Cauchy-Euler Equation:

Form of Cauchy-Euler Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Method of Solution

We try $y = x^m$, since

$$a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)(m-2)\cdots(m-k+1)x^{m-k}$$

$$= a_k m(m-1)(m-2)\cdots(m-k+1)x^m$$

An Auxiliary Equation

For $n = 2$, $y = x^m$, then

$$am(m-1) + bm + c = 0,$$

or

$$am^2 + (b-a)m + c = 0 \qquad (1)$$

Case 1: Distinct Real Roots

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$$y = c_1 x^{m_1} + c_2 x^{m_2} \qquad (2)$$

Example

Solve $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=0$$

Solution:

let $y = x^m$, and sub we get

$$m^2 - 3m - 4 = 0,$$

m = -1, 4

$$y = c_1x^{-1} + c_2x^4$$

Case 2: Repeated Real Roots

Using Reduction of order

$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

$$P(x) = b/ax \qquad and \qquad \int (b/ax)dx = (b/a)\ln x$$

$$y_2 = x^{m_1}\int \frac{e^{-(b/a)\ln x}}{x^{2m_1}}dx = x^{m_1}\int x^{-b/a}x^{-2m_1}dx$$

$$= x^{m_1}\int x^{-b/a}x^{(b-a)/a}dx = x^{m_1}\int \frac{dx}{x} = x^{m_1}\ln x$$

Then

$$y = c_1x^{m_1} + c_2x^{m_1}\ln x \qquad (3)$$

Case 2: Repeated Real Roots

Higher-Order: multiplicity

$$x^{m_1},\ x^{m_1}\ln x,\ x^{m_1}(\ln x)^2,\ \cdots,\ x^{m_1}(\ln x)^{k-1}$$

Example

Solve $4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$

Solution:

let $y = x^m$, and sub we get

$4m^2 + 4m + 1 = 0,$

$m = -½ , -½$

$$y = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x$$

Case 3: Conjugate Complex Roots

$$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta,$$
$$y = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$$

Since

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x} = \cos(\beta \ln x) + i \sin(\beta \ln x)$$
$$x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x)\backslash$$

For $C_1 = C_2 = 1$, $y_1 = x^{\alpha}(x^{i\beta} + x^{-i\beta}) = 2\, x^{\alpha}\cos(\beta \ln x)$

For $C_1 = 1$, $C_2 = -1$, $y_2 = x^{\alpha}(x^{i\beta} - x^{-i\beta}) = 2 i x^{\alpha}\sin(\beta \ln x)$

Since $W(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)) = \beta\, x^{2\alpha-1} \neq 0$

Then

$$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$$
$$= x^{\alpha} [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Example

Solve $4x^2y'' + 17y = 0,\ y(1) = -1,\ y'(1) = -\dfrac{1}{2}$

Solution:

let $y = x^m$, and sub we get

$$4m^2 - 4m + 17 = 0,\ m = \tfrac{1}{2} \pm 2i$$

Apply $y(1)$ = -1, $y'(1)$ = -1/2, then c_1 = -1, c_2 = 0

$$y = x^{1/2}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$$

$$y = -x^{1/2}\cos(2\ln x)$$

Example

Solve $x^3\frac{d^3y}{dx^3}+5x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+8y=0$

Solution:

Let $y = x^m$,

$$\frac{dy}{dx}=mx^{m-1}\ ,\ \frac{d^2y}{dx^2}=m(m-1)x^{m-2}\ ,$$

$$\frac{d^3y}{dx^3}=m(m-1)(m-2)x^{m-3}$$

Then we have $x^m(m + 2)(m^2 + 4) = 0$

$$m = -2,\ m = 2i,\ m = -2i$$

$$y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)$$

Example

Solve $$x^2 y'' - 3xy' + 3y = 2x^4 e^x$$

Solution:

We have $(m-1)(m-3)=0,\ m=1,\ 3$

$y_c = c_1 x + c_2 x^3$, use variation of parameters,

$y_p = u_1 y_1 + u_2 y_2$, where $y_1 = x,\ y_2 = x^3$

Rewrite the DE as

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x$$

Then $f = 2x^2 e^x$

Example

Thus

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3 ,$$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x , W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

We find

$$u_1' = -\frac{2x^5e^x}{2x^3} = -x^2e^x, \quad u_2' = \frac{2x^5e^x}{2x^3} = e^x$$

$$u_1 = -x^2e^x + 2xe^x - 2e^x, \quad u_2 = e^x$$

Example

Then

$$y_p = u_1 y_1 + u_2 y_2 = (-x^2 e^x + 2xe^x - 2e^x)x + e^x x^3$$

$$= 2x^2 e^x - 2xe^x$$

$$y = y_c + y_p = c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x$$