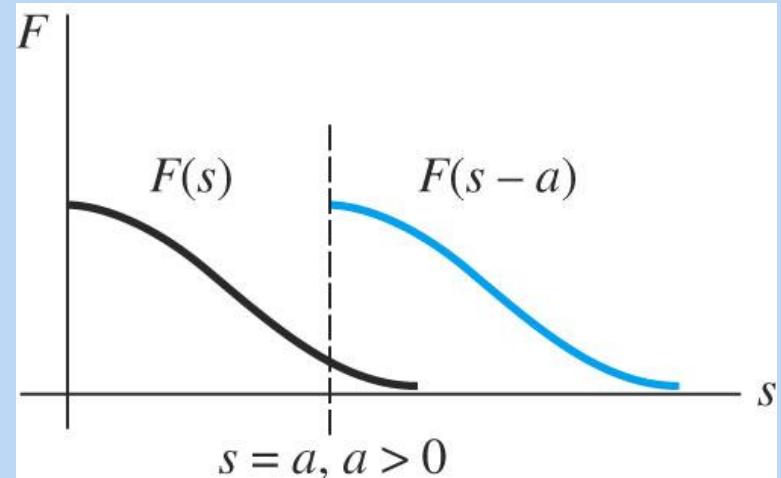
Example 1:

Find the L.T. of

(a) $L\{e^{5t}t^3\}$ (b) $L\{e^{-2t}\cos 4t\}$

Solution

(a) $L\{e^{5t}t^3\} = L\{t^3\}_{s \to s-5} = \left.\dfrac{3!}{s^4}\right|_{s \to s-5} = \dfrac{6}{(s-5)^4}$

(b) $L\{e^{-2t}\cos 4t\} = L\{\cos 4t\}_{s \to s-(-2)}$

$= \left.\dfrac{s}{s^2+16}\right|_{s \to s+2} = \dfrac{s+2}{(s+2)^2+16}$

Inverse Form

$$L^{-1}\{F(s-a)\} = L^{-1}\{F(s)\big|_{s \to s-a}\} = e^{at}f(t) \qquad (1)$$

where $f(t) = L^{-1}\{F(s)\}.$

Example 2:

Find the inverse L.T. of

(a) $L^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$

(b) $L^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$

Solution

(a) $\dfrac{2s+5}{(s-3)^2} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2}$

$2s+5 = A(s-3) + B$

we have $A = 2,\ B = 11$

$$\dfrac{2s+5}{(s-3)^2} = \dfrac{2}{s-3} + \dfrac{11}{(s-3)^2} \tag{2}$$

Example 2 "cont":

And

$$L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2L^{-1}\left\{\frac{1}{s-3}\right\} + 11L^{-1}\left\{\frac{1}{(s-3)^2}\right\} \qquad (3)$$

From (3), we have

$$L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t}t \qquad (4)$$

Example 2 "cont":

(b)
$$\frac{s/2 + 5/3}{s^2 + 4s + 6} = \frac{s/2 + 5/3}{(s+2)^2 + 2} \tag{5}$$

$$L^{-1}\left\{\frac{s/2 + 5/3}{s^2 + 4s + 6}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{s+2}{(s+2)^2 + 2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(s+2)^2 + 2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\left.\frac{s}{s^2 + 2}\right|_{s \to s+2}\right\} + \frac{2}{3\sqrt{2}}L^{-1}\left\{\left.\frac{\sqrt{2}}{s^2 + 2}\right|_{s \to s+2}\right\} \tag{6}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t \tag{7}$$

Example 3:

Solve $\quad y'' - 6y' + 9y = t^2 e^{3t}$, $\quad y(0) = 2$, $\quad y'(0) = 17$

Solution

$$s^2 Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s-3)^2 Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Example 3 "cont":

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$y(t) = 2L^{-1}\left\{\frac{1}{s-3}\right\} + 11L^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}L^{-1}\left\{\frac{4!}{(s-3)^5}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2}\bigg|_{s\to s-3}\right\} = te^{3t}, \qquad\qquad L^{-1}\left\{\frac{4!}{s^5}\bigg|_{s\to s-3}\right\} = t^4 e^{3t}$$

$$y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}$$

Example 4 :

Solve $y''+4y'+6y = 1+e^{-t}$, $\quad y(0) = 0$, $\quad y'(0) = 0$

Solution

$$s^2 Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$(s^2 + 4s + 6)Y(s) = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}$$

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2+5/3}{s^2+4s+6}$$

Example 4 "cont":

$$Y(s) = \frac{1}{6} L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$-\frac{1}{2} L^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} - \frac{2}{3\sqrt{2}} L^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Unit Step Function

Definition

The ***Unit Step Function*** $U(t-a)$ is

$$U(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$
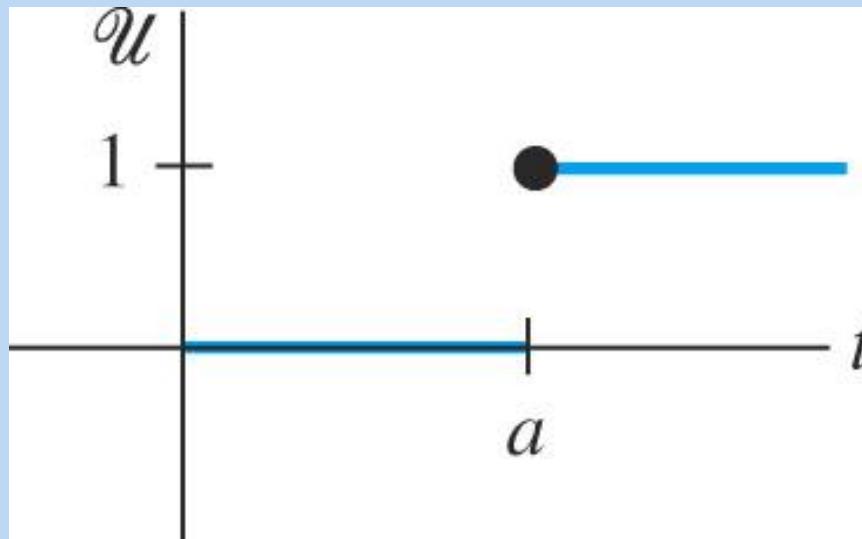
Examples:

❖ Fig. (a) shows the graph of $(2t - 3)\mathcal{U}(t - 1)$.

Considering Fig. (b), it is the same as
$$f(t) = 2 - 3\mathcal{U}(t - 2) + \mathcal{U}(t - 3)$$

(a)

(b)
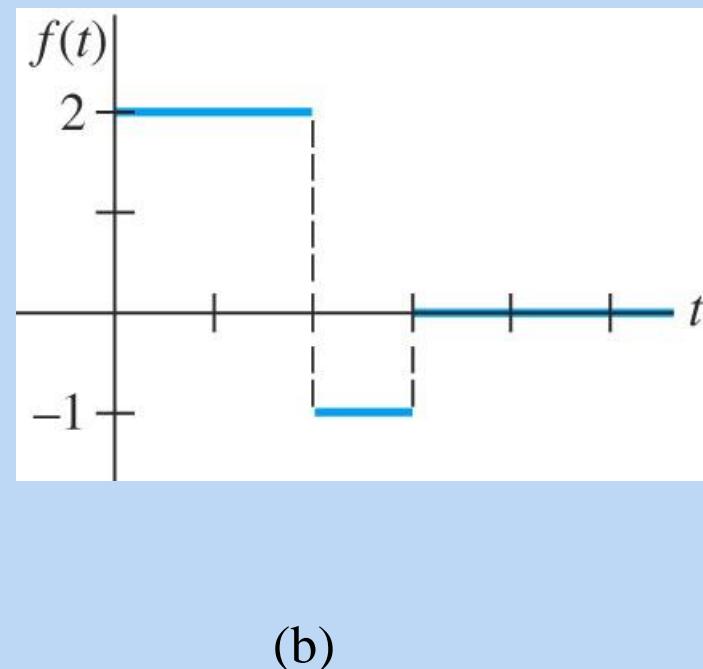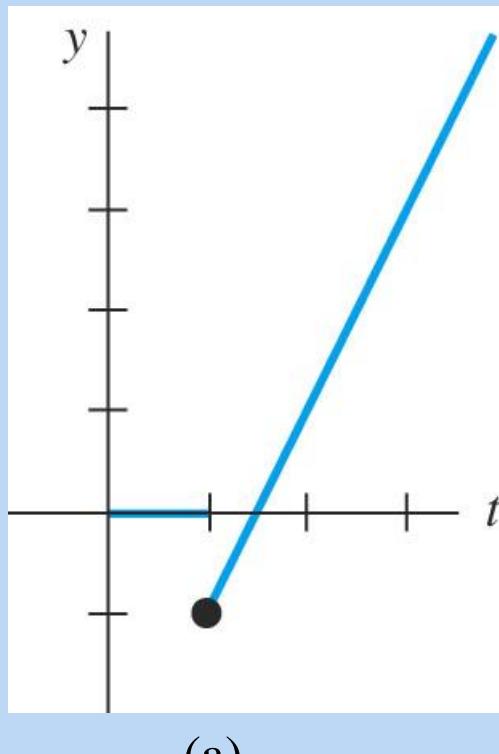
A function of the type

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)U(t-a) + h(t)U(t-a)$$

Similarly, a function of the type

$$f(t) = \begin{cases} 0, & 0 \leq t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b \end{cases}$$

can be written as

$$f(t) = g(t)[U(t-a) - U(t-b)]$$

Example 5:

Express

$$f(t) = \begin{cases} 20t \,, & 0 \le t < 5 \\ 0 \,, & t \ge 5 \end{cases}$$

in terms of $U(t)$.

Solution

$a = 5,\ g(t) = 20t,\ h(t) = 0$

$$f(t) = 20t - 20t\,U(t - 5)$$
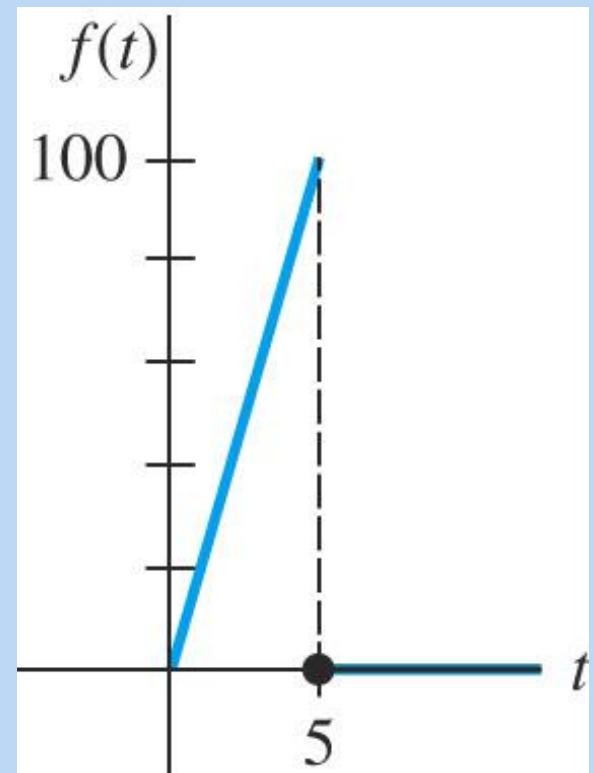
Unit Step Function

❖ Consider the function

$$f(t-a)U(t-a)$$

$$= \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}$$
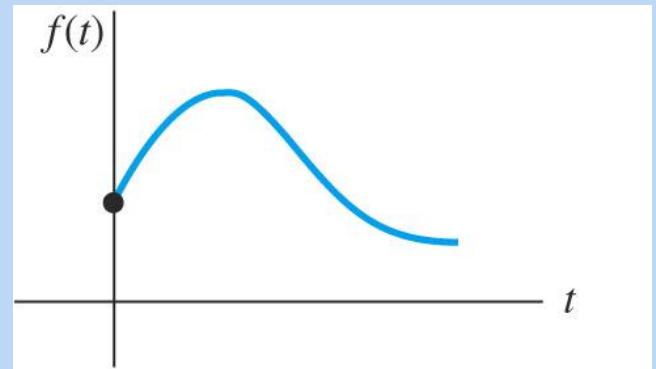
(a) $f(t), t \geq 0$
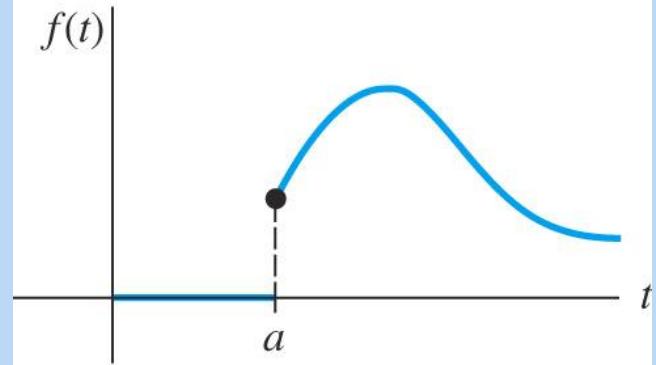
(b) $f(t-a)\,\mathcal{U}(t-a)$

Laplace Form of Unit Step Function

Theorem

If $F(s) = L\{f\}$, and $a > 0$, then

$$L\{f(t-a)U(t-a)\} = e^{-as}F(s)$$

Proof

$L\{f(t-a)U(t-a)\}$

$$= \int_0^a e^{-st} f(t-a)U(t-a)dt + \int_a^\infty e^{-st} f(t-a)U(t-a)dt$$

$$= \int_0^\infty e^{-st} f(t-a)dt$$

Zero
for $0 \le t \le a$

One
for $t \ge a$

Laplace Form of Unit Step Function "cont."

Let $v = t - a,\ dv = dt,$ then

$$L\{f(t-a)U(t-a)\} = \int_0^\infty e^{-s(v+a)} f(v)\,dv$$

$$= e^{-as} \int_0^\infty e^{-sv} f(v)\,dv = e^{-as} L\{f(t)\}$$

If $f(t) = 1$, then $f(t-a) = 1$, $F(s) = 1/s$,

$$L\{U(t-a)\} = \frac{e^{-as}}{s}$$

For example,

$$L\{f(t)\} = 2L\{1\} - 3L\{U(t-2)\} + L\{U(t-3)\}$$

$$= 2\frac{1}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

Laplace Inverse Form of Unit Step Function

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a)$$

Example 6

Find the inverse L.T. of

(a) $\quad L^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$

Solution

(a) $\quad a=2,\; F(s)=1/(s-4),\quad L^{-1}\{F(s)\}=e^{4t}$

then

$$L^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}=e^{4(t-2)}U(t-2)$$

Example 6 "cont.":

(b) $L^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}$

Solution

(b) $a=\pi/2,\ F(s)=s/(s^2+9),\ L^{-1}\{F(s)\}=\cos 3t$

then

$$L^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}=\cos 3\left(t-\dfrac{\pi}{2}\right)U\left(t-\dfrac{\pi}{2}\right)$$

Theorem:

$$L\{g(t)U(t-a)\} = e^{-as}L\{g(t+a)\}$$

Example 7:

Find $L\{\cos t\ U(t - \pi)\}$

Solution

With $g(t) = \cos t$, $a = \pi$, then

$$g(t + \pi) = \cos(t + \pi) = -\cos t$$

$$L\{\cos t\ U(t - \pi)\} = -e^{-\pi s} L\{\cos t\} = -\frac{s}{s^2 + 1} e^{-\pi s}$$

Example 8:

Solve $y' + y = f(t), \quad y(0) = 5$

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 3 \cos t, & t \geq \pi \end{cases}$$

Solution

We find $f(t) = 3 \cos t \, \mathcal{U}(t - \pi)$, then

$$sY(s) - y(0) + Y(s) = -3 \frac{s}{s^2 + 1} e^{-\pi s}$$

$$(s + 1)Y(s) = 5 - \frac{3s}{s^2 + 1} e^{-\pi s}$$

$$Y(s) = \frac{5}{s + 1} - \frac{3}{2} \left[-\frac{1}{s + 1} e^{-\pi s} + \frac{1}{s^2 + 1} e^{-\pi s} + \frac{s}{s^2 + 1} e^{-\pi s} \right]$$

Example 8 "cont":

It follows from (15) with $a = \pi$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}\mathsf{U}(t-\pi) \ , \ \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)\mathsf{U}(t-\pi)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)\mathsf{U}(t-\pi)$$

Thus

$$y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathsf{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathsf{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\mathsf{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \sin t + \cos t]\mathsf{U}(t-\pi)$$

$$= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + 3/2e^{-(t-\pi)} + 3/2\sin t + 3/2\cos t, & t \ge \pi \end{cases}$$

Additional Operational Properties

❖ **Multiplying a Function by t^n**

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]dt = -\int_0^\infty e^{-st}tf(t)dt = -L\{tf(t)\}$$

that is

$$L\{tf(t)\} = -\frac{d}{ds}L\{f(t)\}$$

Similarly

$$L\{t^2 f(t)\} = L\{t \cdot tf(t)\} = -L\{tf(t)\}$$

$$= -\frac{d}{ds}\left(-\frac{d}{ds}L\{f(t)\}\right) = \frac{d^2}{ds^2}L\{f(t)\}$$

Derivatives of Transform

If $F(s) = L\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Find $L\{t \sin kt\}$

Solution

With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, then

$$L\{t \sin kt\} = -\frac{d}{ds} L\{\sin kt\}$$

$$= -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Solve $x''+16x = \cos 4t$, $x(0) = 0$, $x'(0) = 1$

Solution

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16}$$

or

$$X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}$$

From example 1, $\mathcal{L}^{-1}\left\{\dfrac{2ks}{(s^2 + k^2)^2}\right\} = t\sin kt$

Thus

$$x(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\}$$

$$= \frac{1}{4}\sin st + \frac{1}{8}t\sin 4t$$

Convolution

❖ A special product of $f * g$ is defined by

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

and is called the ***convolution*** of f and g.

❖ The convolution is a function of t, eg:

$$e^t * \sin t = \int_0^t e^\tau \sin(t - \tau) d\tau$$

$$= \frac{1}{2}(-\sin t - \cos t + e^t)$$

❖ ***Note:*** $f * g = g * f$

Convolution Theorem

Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$L\{f * g\} = L\{f(t)\}L\{g(t)\} = F(s)G(s)$$

Proof

$$F(s)G(s) = (\int_0^\infty e^{-s\tau} f(\tau)d\tau)(\int_0^\infty e^{-s\beta} f(\beta)d\beta)$$

$$= \int_0^\infty\int_0^\infty e^{-s(\tau+\beta)} f(\tau)f(\beta)d\tau d\beta$$

$$= \int_0^\infty f(\tau)d\tau \int_0^\infty e^{-(\tau+\beta)} g(\beta)d\beta$$

Holding τ fixed, let $t = \tau + \beta$, $dt = d\beta$

$$F(s)G(s) = \int_0^\infty f(\tau)d\tau \int_\tau^\infty e^{-st} g(t-\tau)dt$$

Since f and g are piecewise continuous on $[0,\infty)$ and of exponential order, it is possible to interchange the order of integration:

$$F(s)G(s) = \int_0^\infty e^{-st} dt \int_0^t f(\tau) g(t-\tau) d\tau$$

$$= \int_0^\infty e^{-st} \left\{ \int_0^t f(\tau) g(t-\tau) d\tau \right\} dt$$

$$= L\{f * g\}$$
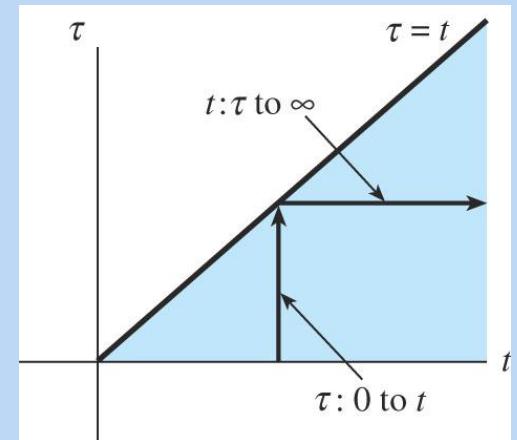
Example 3:

Find $L\left\{\int_0^t e^\tau \sin(t-\tau)\, d\tau\right\}$

Solution

$$L\left\{\int_0^t e^\tau \sin(t-\tau)\, d\tau\right\} = L\{e^t * \sin t\}$$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Inverse Transform of Convolution:

$$L^{-1}\{F(s)\,G(s)\} = f * g$$

Example 4

Find $L^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

Solution

Let

$$F(s)=G(s)=\frac{1}{s^2+k^2}$$

$$f(t)=g(t)=\frac{1}{k}L^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\frac{1}{k}\sin kt$$

Then

$$L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{k^2}\int_0^t \sin k\,\tau \sin k\,(t-\tau)d\,\tau$$

Example 4 "cont."

Now recall that

$$\sin A \sin B = (1/2)\,[\cos(A-B) - \cos(A+B)]$$

If we set $A = k\tau$, $B = k(t - \tau)$, then

$$L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^2}\int_0^t [\cos k\,(2\tau - t) - \cos kt\,]d\,\tau$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k\,(2\tau - t) - \tau \cos kt\,\right]\Bigg|_0^t$$

$$= \frac{\sin kt - kt \cos kt}{2k^3}$$

Transform of an Integral

❖ When $g(t) = 1$, $G(s) = 1/s$, then

$$\int_0^t f(\tau)d\tau = L^{-1}\left\{\frac{F(s)}{s}\right\}$$

❖ Examples:

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau d\tau = 1 - \cos t$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau)d\tau = t - \sin t$$

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (\tau - \sin\tau)d\tau = \frac{1}{2}t^2 - 1 + \cos t$$

Volterra Integral Equation

$$f(t) = g(t) + \int_0^t f(\tau) h(t - \tau) d\tau$$

Example 5:

Solve

$$f(t) = 3t^2 - e^{-t} + \int_0^t f(\tau)e^{t-\tau}\,d\tau \qquad \text{for } f(t)$$

Solution

First, $h(t-\tau) = e^{(t-\tau)}$, $h(t) = e^t$.

$$F(s) = 3 \cdot \frac{2}{s^3} - \frac{1}{s+1} - F(s) \cdot \frac{1}{s-1}$$

Solving for $F(s)$ and using partial fractions

Example 5 "cont.":

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$f(t) = 3L^{-1}\left\{\frac{2!}{s^3}\right\} - L^{-1}\left\{\frac{3!}{s^4}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 3t^2 - t^3 + 1 - 2e^{-t}$$

Periodic Function

Theorem: Transform of a Periodic Function

If $f(t)$ is a periodic function with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Periodic Function

Proof

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

$$Let\ \ t = u + T$$

$$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

$$\int_T^\infty e^{-st} f(t)\,dt = e^{-sT} L\{f(t)\}$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT} L\{f(t)\}$$

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$