Reduction of Order

Introduction:

We know the general solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

is $$y = c_1y_1 + c_2y_2.$$

Suppose $y_1(x)$ denotes a known solution of (1). We assume the other solution y_2 has the form $y_2 = uy_1$.

Our goal is to find a $u(x)$ and this method is called ***reduction of order.***

Example 1:

Given $y_1 = e^x$ is a solution of $y'' - y = 0$, find a second solution y_2 by the method of reduction of order.

Solution:

If $y = u(x)\, y_1(x) = u(x)\, e^x$, then

$$y' = ue^x + e^x u' \,,\; y'' = ue^x + 2e^x u' + e^x e''$$

And
$$y'' - y = e^x(u'' + 2u') = 0$$

Since $e^x \neq 0$, we let $w = u'$, then

$$w = c_1 e^{-2x} = u'$$

$$u = -1/2\, c_1 e^{-2x} + c_2$$

Example 1 : "cont."

Thus

$$y = u(x)e^{x} = -\frac{c_1}{2}e^{-x} + c_2 e^{x} \qquad (2)$$

Choosing $c_1 = 0$, $c_2 = -2$, we have $y_2 = e^{-x}$.

Because $W(e^{x}, e^{-x}) \neq 0$ for every x, they are independent.

General Case

Rewrite (1) as the ***standard form***

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

Let $y_1(x)$ denotes a known solution of (3) and $y_1(x) \neq 0$ for every x in the interval.

If we define $y = uy_1$, then we have

$$y' = uy_1' + y_1u' \,,\; y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$y'' + Py' + Qy$$

$$= u\underbrace{[y_1'' + Py_1' + Qy_1]}_{\text{zero}} + y_1u'' + (2y_1' + Py_1)u' = 0$$

General Case

This implies that

$$y_1 u'' + (2y_1' + Py_1)u' = 0$$

or

$$y_1 w' + (2y_1' + Py_1)2w = 0 \qquad (4)$$

where we let $w = u'$.

Solving (4), we have

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

$$\ln|wy_1^2| = -\int Pdx + c \quad \text{or} \quad wy_1^2 = c_1 e^{-\int Pdx}$$

General Case

then

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2$$

Let $c_1 = 1$, $c_2 = 0$, we find

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \qquad (5)$$

Example 2:

The function $y_1 = x^2$ is a solution of

$$x^2 y'' - 3xy' + 4y = 0$$

Find the general solution on $(0, \infty)$.

Solution:

The standard form is

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$$

From (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx = x^2 \int \frac{x^3}{x^4} dx = x^2 \ln x$$

The general solution is

$$y = c_1 x^2 + c_2 x^2 \ln x$$