Variation of Parameters

Some Assumptions

For the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \qquad (1)$$

we put (1) in the form

$$y'' + P(x)y' + Q(x)y = f(x) \qquad (2)$$

where P, Q, f are continuous on I.

Method of Variation of Parameters

We try

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (3)$$

After we obtain y'_p, y''_p we put them into (2), then

$$y''_p + P(x)y'_p + Q(x)y_p$$

$$= u_1[y''_1 + Py'_1 + Qy_1] + u_2[y''_2 + Py'_2 + Qy_2]$$

$$+ y_1u''_1 + u'_1y'_1 + y_2u''_2 + u'_2y'_2 + P[y_1u'_1 + y_2u'_2] + y'_1u'_1 + y'_2u'_2$$

$$= \frac{d}{dx}[y_1u'_1] + \frac{d}{dx}[y_2u'_2] + P[y_1u'_1 + y_2u'_2] + y'_1u'_1 + y'_2u'_2$$

$$= \frac{d}{dx}[y_1u'_1 + y_2u'_2] + P[y_1u'_1 + y_2u'_2] + y'_1u'_1 + y'_2u'_2 = f(x) \qquad (4)$$

Method of Variation of Parameters

Making further assumptions:

$$y_1 u'_1 + y_2 u'_2 = 0, \text{ then from (4),}$$

$$y'_1 u'_1 + y'_2 u'_2 = f(x)$$

Express the above in terms of determinants

$$u'_1 = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u'_2 = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \qquad (5)$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix} \qquad (6)$$

Example 1

Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

Solution:

$$m^2 - 4m + 4 = 0, \ m = 2, 2$$

$$y_1 = e^{2x}, \ y_2 = xe^{2x},$$

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \neq 0$$

Since $f(x) = (x+1)e^{2x}$, then

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & (2x+1)e^{2x} \end{vmatrix} = -(x+1)xe^{4x}, \ W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Example 1 "cont."

From (5),

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x \, , \; u_2' = -\frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

Then

$$u_1 = (-1/3)x^3 - ½\, x^2, \quad u_2 = ½\, x^2 + x$$

And

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

Example 2

Solve $4y''+36y=\csc 3x$

Solution:

$$y''+9y=(1/4)\csc 3x$$

$$m^2+9=0,\ m=3i,\ -3i$$

$$y_1=\cos 3x,\ y_2=\sin 3x,\ f=(1/4)\csc 3x$$

Since

$$W(\cos 3x\,,\sin 3x)=\begin{vmatrix}\cos 3x & \sin 3x\\ -3\sin 3x & 3\cos 3x\end{vmatrix}=3$$

$$W_1=\begin{vmatrix}0 & \sin 3x\\ 1/4\csc 3x & 3\cos 3x\end{vmatrix}=-\frac{1}{4},\ W_2=\begin{vmatrix}\cos 3x & 0\\ -3\sin 3x & 1/4\csc 3x\end{vmatrix}=\frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

Example 2 “cont.”

$$u_1' = \frac{W_1}{W} = -\frac{1}{12}$$

$$u_2' = \frac{W_2}{W} = \frac{1}{12}\,\frac{\cos 3x}{\sin 3x}$$

Then $\quad u_1 = -1/12\; x, \qquad u_2 = 1/36 \ln|\sin 3x|$

And $\quad y_p = -\frac{1}{12} x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12} x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Example 3

Solve $y'' - y = \frac{1}{x}$

Solution:

$m^2 - 1 = 0$, $m = 1, -1$

$y_1 = e^x$, $y_2 = e^{-x}$, $f = 1/x$, and $W(e^x, e^{-x}) = -2$

Then

$$u_1' = -\frac{e^{-x}(1/x)}{-2}, u_1 = \frac{1}{2}\int_{x_o}^{x} \frac{e^{-t}}{t} dt$$

$$u_2' = -\frac{e^{x}(1/x)}{-2}, u_2 = -\frac{1}{2}\int_{x_o}^{x} \frac{e^{t}}{t} dt$$

The low and up bounds of the integral are x_0 and x, respectively.

Example 3 "cont."

$$y_p = \frac{1}{2}e^{x}\int_{x_o}^{x}\frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_o}^{x}\frac{e^{t}}{t}dt$$

$$y = y_c + y_p = c_1e^{x} + c_2e^{-x} + \frac{1}{2}e^{x}\int_{x_o}^{x}\frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_o}^{x}\frac{e^{t}}{t}dt$$

Higher-Order Equations

For the DEs of the form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x) \qquad (8)$$

then $y_p = u_1y_1 + u_2y_2 + \ldots + u_ny_n$, where y_i, $i = 1, 2, \ldots, n$, are the elements of y_c. Thus we have

$$\begin{aligned} y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0 \\ y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0 \\ \vdots \qquad\qquad\qquad &\vdots \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' &= f(x) \end{aligned} \qquad (9)$$

and $u'_k = W_k/W$, $k = 1, 2, \ldots, n$.

Higher-Order Equations

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}, W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$