Homogeneous Linear Equation with Constant Coefficients

Introduction:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \qquad (1)$$

where a_i are constants, $a_n \neq 0$.

Auxiliary Equation:

For $n = 2$,

$$ay'' + by' + cy = 0 \qquad (2)$$

Try $y = e^{mx}$, then

$$e^{mx}(am^2 + bm + c) = 0$$

$$am^2 + bm + c = 0 \qquad (3)$$

is called an ***auxiliary equation.***

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From (3) the two roots are

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

(1) $b^2 - 4ac > 0$: *two distinct real numbers.*

(2) $b^2 - 4ac = 0$: *two equal real numbers.*

(3) $b^2 - 4ac < 0$: *two conjugate complex numbers.*

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Case 1: Distinct real roots

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad (4)$$

Case 2: Repeated real roots

The general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

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Case 3: Conjugate complex roots

We write $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$, a general solution is

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

From Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x$$

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Since $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$ is a solution then set $C_1 = C_1 = 1$ and $C_1 = 1$, $C_2 = -1$, we have two solutions:

$$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$

$$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x$$

So, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are a fundamental set of solutions, that is, the general solution is

$$\begin{aligned} y &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

Example 1:

Solve the following DEs:

(a) $2y''-5y'-3y=0$

$$2m^2-5m-3=(2m+1)(m-3)\ ,\ \ m_1=-1/2\ ,\ \ m_2=3$$

$$y=c_1e^{-x/2}+c_2e^{3x}$$

(b) $y''-10y'+25y=0$

$$m^2-10m+25=(m-5)^2\ ,\ \ m_1=m_2=5$$

$$y=c_1e^{5x}+c_2xe^{5x}$$

(c) $y''+4y'+7y=0$

$$m^2+4m+7=0\ ,\ \ m_1=-2+\sqrt{3}i\ ,\ \ m_2=-2-\sqrt{3}i$$

$$\alpha=-2\ ,\ \ \beta=\sqrt{3}\ ,\ \ y=e^{-2x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)$$

Example 2:

Solve $4y''+4y'+17y=0,\ y(0)=-1,\ y'(0)=2$

Solution*:*

$$4m^2+4m+17=0, \qquad m=-1/2\pm 2i$$

$$y=e^{-x/2}(c_1\cos 2x+c_2\sin 2x)$$

$$y(0)=-1,\ c_1=-1,\ \text{and}\ \ y'(0)=2,\ c_2=3/4$$

Hence the solution of the IVP is

$$y=e^{-x/2}(-\cos 2x+3/4\sin 2x)$$

Higher-Order Equations

Given

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

we have

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$$

as an ***auxiliary equation***.

Example 3:

Solve $y''' + 3y'' - 4y = 0$

Solution:

$$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4) = (m-1)(m+2)^2$$

$$m_1 = 1, m_2 = m_3 = -2$$

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$$

Example 4:

Solve

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Solution:

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$$

$$m_1 = m_3 = i,\, m_2 = m_4 = -i$$

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}$$

$$= c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$