Exact Differential Equation:

Definition: Exact Equation

If An expression $M(x,y)dx+N(x,y)dy$ is an exact differential in a region R corresponding to the differential of some function $f(x, y)$.

A first-order DE of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is said to be an **exact equation**, if the left side is an exact differential.

Exact Differential Equation:

Definition: Exact Equation

If An expression $M(x,y)dx+N(x,y)dy$ is an exact differential in a region R corresponding to the differential of some function $f(x, y)$.

A first-order DE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation**, if the left side is an exact differential.

Remark

If the differential of $f(x, y)$ is $M(x, y)\,dx + N(x, y)\,dy$, then $f(x, y) = c$ is an implicit solution to the DE $M(x, y)\,dx + N(x, y)\,dy = 0$

Exact Equation:

Theorem:

Let $M(x, y)$ and $N(x, y)$ be continuous with continuous first partial derivatives on a rectangular region R of the xy-plane. Then, a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equation:

Theorem:

Let $M(x, y)$ and $N(x, y)$ be continuous with continuous first partial derivatives on a rectangular region R of the xy-plane. Then, a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Facts

- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ implies exactness.
- exactness implies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution Method:

Solution Method:

Since $\partial f/\partial x = M(x, y)$, we have.

$$f(x,\ y) = \int M(x,y)dx + g(y) \qquad (i)$$

Solution Method:

Since $\partial f/\partial x = M(x, y)$, we have.

$$f(x,\ y) = \int M(x,y)dx + g(y) \qquad (i)$$

Differentiating *(i)* with respect to *y* and assume

$$\partial f/\partial y = N(x, y)$$

Then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,\ y)dx + g'(y) = N(x,\ y)$$

And

$$g'(y) = N(x,\ y) - \frac{\partial}{\partial y}\int M(x,\ y)dx \qquad (ii)$$

Solution Method:

Since $\partial f/\partial x = M(x, y)$, we have.

$$f(x, y) = \int M(x, y)dx + g(y) \qquad (i)$$

Differentiating *(i)* with respect to *y* and assume

$$\partial f/\partial y = N(x, y)$$

Then
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)dx + g'(y) = N(x, y)$$

And
$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx \qquad (ii)$$

Integrate *(ii)* with respect to *y* to get *g(y)*, and substitute the result into *(i)* to obtain the implicit solution *f(x, y) = c*.

Example 1

Solve the DE $(x^2 + y^2)\, dx + (x^2 - xy)\, dy = 0$.

Example 1

Solve the DE $(x^2 + y^2)\, dx + (x^2 - xy)\, dy = 0.$

With $M(x, y) = 2xy,$ $N(x, y) = x^2 - 1,$

we have $\partial M/\partial y = 2x = \partial N/\partial x$

Thus it is exact.

Example 1

Solve the DE $(x^2 + y^2)\, dx + (x^2 - xy)\, dy = 0.$

With $M(x, y) = 2xy,$ $N(x, y) = x^2 - 1,$
we have $\partial M/\partial y = 2x = \partial N/\partial x$
Thus it is exact.

There exists a function f such that

$$df = Mdx + Ndy$$
$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Example 1

Solve the DE $(x^2 + y^2)\, dx + (x^2 - xy)\, dy = 0$.

With $M(x, y) = 2xy$, $N(x, y) = x^2 - 1$,

we have $\partial M/\partial y = 2x = \partial N/\partial x$

Thus it is exact.

There exists a function f such that

$$df = Mdx + Ndy$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

So

$M = \partial f/\partial x = 2xy$, $N = \partial f/\partial y = x^2 - 1$

Example 1

Then $f(x, y) = x^2y + g(y)$

$\partial f/\partial y = x^2 + g'(y) = x^2 - 1$

$g'(y) = -1,$

$g(y) = -y$

Example 1

Then $f(x, y) = x^2y + g(y)$

$$\partial f/\partial y = x^2 + g'(y) = x^2 - 1$$

$$g'(y) = -1,$$

$$g(y) = -y$$

Hence $f(x, y) = x^2y - y$, and the solution is

$$x^2y - y = c$$

Example 2

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$

Example 2

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$

Rewrite the DE in the form

$$(\cos x \sin x - xy^2)\, dx + y(1 - x^2)\, dy = 0$$

Example 2

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$

Rewrite the DE in the form

$$(\cos x \sin x - xy^2)\, dx + y(1 - x^2)\, dy = 0$$

Since

$$\partial M/\partial y = -2xy = \partial N/\partial x$$ (This DE is exact)

There exists a function f such that

$$df = Mdx + Ndy$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Example 2

So

$M = \partial f/\partial x = \cos x \sin x - xy^2$, $\quad N = \partial f/\partial y = y(1 - x^2)$

Example 2

So

$M = \partial f/\partial x = \cos x \sin x - xy^2$, $\quad N = \partial f/\partial y = y(1 - x^2)$

Then $\partial f/\partial y = y(1 - x^2)$

$$f(x, y) = \frac{1}{2} y^2(1 - x^2) + h(x)$$

$$\partial f/\partial x = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

Example 2

So

$M = \partial f/\partial x = \cos x \sin x - xy^2$, $N = \partial f/\partial y = y(1 - x^2)$

Then $\partial f/\partial y = y(1 - x^2)$

$$f(x, y) = \frac{1}{2} y^2(1 - x^2) + h(x)$$

$$\partial f/\partial x = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

We have

$$h'(x) = \cos x \sin x$$

$$h(x) = -\frac{1}{2} \cos^2 x$$

Example 2

So

$M = \partial f/\partial x = \cos x \sin x - xy^2$, $\quad N = \partial f/\partial y = y(1 - x^2)$

Then $\partial f/\partial y = y(1 - x^2)$

$$f(x, y) = \frac{1}{2} y^2(1 - x^2) + h(x)$$

$$\partial f/\partial x = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

We have

$$h'(x) = \cos x \sin x$$

$$h(x) = -\frac{1}{2} \cos^2 x$$

Thus $\quad \frac{1}{2} y^2(1 - x^2) - \frac{1}{2} \cos^2 x = c_1$

or $\quad y^2(1 - x^2) - \cos^2 x = c$

where $c = 2c_1$.

Example 2

Now $y(0) = 2$, so $c = 3$.

The solution is

$$y^2(1 - x^2) - cos^2 x = 3$$

Integrating Factors:

Introduction:

It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$M(x,y)dx + N(x,y)dy = 0$$

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

For this equation to be exact, we need

$$\left(\mu M\right)_y = \left(\mu N\right)_x$$

$$M\mu_y + M_y\mu = N\mu_x + N_x\mu$$

$$N\mu_x - M\mu_y = \mu\left(M_y - N_x\right)$$

Integrating Factors:

- μ is a function of x only

If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only.

Integrating Factors:

μ is a function of x only

If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only.

μ is a function of y only

If μ is a function of y alone, then $\mu_x = 0$ and hence we solve

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu,$$

provided right side is a function of y only.

Integrating Factors:

Summary

We summarize the results for $M(x, y)\,dx + N(x, y)\,dy = 0$

- If $\dfrac{M_y - N_x}{N}$ depends only on x, then $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$

Integrating Factors:

Summary

We summarize the results for *M(x, y) dx + N(x, y) dy = 0*

- If $\dfrac{M_y - N_x}{N}$ depends only on x, then $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$
- If $\dfrac{N_x - M_y}{M}$ depends only on y, then $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

Integrating Factors:

Summary

We summarize the results for *M(x, y) dx + N(x, y) dy = 0*

- If $\dfrac{M_y - N_x}{N}$ depends only on x, then $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$
- If $\dfrac{N_x - M_y}{M}$ depends only on y, then $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$
- If we can write $M = yf(xy)$ $\quad N = xg(xy)$ then

$$\mu(x, y) = \frac{1}{xM - yN}$$

Example 3

The nonlinear DE: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.

Since $M = xy,$ $N = 2x^2 + 3y^2 - 20$

$M_y = x,$ $N_x = 4x.$

Example 3

The nonlinear DE: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.

Since $M = xy$, $N = 2x^2 + 3y^2 - 20$

$M_y = x$, $N_x = 4x$.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$ depends on both x and y.

Example 3

The nonlinear DE: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.

Since $M = xy$, $N = 2x^2 + 3y^2 - 20$

$M_y = x$, $N_x = 4x$.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$ depends on both x and y.

$$\frac{N_x - M_y}{M} = \frac{3}{y}$$ depends only on y.

Example 3

The nonlinear DE: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.

Since $M = xy$, $N = 2x^2 + 3y^2 - 20$

$M_y = x$, $N_x = 4x$.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$ depends on both x and y.

$$\frac{N_x - M_y}{M} = \frac{3}{y}$$ depends only on y.

The integrating factor is $e^{\int \frac{3}{y} dy} = e^{3\ln y} = y^3 = \mu(y)$

Example 3

The nonlinear DE: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.

Since $M = xy$, $N = 2x^2 + 3y^2 - 20$

$M_y = x$, $N_x = 4x$.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$ depends on both x and y.

$$\frac{N_x - M_y}{M} = \frac{3}{y}$$ depends only on y.

The integrating factor is $e^{\int \frac{3}{y} dy} = e^{3 \ln y} = y^3 = \mu(y)$

then the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

Try to show that the solution is:

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$