What are Differential Equations ?

Definition

An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables is said to be a **differential equation.(DE)**

Classification of DEs

Differential Equations are classified by

- type
- order
- linearity

Classification by Type

Differential equations can be classified into two distinct types:

- **Ordinary Differential Equations :** are equations which have a single independent variable. **(ODE).**

- **Partial Differential Equations :** are equations with multiple independent variables. **(PDE).**

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Examples

$$\frac{dy}{dx}+5y=e^x,\quad \frac{d^2y}{dx^2}-\frac{dy}{dx}+6y=0,\quad \frac{dx}{dt}+\frac{dy}{dt}=2x+y \text{ are ODE}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}-2\frac{\partial u}{\partial t},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \text{ are PDE}$$

Classification by Order

The order of a DE is the highest order derivative appearing in the equation.

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Examples

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

2^{nd} order

$$y' + \sin y = x^2$$

1^{st} order

Classification by linearity

A differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Notice the following properties, which make this linear.

- All coefficients depend only on x.
- The function y and its derivatives appear to at most the first power.
- The function g(x) depends only on x.

Examples

$(y-x)dx+4xdy=0$	linear 1^{st} order ODE
$(1-y)y^{'}+2y=e^{x}$	nonlinear 1^{st} order ODE
$\frac{d^{3}y}{dx^{3}}+3x\frac{dy}{dx}-5y=e^{x}$	linear 3^{rd} order ODE
$\frac{d^{2}y}{dx^{2}}+\sin(y)=0$	nonlinear 2^{nd} order ODE

Solution of a Differential Equation

Definition

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

The graph of a solution ϕ if an ODE is called an **integral curve.**

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Example

Verify that the function

$$y = e^{-x^2} \int_o^x e^{t^2} \text{ is a solution to the ODE } y' + 2xy = 1$$

Solution of a Differential Equation

General Solution:

Solutions obtained from integrating the differential equations are called general solutions. The general solution of a nth order ordinary differential equation contains n arbitrary constants resulting from integrating times.

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Particular Solution:

Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.

Solution of a Differential Equation

Singular Solutions:

Solutions that can not be expressed by the general solutions are called singular solutions.

Explicit Solution

Definition

A solution in which the dependent variable is expressed in terms of the independent variable and constants is said to be an explicit solution. i.e $y=f(x)$

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Implicit Solution

Definition

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the **differential equation** on I.

Example

$x^2 + y^2 = 25$ is an implicit solution to the ODE $y' = -\dfrac{x}{y}$

The Existence and Uniqueness Theorem

THEOREM: Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\partial f / \partial y$ are continuous on R, Then there exists some interval $I_0 : x_0 - h < x < x_0 + h,\ \ h > 0$ contained in $a \le x \le b$ and a unique function $y(x)$ defined on I_0 that is a solution of the initial value problem.