Linear Differential Equation:

Definition: Linear Equation of order *n*

A differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Definition: Linear Equation of the first order

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation in the dependent variable *y*.

Linear Differential Equation:

if $a_1(x) \neq 0$, we can write this differential equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where ,

$$P(x) = \frac{a_0(x)}{a_1(x)} \qquad Q(x) = \frac{g(x)}{a_1(x)}$$

$P(x)$ and $Q(x)$ are continuous on an interval.

Linear Differential Equation:

Let’s express linear equation in the differential form

$$[P(x)y - Q(x)]dx + dy = 0 \quad (1)$$

If we test this equation for exactness, we find

$$\frac{\partial M}{\partial y} = P(x) \quad and \quad \frac{\partial N}{\partial x} = 0$$

Consequently, equation(1) is exact only when $P(x) = 0$. It turns out that an integrating factor μ, which depends only on x, can easily obtained the general solution of (1).

Linear Differential Equation:

Multiply (1) by a function $\mu(x)$ and try to determine $\mu(x)$ so that the resulting equation

$$\left[\mu(x)P(x)y-\mu(x)Q(x)\right]dx+\mu(x)dy=0 \quad (2)$$

is exact.

$$\frac{\partial M}{\partial y}=\mu(x)P(x) \quad and \quad \frac{\partial N}{\partial x}=\frac{d\mu}{dx}(x)$$

We see that (2) is exact if μ satisfies the DE

$$\frac{d\mu}{dx}(x)=P(x)\mu(x) \qquad (3)$$

$$\mu(x)=\exp\left(\int P(x)dx\right) \quad (4)$$

Which is our desired IF

Linear Differential Equation:

we multiply by $\mu(x)$ defined in (4) to obtain

$$\mu(x)\frac{dy}{dx}+P(x)\mu(x)y=\mu(x)Q(x) \quad (5)$$

We know from (3)

$$P(x)\mu(x)=\frac{d\mu}{dx}$$

and so (5) can be written in the form

$$\underbrace{\mu(x)\frac{dy}{dx}+\frac{d\mu}{dx}(x)y}_{\frac{d}{dx}\left(\mu(x)y\right)} = \mu(x)Q(x)$$

$$\frac{d}{dx}\left(\mu(x)y\right)=\mu(x)Q(x) \quad (6)$$

Linear Differential Equation:

Integrating (6) w.r.t. x gives

$$\mu(x)y = \int \mu(x)Q(x)dx + C$$

and solving for y yields

$$y = \left(\exp\left(-\int P(x)dx\right)\right)\left(\int \mu(x)Q(x)dx + C\right)$$

Solution Method

1. Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Calculate the IF μ(x) by the formula

$$\mu(x) = \exp\left(\int P(x)dx\right)$$

3. Multiply the equation in standard form by $\mu(x)$ nd recalling that the LHS is just $\frac{d}{dx}\left(\mu(x)y\right)$,

obtain $\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)Q(x)$

4. Integrate the last equation and solve for y by dividing by $\mu(x)$.

Example 1

Solve the

$$x\cos x\left(\frac{dy}{dx}\right)+y\left(x\sin x+\cos x\right)=1$$

Dividing by *xcosx*, throughout, we get

$$\frac{dy}{dx}+\left(\tan x+\frac{1}{x}\right)y=\frac{\sec x}{x}$$

$$\mu(x)=e^{\int P(x)dx}=e^{\int\left(\tan x+\frac{1}{x}\right)dx}$$

$$\mu(x)=e^{\ln\sec x+\ln x}=e^{\ln\sec x}\,e^{\ln x}=x\sec x$$

Multiply by

$$x\sec x$$

Example 1

$$\frac{d}{dx}\left(y\,x\sec x\right)=\frac{\sec x}{x}\,x\sec x$$

Integrate both side we get

$$yx\sec x=\int \sec^2 x\,dx+C=\tan x+C$$

Linear Equation in x:

The usual notation $\frac{dy}{dx}$ implies that x is independent variable & y the dependent variable. Sometimes it is helpful to replace x by y and y by x & work on the resulting equation.

* When diff equation is of the form

$$\frac{dx}{dy} + P(y).x = Q(y)$$

$$IF = e^{\int P(y)dy} \quad \& \quad solution \quad is \quad x \times IF = \int Q \times IF \, dy + c$$

Example 2

Solve the $y - xy' = y' y^2 e^y$

$$=> y\frac{dx}{dy} = x + y^2 e^y$$

$$\frac{dx}{dy} - \frac{1}{y}x = y e^y$$

$$IF = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$\therefore x\frac{1}{y} = \int y e^y \ \frac{1}{y} dy = \int e^y \, dy$$

Example 2

$$=> \frac{x}{y} = e^{y} + c$$

$$=> x = y\,e^{y} + cy$$