

Introduction

If we want to solve

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \quad (1)$$

we have to find $y = y_c + y_p$. Thus we introduce the method of

undetermined coefficients.

Example 1

$$\text{Solve } y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Solution:

We can get y_c by

$$m^2 + 4m - 2 = 0$$

$$m = -2 \pm \sqrt{6}$$

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{-(2-\sqrt{6})x}$$

Now, we want to find y_p .

Example 1 “cont.”

Since the right side of the DE is a polynomial,
we set

$$y_p = Ax^2 + Bx + C, \quad y_p' = 2Ax + B, \quad y_p'' = 2A$$

After substitution,

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

Then

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6$$

$$A = -1, \quad B = -5/2, \quad C = -9$$

$$y_p = -x^2 - \frac{5}{2}x - 9$$

Example 2

Find a particular solution of

$$y'' - y' + y = 2\sin 3x$$

Solution:

Let $y_p = A \cos 3x + B \sin 3x$

After substitution,

$$(-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

Then

$$A = 6/73, B = -16/73$$

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example 3

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

Solution:

We can find $y_c = c_1e^{-x} + c_2e^{3x}$

Let

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

After substitution,

$$\begin{aligned} & -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} \\ & = 4x - 5 + 6xe^{2x} \end{aligned}$$

Then

$$A = -4/3, B = 23/9, C = -2, E = -4/3$$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$$

Example 4

Find y_p of $y'' - 5y' + 4y = 8e^x$

Solution:

First let $y_p = Ae^x$

After substitution, $0 = 8e^x$, (***wrong guess***)

Let $y_p = Axe^x$

After substitution, $-3Ae^x = 8e^x$

Then $A = -8/3$, $y_p = (-8/3)xe^x$

Rule of Case 1:

No function in the assumed y_p is part of y_c

Trial particular solutions.

$g(x)$	Form of y_p
1. 1(any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Example 5

Find the form of y_p of

$$(a) \quad y'' - 8y' + 25y = 5x^3 e^{-x} - 7e^{-x}$$

Solution:

We have

$$g(x) = (5x^3 - 7)e^{-x} \text{ and try}$$

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}$$

There is no duplication between y_p and y_c .

$$(b) \quad y'' + 4y = x \cos x$$

Solution:

$$\text{We try } x_p = (Ax + B)\cos x + (Cx + E)\sin x$$

There is also no duplication between y_p and y_c .

Example 6

Find the form of y_p of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Solution:

For $3x^2$: $y_{p_1} = Ax^2 + Bx + C$

For $-5\sin 2x$: $y_{p_2} = E\cos 2x + F\sin 2x$

For $7xe^{6x}$: $y_{p_3} = (Gx + H)e^{6x}$

No term in $y_p = y_{p_1} + y_{p_2} + y_{p_3}$ duplicates a term in y_c

Rule of Case 2:

If any term in y_p duplicates a term in y_c , it should be multiplied by x^n , where n is the smallest positive integer that eliminates that duplication.

Example 7

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$

Solution:

$$y_c = c_1 \cos x + c_2 \sin x$$

First trial: $y_p = Ax + B + C \cos x + E \sin x$

However, duplication occurs. Then we try

$$y_p = Ax + B + Cx \cos x + Ex \sin x$$

After substitution and simplification,

$$A = 4, B = 0, C = -5, E = 0$$

Then $y = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x$

Using $y(\pi) = 0$, $y'(\pi) = 2$, we have

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

Example 8

$$\text{Solve } y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

Solution:

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p1}} + \underbrace{Ex^2 e^{3x}}_{y_{p2}}$$

After substitution and simplification,

$$A = 2/3, B = 8/9, C = 2/3, E = -6$$

Then

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}$$

Example 9

Solve $y''' + y'' = e^x \cos x$

Solution:

$$m^3 + m^2 = 0, m = 0, 0, -1$$

$$y_c = c_1 + c_2x + c_3e^{-x}$$

$$y_p = Ae^x \cos x + Be^x \sin x$$

After substitution and simplification,

$$A = -1/10, B = 1/5$$

Then

$$y = y_c + y_p = c_1 + c_2x + c_3e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x$$

Example 10

Find the form of y_p of

$$y^{(4)} + y''' = 1 - x^2 e^{-x}$$

Solution:

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

Normal trial:

$$y_p = \underbrace{A}_{y_{p1}} + \underbrace{Bx^2 e^{-x} + Cx e^{-x} + Ee^{-x}}_{y_{p2}}$$

Multiply A by x^3 and $(Bx^2 e^{-x} + Cx e^{-x} + Ee^{-x})$ by x

Then

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$