Ricatti Equation:

Definition:

The nonlinear differential equation

$$\frac{dy}{dx} = P(x) + Q(x)\,y + R(x)\,y^2$$

is called ***Ricatti's equation***.

Note:

Note for n = 0 and n = 1, (1) is linear, otherwise, let

$$u = y^{1-n}$$

to reduce (1) to a linear equation.

Solving a Ricatti Equation

1. Find a particular solution y_1. (This may be given.)

2. Use the substitution $y = y_1 + u$ and

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

Substitution into the Riccati equation given us

$$\frac{dy}{dx} = P(x) + Q(x)\,y + R(x)\,y^2$$

$$y_1' + u' = P(x) + Q(x)\left(y_1 + u\right) + R(x)\left(y_1 + u\right)^2$$

$$y_1' + u' = P(x) + Q(x)\,y_1 + Q(x)u + R(x)\,y_1^2 + 2R(x)y_1u + R(x)u^2$$

$$u' = \quad Q(x)u + 2R(x)y_1u + R(x)u^2 \qquad\qquad y_1 \text{ is a solution}$$

$$u' - \ (Q(x) + 2R(x)y_1)u = R(x)u^2 \qquad\qquad \text{Bernoulli in } u$$

3. Solve the Bernoulli equation.

Example 1

Solve

$$y' = e^x + y - e^{-x}y^2 \qquad \text{If } y_1 = -e^x$$

Solution:

$$y_1 = -e^x \qquad P(x) = e^x \qquad Q(x) = 1 \qquad R(x) = e^{-x}$$

$$y = y_1 + u \to y' = -e^x + u'$$

$$y_1' + u' = e^x + 1\left(-e^x + u\right) + e^{-x}\left(-e^x + u\right)^2$$

$$-e^x + u' = e^x - e^x + u + e^{-x}\left(-e^{2x} - 2e^x u + u^2\right)$$

$$= -u - e^x + e^{-x}u^2$$

$$u' + u = e^{-x}u^2 \qquad \text{Bernoulli in } u \text{ with } n = 2$$

$$v = u^{-1} \to \frac{dv}{dx} = \frac{-1}{u^2}\frac{du}{dx} \to -u^2\frac{dv}{dx} = \frac{du}{dx}$$

Example 1

$$-u^2\frac{dv}{dx}+u=e^{-x}u^2$$

$$\frac{dv}{dx}-v=-e^{-x}$$

Linear in v

$$\mu=e^{\int -dx}=e^{-x}$$

$$e^{-x}\frac{dv}{dx}-e^{-x}v=-e^{-x}e^{-x}$$

$$\left(e^{-x}v\right)'=-e^{-2x}$$

$$e^{-x}v=\frac{e^{-2x}}{2}+c$$

$$e^{-x}u^{-1}=\frac{e^{-2x}}{2}+c$$

Example 1

$$u^{-1} = \left(\frac{e^{-2x}}{2} + c \right) e^{x}$$

$$u = \left(\left(\frac{e^{-2x}}{2} + c \right) e^{x} \right)^{-1}$$

$$y = \left(\left(\frac{e^{-2x}}{2} + c \right) e^{x} \right)^{-1} + -e^{x}$$