Preliminary Theory: Linear Equation:

Initial-value Problem

An *n*th-order initial problem is

Solve:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to:

$$y(x_0)=y_0,\ y'(x_0)=y_1,\ \cdots,\ y^{(n-1)}(x_0)=y_{n-1}$$

with *n* initial conditions.

Existence and Uniqueness

Theorem

Let $a_n(x)$, $a_{n-1}(x)$, …, $a_0(x)$, and $g(x)$ be continuous on I, $a_n(x) \neq 0$ for all x on I. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of (1) exists on the interval and is unique.

Example

The problem

$$3y''' + 5y'' + y' + 7y = 0, \; y(1) = 0 \,, \; y'(1) = 0, \; y''(1) = 0$$

possesses the trivial solution $y = 0$. Since this DE with constant coefficients, ;

hence $y = 0$ is the only one solution on any interval containing x = 1.

Homogeneous and Non homogeneous Equations

The following DE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

is said to be ***homogeneous;***

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

with $g(x)$ not zero, is ***nonhomogeneous***.

Differential Operators

Let $dy/dx = Dy$. This symbol D is called a ***differential operator***.

We define an nth-order differential operator as

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

In addition, we have

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x))$$

so the differential operator L is a ***linear operator***.

Differential Equations

We can simply write the DEs as

$$L(y) = 0 \text{ and } L(y) = g(x)$$

Superposition Principles – Homogeneous Equations

Theorem

Let $y_1, y_2, \ldots, y_k$ be a solutions of the homogeneous nth-order differential equation on an interval I.

Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x)$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

Superposition Principles – Homogeneous Equations

Corollary

a) $y = cy_1$ is also a solution if y_1 is a solution.

b) A homogeneous linear DE always possesses the trivial solution

$y = 0$.

Linear Dependence and Linear Independence

Definitions

A set of $f_1(x), f_2(x), \ldots, f_n(x)$ is ***linearly dependent*** on an interval I, if there exists constants $c_1, c_2, \ldots, c_n$, ***not all zero,*** such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

If not linearly dependent, it is ***linearly independent***.

In other words, if the set is linearly independent, when

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

then $c_1 = c_2 = \ldots = c_n = 0$

Example

The functions

$f_1 = \cos^2 x,\ \ f_2 = \sin^2 x,\ \ f_3 = \sec^2 x,\ f_4 = \tan^2 x$ a

re linearly dependent on the interval (-π/2, π/2)

since

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

when $c_1 = c_2 = 1,\ c_3 = -1,\ c_4 = 1.$

Example

The functions

$$f_1 = x^{1/2} + 5,\;\; f_2 = x^{1/2} + 5x,\;\; f_3 = x - 1,\; f_4 = x^2$$

are linearly dependent on the interval $(0, \infty)$, since

$$f_2 = 1 \cdot f_1 + 5 \cdot f_3 + 0 \cdot f_4$$

Wronskian

Definitions

Suppose each of the functions $f_1(x)$, $f_2(x)$, …, $f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1,\ldots,f_n)=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the ***Wronskian*** of the functions.

Theorems

Theorem: Criterion for Linear Independence

Let $y_1(x)$, $y_2(x)$, …, $y_n(x)$ be solutions of the nth-order homogeneous DE (6) on an interval I. This set of solutions is ***linearly independent*** if and only if

$W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Definition: Fundamental Set of a Solution

Any set $y_1(x)$, $y_2(x)$, …, $y_n(x)$ of n linearly independent solutions is said to be a ***fundamental*** set of solutions.

Theorems

Theorem: Existence of a Fundamental Set

There exists a fundamental set of solutions for DE on an interval I.

Definition: General Solution – Homogeneous Equations

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be a fundamental set of solutions of homogeneous DE (6) on an interval I. Then the general solution is

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where c_i are arbitrary constants.

Example

The functions $y_1 = e^{3x}$, $y_2 = e^{-3x}$ are solutions of

$$y'' - 9y = 0 \text{ on } (-\infty, \infty)$$

Now

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x.

So $y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution.

Example

The functions $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ are solutions of $y''' - 6y''$ $+ 11y' - 6y = 0$ on (-∞, ∞).

Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every real value *of x.*

So $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ is the general solution on (-∞, ∞).

General Solution – Nonhomogeneous Equations

Theorem

Let y_p *be any particular solution of* nonhomegeneous on ***I***, and $y_1(x), y_2(x), \ldots, y_k(x)$ be a fundamental set of solutions of homogeneous equation. Then the general solution is

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p$$

Example

The function y_p = -(11/12) – ½ x is a particular solution of

$$y''' - 6y'' + 11y' - 6y = 3x$$

The general solution is

$$y = y_c + y_p = c_1e^{x} + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

Theorem

Given

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

where $i = 1, 2, \ldots, k.$

If y_{pi} denotes a particular solution corresponding to the DE , then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \cdots + g_k(x)$$

Example

We find

$y_{p1} = -4x^2$ is a particular solution of

$$y''-3y'+4y=-16x^2+24x-8$$

$y_{p2} = e^{2x}$ is a particular solution of

$$y''-3y'+4y=2e^{2x}$$

$y_{p3} = xe^x$ is a particular solution of

$$y''-3y'+4y=2xe^x-e^x$$

$, y_p = y_{p_1} + y_{p_2} + y_{p_3}$ is a solution of

$$y''-3y'+4y=\underbrace{-16x^2+24x-8}_{g_1(x)}+\underbrace{2e^{2x}}_{g_2(x)}+\underbrace{2xe^x-e^x}_{g_3(x)}$$