Homogeneous Equation:

Definition: Homogeneous Function

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Example

$f(x, y) = x^3 + y^3$ is homogeneous of degree 3,

$f(\lambda x, \lambda y) = (\lambda x)^3 + (\lambda y)^3 = \lambda^3 f(x, y)$

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A first-order DE:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is said to be homogeneous, if both *M* and *N* are homogeneous of the same degree, that is, if

$$M(\lambda x, \lambda y) = \lambda^{\alpha} M(x, y),\ N(\lambda x, \lambda y) = \lambda^{\alpha} N(x, y)$$

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Examples

$(x^2-3xy)dx + (7x^2-y^2)dy=0$ homogeneous DE of order 2

$(x+5y)dx - (x^2+4y^2)dy=0$ not a homogeneous DE

Facts about Homogeneous Equation:

Fact 1

If M and N are homogeneous of the same degree, then $\frac{M}{N}$ is homogeneous of degree 0.

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Proof

$$\frac{M(\lambda x,\lambda y)}{N(\lambda x,\lambda y)}=\frac{\lambda^{a}M(x,y)}{\lambda^{a}N(x,y)}=\frac{M(x,y)}{N(x,y)}=\lambda^{0}\,\frac{M(x,y)}{N(x,y)}$$

Facts about homogeneous equation

Fact 2

If f is homogeneous of degree 0, then f can be expressed as a function of $\frac{y}{x}$.

$$f(x,y)=\phi(v), \qquad v=\frac{y}{x}$$

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$(x^2 + u^2x^2)\, dx + (x^2 - ux^2)(u\, dx + x\, du) = 0$

$$\frac{1-u}{1+u}\, du + \frac{dx}{x} = 0$$

$$\left[-1 + \frac{2}{1+u}\right] du + \frac{dx}{x} = 0$$

Example 1

Then

$$-u + 2\ln|1+u| + \ln|x| = \ln|c|$$

$$-\frac{y}{x} + 2\ln\left|1+\frac{y}{x}\right| + \ln|x| = \ln|c|$$

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Simplify to get

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x} \quad \text{or} \quad (x+y)^2 = cxe^{y/x}$$

* Note: We may also try x = vy.