Chapter Two: Kinematics

2.1 Mechanism:

A mechanical manipulator can be modelled as an open-loop articulated chain with several rigid bodies (links) connected in series by either revolute or prismatic joints driven by actuators.

One end of the chain is attached to a supporting base while the other end is free and attached with a tool (the end-effector) to manipulate objects or perform assembly tasks. The relative motion of the joints results in the motion of the links that positions the hand in a desired orientation.

In most robotic applications, one is interested in the spatial description of the end-effector of the manipulator with respect to a fixed reference coordinate system.

Robot arm kinematics deals with the analytical study of the geometry of motion of a robot arm with respect to a fixed reference coordinate system as a function of time without regard to the forces/moments that cause the motion. Thus, it deals with the analytical description of the spatial displacement of the robot as a function of time, in particular the relations between the joint-variable space and the position and orientation of the end-effector of a robot arm.

Theoretical and practical interest in robot arm kinematics:

1. For a given manipulator, given the joint angle vector $q(t) = (q_1(t), q_2(t), \ldots q_n(t))^T$ and the geometric link parameters, where n is the number of degrees of freedom, what is the position and orientation of the end-effector of the manipulator with respect to a reference coordinate system?

2. Given a desired position and orientation of the end-effector of the manipulator and the geometric link parameters with respect to a reference coordinate system, can the manipulator reach the desired prescribed manipulator hand position and orientation? And if it can, how many different manipulator configurations will satisfy the same condition?

The first question is usually referred to as the ***direct (or forward) kinematics*** problem, while the second question is the ***inverse kinematics (or arm solution) problem***.
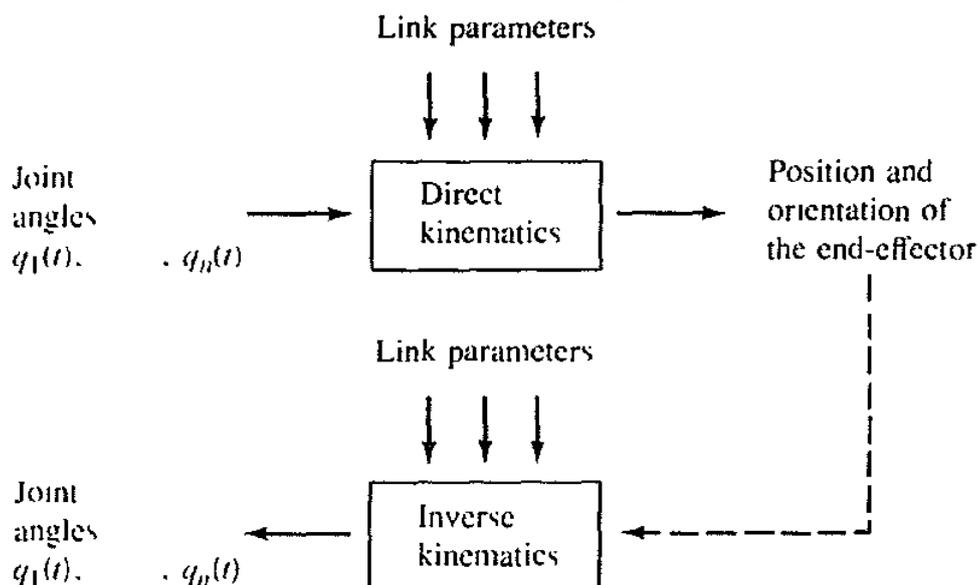

Fig. 2.1: The direct and inverse kinematics problems.

2.2 Matrix representation:

2.2.1 Rotation Matrix:

Let the point p represented by its coordinates with respect to the OUVW and OXYZ coordinate systems, respectively, as

$p_{uvw} = (p_u, p_v, p_w)$ and $p_{xyz} = (p_x, p_y, p_z)$ (1)

We would like to find a 3 x 3 transformation matrix R that will transform the coordinates of p_{uvw} to the coordinates expressed with respect to the OXYZ coordinate system, after the OUVW coordinate system has been rotated. That is,

$p_{xyz} = R p_{uvw}$ (2)

Note that physically the point p_{uvw} has been rotated together with the OUVW coordinate system.
Recalling the definition of the components of a vector, we have

$p_{uvw} = p_u\, i_u + p_v\, j_v + p_w\, k_w$ (3)

where p_x, p_y, and p_z represent the components of p along the OX, OY, and OZ axes, respectively, or the projections of p onto the respective axes. Thus, using the definition of a scalar product and Eq.(3)

$p_x = i_x \cdot p = i_x \cdot i_u\, p_u + i_x \cdot j_v\, p_v + i_x \cdot k_w\, p_w$

$p_y = j_y \cdot p = j_y \cdot i_u\, p_u + j_y \cdot j_v\, p_v + j_y \cdot k_w\, p_w$ (4)

$p_x = k_z \cdot p = k_z \cdot i_u\, p_u + k_z \cdot j_v\, p_v + k_z \cdot k_w\, p_w$

or expressed in matrix form,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$ (5)

Using this notation, the matrix R in Eq. (2) is given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix}$$ (6)

Similarly, one can obtain the coordinates of p_{uvw} from the coordinates of p_{xyz}:

$p_{uvw} = Q\, p_{xyz}$ (7)

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \mathbf{i}_u \cdot \mathbf{i}_x & \mathbf{i}_u \cdot \mathbf{j}_y & \mathbf{i}_u \cdot \mathbf{k}_z \\ \mathbf{j}_v \cdot \mathbf{i}_x & \mathbf{j}_v \cdot \mathbf{j}_y & \mathbf{j}_v \cdot \mathbf{k}_z \\ \mathbf{k}_w \cdot \mathbf{i}_x & \mathbf{k}_w \cdot \mathbf{j}_y & \mathbf{k}_w \cdot \mathbf{k}_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$ (8)

Since dot products are commutative, one can see from Eqs. (6) to (8) that

$Q = R^{-1} = R^T$ (9)

$RQ = RR^T = RR^{-1} = I_3$ (10)

where I_3 is a 3 x 3 identity matrix. The transformation given in Eq. (2) or (7) is called an *orthogonal* transformation and since the vectors in the dot products are all unit vectors, it is also called an *orthonormal* transformation.

The primary interest in developing the above ***transformation matrix*** is to find ***the rotation matrices*** that represent rotations of the OUVW coordinate system about each of the three principal axes of the reference coordinate system OXYZ. If the OUVW coordinate system is rotated an α angle about the OX axis to arrive at a new location in the space, then the point p_{uvw} having coordinates (p_u, p_v, p_w)^T with respect to the OUVW system will have different coordinates (p_x, p_y, p_z)^T with respect to the reference system OXYZ.

The necessary transformation matrix $R_{x,\alpha}$ is called the rotation matrix about the OX axis with α angle. $R_{x,\alpha}$ can be derived from the above transformation matrix concept, that is

$$p_{xyz} = R_{x,\alpha}\, p_{uvw} \qquad\qquad (11)$$

with $i_x = i_u$

$$\mathbf{R}_{x,\alpha} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$(12)$$

Similarly, the 3 x 3 rotation matrices for rotation about the OY axis with ϕ angle and about the OZ axis with θ angle are, respectively (see Fig. 2.2),

$$\mathbf{R}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(13)$$

The matrices $R_{x,\alpha}$, $R_{y,\phi}$, and $R_{z,\theta}$ are called the basic rotation matrices. Other finite rotation matrices can be obtained from these matrices.
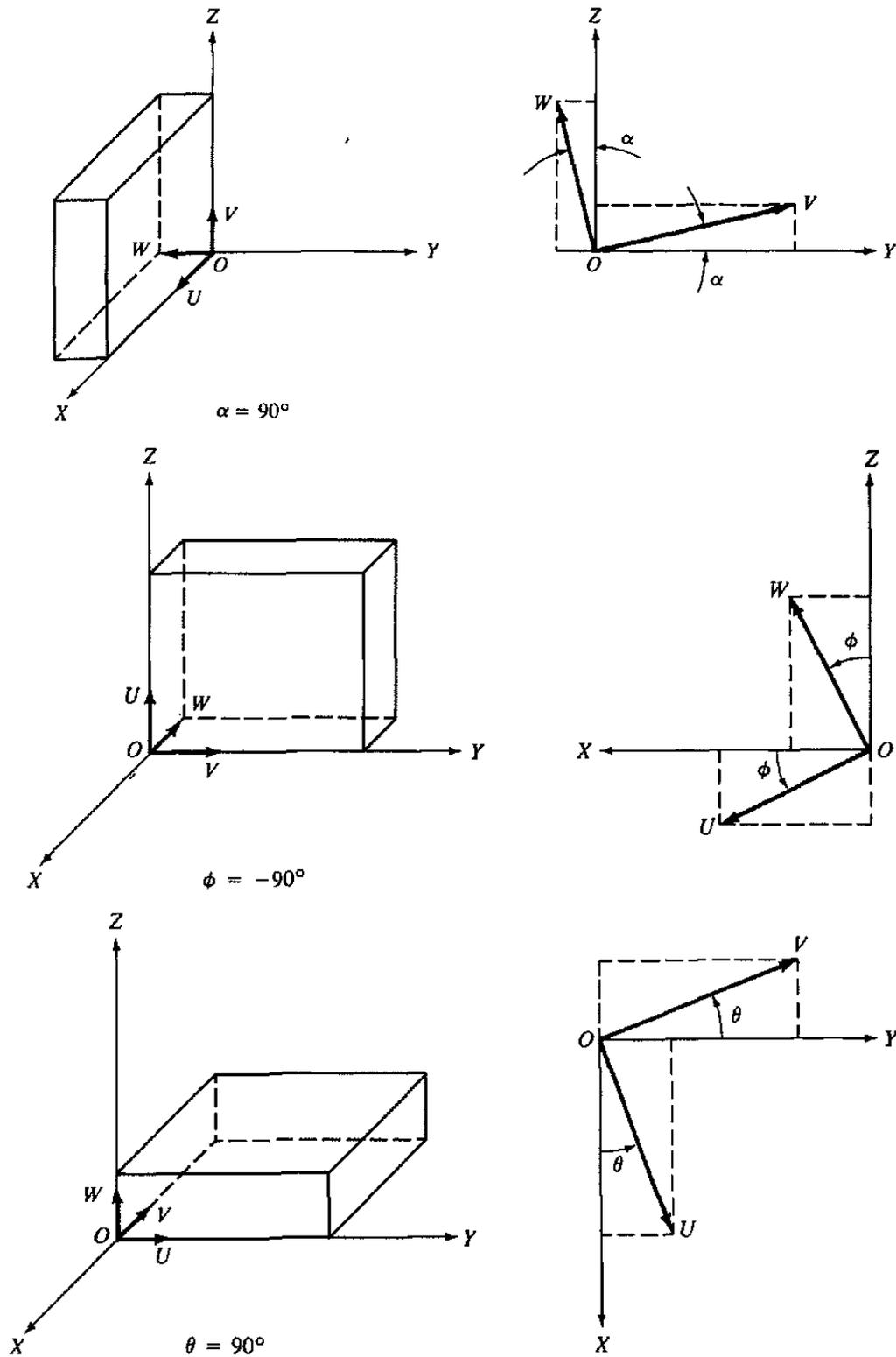
Fig. 2.2: Rotating coordinate systems.

Example 2.1: Given two points $a_{uvw} = (4, 3, 2)^T$ and $b_{uvw} = (6, 2, 4)^T$ with respect to the rotated OUVW coordinate system, determine the corresponding points a_{xyz}, b_{xyz}, with respect to the reference coordinate system if it has been rotated 60° about the OZ axis.

Solution: $a_{xyz} = R_{z,60} \, a_{uvw}$ and $b_{xyz} = R_{z,60} \, b_{uvw}$

$$
\mathbf{a}_{xyz} = \begin{bmatrix} 0.500 & -0.866 & 0 \\ 0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4(0.5)+3(-0.866)+2(0) \\ 4(0.866)+3(0.5)+2(0) \\ 4(0)+3(0)+2(1) \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2.0 \end{bmatrix}
$$

$$
\mathbf{b}_{xyz} = \begin{bmatrix} 0.500 & -0.866 & 0 \\ 0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.268 \\ 6.196 \\ 4.0 \end{bmatrix}
$$

Thus, a_{xyz} and b_{xyz} are equal to $(-0.598, 4.964, 2.0)^T$ and $(1.268, 6.196, 4.0)^T$, respectively, when expressed in terms of the reference coordinate system.

Example 2.2: If $a_{xyz} = (4, 3, 2)^T$ and $b_{xyz} = (6, 2, 4)^T$ are the coordinates with respect to the reference coordinate system, determine the corresponding points a_{uvw}, b_{uvw} with respect to the rotated OUVW coordinate system if it has been rotated 60° about the OZ axis.

Solution: $a_{uvw} = (R_{z,60})^T \, a_{xyz}$, $b_{uvw} = (R_{z,60})^T \, b_{xyz}$

$$
\mathbf{a}_{uvw} = \begin{bmatrix} 0.500 & 0.866 & 0 \\ -0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(0.5)+3(0.866)+2(0) \\ 4(-0.866)+3(0.5)+2(0) \\ 4(0)+3(0)+2(1) \end{bmatrix}
$$

$$
= \begin{bmatrix} 4.598 \\ -1.964 \\ 2.0 \end{bmatrix}
$$

$$
\mathbf{b}_{uvw} = \begin{bmatrix} 0.500 & 0.866 & 0 \\ -0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4.732 \\ -4.196 \\ 4.0 \end{bmatrix}
$$

2.2.2 Composite Rotation Matrix:

Basic rotation matrices can be multiplied together to represent a sequence of finite rotations about the principal axes of the OXYZ coordinate system. Since matrix multiplications do not commute, the order or sequence of performing rotations is important

In addition to rotating about the principal axes of the reference frame OXYZ, the rotating coordinate system OUVW can also rotate about its own principal axes. In this case, the resultant or composite rotation matrix may be obtained from the following simple rules:

1. Initially both coordinate systems are coincident; hence, the rotation matrix is a 3 x 3 identity matrix, I_3.
2. If the rotating coordinate system OUVW is rotating about one of the principal axes of the OXYZ frame, then **_premultiply_** the previous (resultant) rotation matrix with an appropriate basic rotation matrix.
3. If the rotating coordinate system OUVW is rotating about its own principal axes, then **_postmultiply_** the previous (resultant) rotation matrix with an appropriate basic rotation matrix.

Example 2.3: Find the resultant rotation matrix that represents a rotation of ϕ angle about the OY axis followed by a rotation of θ angle about the OW axis followed by a rotation of α angle about the OU axis.

SOLUTION:

$R = R_{y,\phi} \, I_3 \, R_{w,\theta} \, R_{u,\alpha} = R_{y,\phi} \, R_{w,\theta} \, R_{u,\alpha}$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

2.2.3 Rotation Matrix with Euler Angles Representation

The matrix representation for rotation of a rigid body simplifies many operations, but it needs nine elements to completely describe the orientation of a rotating rigid body. It does not lead directly to a complete set of generalized coordinates. Such a set of generalized coordinates can describe the orientation of a rotating rigid body with respect to a reference coordinate frame. They can be provided by three angles called Euler angles ϕ, θ, and ψ. Although Euler angles describe the orientation of a rigid body with respect to a fixed reference frame, there are many different types of Euler angle representations. The three most widely used Euler angles representations are tabulated in Table 2.1.

Table 2.1 Three types of Euler angle representations

	Eulerian angles system I	Euler angles system II	Roll, pitch, and yaw system III
Sequence of rotations	ϕ about OZ axis θ about OU axis ψ about OW axis	ϕ about OZ axis θ about OV axis ψ about OW axis	ψ about OX axis θ about OY axis ϕ about OZ axis

The first Euler angle representation in Table 2.1 is usually associated with gyroscopic motion. This representation is usually called the eulerian angles, and corresponds to the following sequence of rotations

1. A rotation of ϕ angle about the OZ axis ($R_{z,\phi}$)

2. A rotation of θ angle about the rotated OU axis ($R_{u,\theta}$)

3. Finally a rotation of ψ angle about the rotated OW axis ($R_{w,\psi}$)

The resultant eulerian rotation matrix is

$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{u,\theta}\,\mathbf{R}_{w,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -C\phi S\psi - S\phi C\theta C\psi & S\phi S\theta \\ S\phi C\psi + C\phi C\theta S\psi & -S\phi S\psi + C\phi C\theta C\psi & -C\phi S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix}$$

Another set of Euler angles ϕ, θ, and ψ representation corresponds to the following sequence of rotations:

1. A rotation of ϕ angle about the OZ axis ($R_{z,\phi}$)

2. A rotation of θ angle about the rotated OV axis ($R_{v,\theta}$)

3. Finally a rotation of ψ angle about the rotated OW axis ($R_{w,\psi}$)

The resultant rotation matrix is

$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{v,\theta}\,\mathbf{R}_{w,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta \\ -S\theta C\psi & S\theta S\psi & C\theta \end{bmatrix}$$

Another set of Euler angles representation for rotation is called *roll*, *pitch*, and *yaw* (RPY). This is mainly used in aeronautical engineering in the analysis of space vehicles. They correspond to the following rotations in sequence:

1. A rotation of ψ about the OX axis ($R_{x,\psi}$)-yaw

2. A rotation of θ about the OY axis ($R_{y,\theta}$)-pitch

3. A rotation of ϕ about the OZ axis ($R_{z,\phi}$)-roll

The resultant rotation matrix is
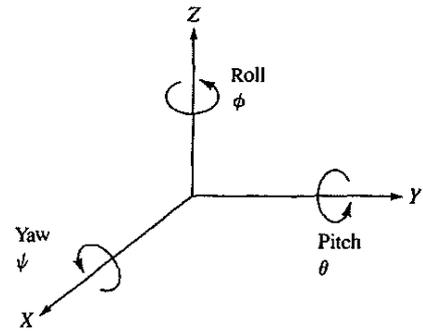
$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\psi} = \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & -S\psi \\ 0 & S\psi & C\psi \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\ -S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix}$$

2.3 Homogenous transformation:

The homogeneous transformation matrix is a 4 x 4 matrix which maps a position vector expressed in homogeneous coordinates from one coordinate system to another coordinate system. A homogeneous transformation matrix can be considered to consist of four submatrices:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R}_{3\times3} & | & \mathbf{p}_{3\times1} \\ - & | & - \\ \mathbf{f}_{1\times3} & | & 1\times1 \end{bmatrix} = \begin{bmatrix} \text{rotation matrix} & | & \text{position vector} \\ - & | & - \\ \text{perspective transformation} & | & \text{scaling} \end{bmatrix} \qquad (14)$$

The upper-left 3 x 3 submatrix represents the rotation matrix; the upper right 3 x 1 submatrix represents the position vector of the origin of the rotated coordinate system with respect to the reference system; the lower left 1 x 3 submatrix represents perspective transformation; and the fourth diagonal element is the global scaling factor. The homogeneous transformation matrix can be used to explain the geometric relationship between the bodies attached frame OUVW and the reference coordinate system OXYZ.

If a position vector p in a three-dimensional space is expressed in homogeneous coordinates [i.e., $\acute{p} = (p_x, p_y, p_z, 1)^T$], then using the transformation matrix concept, a 3 x 3 rotation matrix can be extended to a 4 x 4 homogeneous transformation matrix Trot for pure rotation operations. Thus, Eqs.(12) and (13), expressed as homogeneous rotation matrices, become

$$T_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(15)

These 4 x 4 rotation matrices are called the basic homogeneous rotation matrices.

The upper right 3 x 1 submatrix of the homogeneous transformation matrix has the effect of translating the OUVW coordinate system which has axes parallel to the reference coordinate system OXYZ but whose origin is at (dx, dy, dz) of the reference coordinate system:

$$T_{tran} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(16)

This 4 x 4 transformation matrix is called the basic homogeneous translation matrix.

The principal diagonal elements of a homogeneous transformation matrix produce local and global scaling. The first three diagonal elements produce local stretching or scaling, as in

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

(17)

Thus, the coordinate values are stretched by the scalars a, b, and c, respectively. Note that the basic rotation matrices, T_{rot}, do not produce any local scaling effect. The fourth diagonal element produces global scaling as in

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix}$$

(18)

where s > 0. The physical Cartesian coordinates of the vector are

$$p_x = \frac{x}{s} \qquad p_y = \frac{y}{s} \qquad p_z = \frac{z}{s} \qquad w = \frac{s}{s} = 1$$

(19)

Therefore, the fourth diagonal element in the homogeneous transformation matrix has the effect of globally reducing the coordinates if s > 1 and of enlarging the coordinates if 0 < s < 1.

In summary, a 4 X 4 homogeneous transformation matrix maps a vector expressed in homogeneous coordinates with respect to the OUVW coordinate system to the reference coordinate system OXYZ. That is, with w = 1,

$P_{xyz} = T\ p_{uvw}$ (20)

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

                                                                        (21)

In general, the inverse of a homogeneous transformation matrix can be found to be

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{n}^T\mathbf{p} \\ s_x & s_y & s_z & -\mathbf{s}^T\mathbf{p} \\ a_x & a_y & a_z & -\mathbf{a}^T\mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & -\mathbf{n}^T\mathbf{p} \\ & R^T_{3\times3} & & -\mathbf{s}^T\mathbf{p} \\ & & & -\mathbf{a}^T\mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

                                                                              (22)

The homogeneous rotation and translation matrices can be multiplied together to obtain a composite homogeneous transformation matrix (we shall call it the T matrix). However, since matrix multiplication is not commutative, careful attention must be paid to the order in which these matrices are multiplied. The following rules are useful for finding a composite homogeneous transformation matrix:

1. Initially both coordinate systems are coincident, hence the homogeneous transformation matrix is a 4 x 4 identity matrix, I_4.
2. If the rotating coordinate system OUVW is rotating/translating about the principal axes of the OXYZ frame, then ***premultiply*** the previous (resultant) homogeneous transformation matrix with an appropriate basic homogeneous rotation/translation matrix.
3. If the rotating coordinate system OUVW is rotating/translating about its own principal axes, then ***postmultiply*** the previous (resultant) homogeneous transformation matrix with an appropriate basic homogeneous rotation/translation matrix.

Example 2.4: Two points $a_{uvw} = (4, 3, 2)^T$ and $b_{uvw} = (6, 2, 4)^T$ are to be translated a distance +5 units along the OX axis and -3 units along the OZ axis. Using the appropriate homogeneous transformation matrix, determine the new points a_{xyz} and b_{xyz}.

Solution:

$$\hat{\mathbf{a}}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 1(5) \\ 3(1) + 1(0) \\ 2(1) + 1(-3) \\ 1(1) \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\hat{\mathbf{b}}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

The translated points are $a_{xyz} = (9, 3, -1)^T$ and $b_{xyz.} = (11, 2, 1)^T$.

Example 2.5: A T matrix is to be determined that represents a rotation of a angle about the OX axis, followed by a translation of b units along the rotated OV axis.
Solution:
In the orthodox approach, following the rules as stated earlier, one should realize that since the $T_{x,\alpha}$, matrix will rotate the OY axis to the OV axis, then translation along the OV axis will accomplish the same goal, that is,

$$T = T_{x,\,\alpha}\,T_{v,\,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & b\cos\alpha \\ 0 & \sin\alpha & \cos\alpha & b\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Find a homogeneous transformation matrix T that represents a rotation of α angle about the OX axis, followed by a translation of a units along the OX axis, followed by a translation of d units along the OZ axis, followed by a rotation of θ angle about the OZ axis.
SOLUTION:
$T = T_{z,\theta}\,T_{z,d}\,T_{x,a}\,T_{x,\alpha}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\cos\alpha\sin\theta & \sin\alpha\sin\theta & a\cos\theta \\ \sin\theta & \cos\alpha\cos\theta. & -\sin\alpha\cos\theta & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \square$$

2.4 D-H representation:

A mechanical manipulator consists of a sequence of rigid bodies, called links, connected by either revolute or prismatic joints (see Fig. 2.3). Each joint-link pair constitutes 1 degree of freedom. Hence, for an N degree of freedom manipulator, there are N joint-link pairs with link 0 (not considered part of the robot) attached to a supporting base where an inertial coordinate frame is usually established for this dynamic system, and the last link is attached with a tool. The joints and links are numbered outwardly from the base; thus, joint 1 is the point of connection between link 1 and the supporting base. Each link is connected to, at most, two others so that no closed loops are formed.

A joint axis (for joint i) is established at the connection of two links (see Fig. 2.4). This joint axis will have two normal connected to it, one for each of the links. The relative position of two such connected links (link i - 1 and link i) is given by d_i which is the distance measured along the joint axis between the normal. The joint angle θ_i between the normal is measured in a plane normal to the joint axis. Hence, d_i and θ_i may be called *the distance and the angle* between the adjacent links, respectively. They determine the relative position of neighbouring links.

A link i (i = 1, ... , 6) is connected to, at most, two other links (e.g., link i - 1 and link i + 1); thus, two joint axes are established at both ends of the connection. The significance of links, from a kinematic perspective, is that they maintain a fixed configuration between their joints, which can be characterized by two parameters: a_i and α_i. The parameter a_i is the shortest distance measured along the common normal between the joint axes (i.e., the z_{i-1} and z_i axes for joint i and joint i+1, respectively), and α_i is the angle between the joint axes measured in a plane perpendicular to ai. Thus, a_i and α_i may be called the *length and the twist angle* of the link i, respectively. They determine the structure of link i.
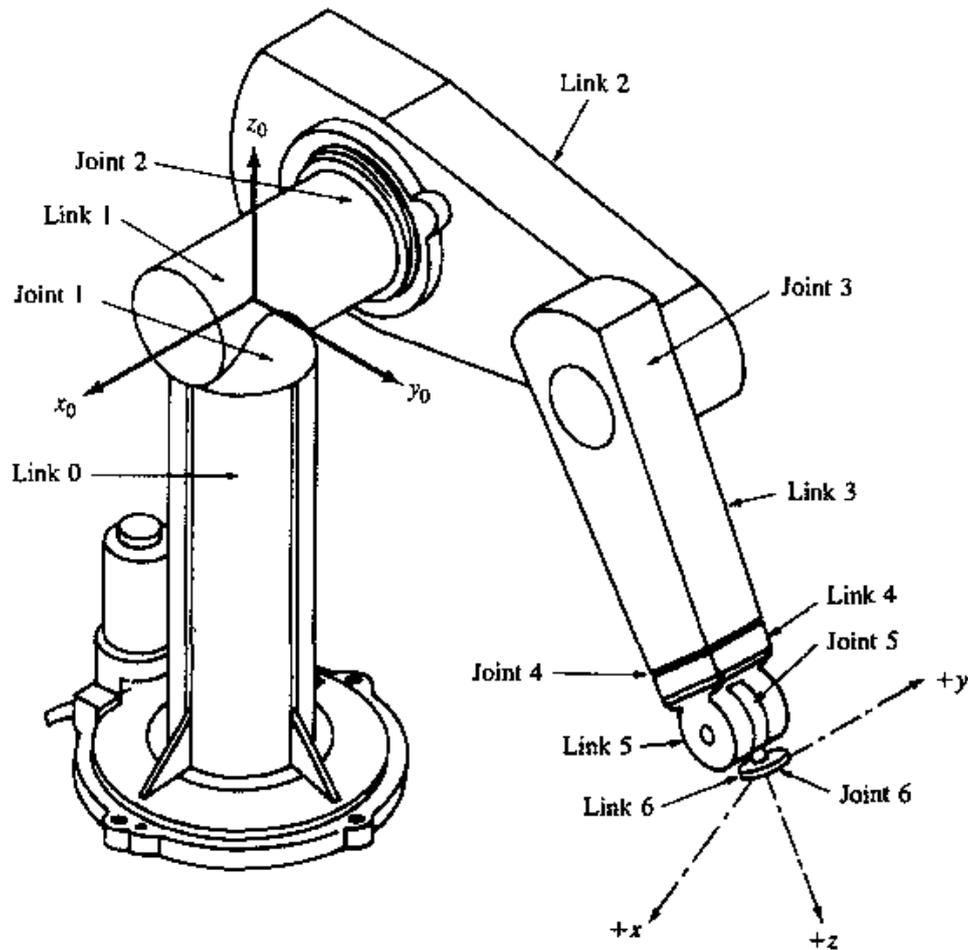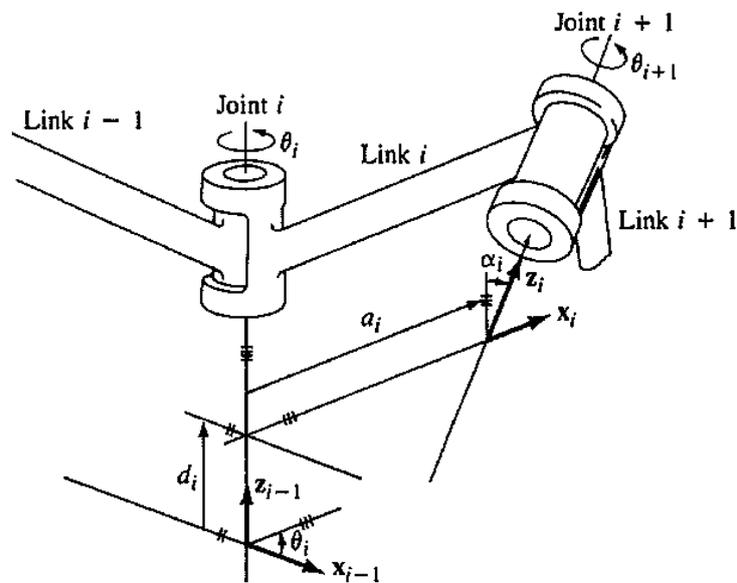
Fig. 2.3: A PUMA robot arm illustrating joints and links.

Fig. 2.4 Link coordinate system and its parameters.

In summary, four parameters, a_i, α_i, d_i, and θ_i, are associated with each link of a manipulator. If a sign convention for each of these parameters has been established, then these parameters constitute a sufficient set to completely determine the kinematic configuration of each link of a robot arm. Note that these four parameters come in pairs: the link parameters (a_i, α_i) which determine the structure of the link and the joint parameters (d_i, θ_i) which determine the relative position of neighbouring links.

The Denavit-Hartenberg (D-H) representation results in a 4 x 4 homogeneous transformation matrix representing each link's coordinate system at the joint with respect to the previous link's coordinate system. Thus, through sequential transformations, the end-effector expressed in the "hand coordinates" can be transformed and expressed in the "base coordinates" which make up the inertial frame of this dynamic system.

An orthonormal Cartesian coordinate system (x_i, y_i, z_i)t can be established for each link at its joint axis, where i = 1 , 2, ... ,n (n = number of degrees of freedom) plus the base coordinate frame. Since a rotary joint has only 1 degree of freedom, each (x_i, y_i, z_i) coordinate frame of a robot arm corresponds to joint i + 1 and is fixed in link i. When the joint actuator activates joint i, link i will move with respect to link i - 1. Since the ith coordinate system is fixed in link i, it moves together with the link i. Thus, the nth coordinate frame moves with the hand (link n). The base coordinates are defined as the 0th coordinate frame (x_o, y_o, z_o) which is also the inertial coordinate frame of the robot arm. Thus, for a six-axis PUMA-like robot arm, we have seven coordinate frames, namely, (x_o, y_o, z_o), (x_1, y_1, z_1), ... , (x_6, y_6, z_6).

Every coordinate frame is determined and established on the basis of three rules:
1. The z_{i-1} axis lies along the axis of motion of the ith joint.
2. The x_i axis is normal to the z_{i-1} axis, and pointing away from it.
3. The y_i axis completes the right-handed coordinate system as required.

The D-H representation of a rigid link depends on four geometric parameters associated with each link. These four parameters completely describe any revolute or prismatic joint. Referring to Fig. 2.4, these four parameters are defined as follows:

θ_i is the joint angle from the x_{i-1} axis to the x_i axis about the z_{i-1} axis (using the right-hand rule).

d_i is the distance from the origin of the (i -1)th coordinate frame to the intersection of the z_{i-1} axis with the x_i axis along the z_{i-1} axis.

a_i is the offset distance from the intersection of the z_{i-1} axis with the x_i axis to the origin of the ith frame along the x_i axis (or the shortest distance between the z_{i-1} and z_i axes).

α_i is the offset angle from the z_{i-1} axis to the z_i axis about the x_i axis (using the right-hand rule).

For a rotary joint, d_i, a_i, and α_i are the joint parameters and remain constant for a robot, while θ_i is the joint variable that changes when link i moves (or rotates) with respect to link i - 1. For a prismatic joint, θ_i, a_i, and α_i are the joint parameters and remain constant for a robot, while d_i is the joint variable.

Examples of a six-axis PUMA-like robot arm and a Stanford arm are given in Figs. 2.5 and 2.6, respectively.
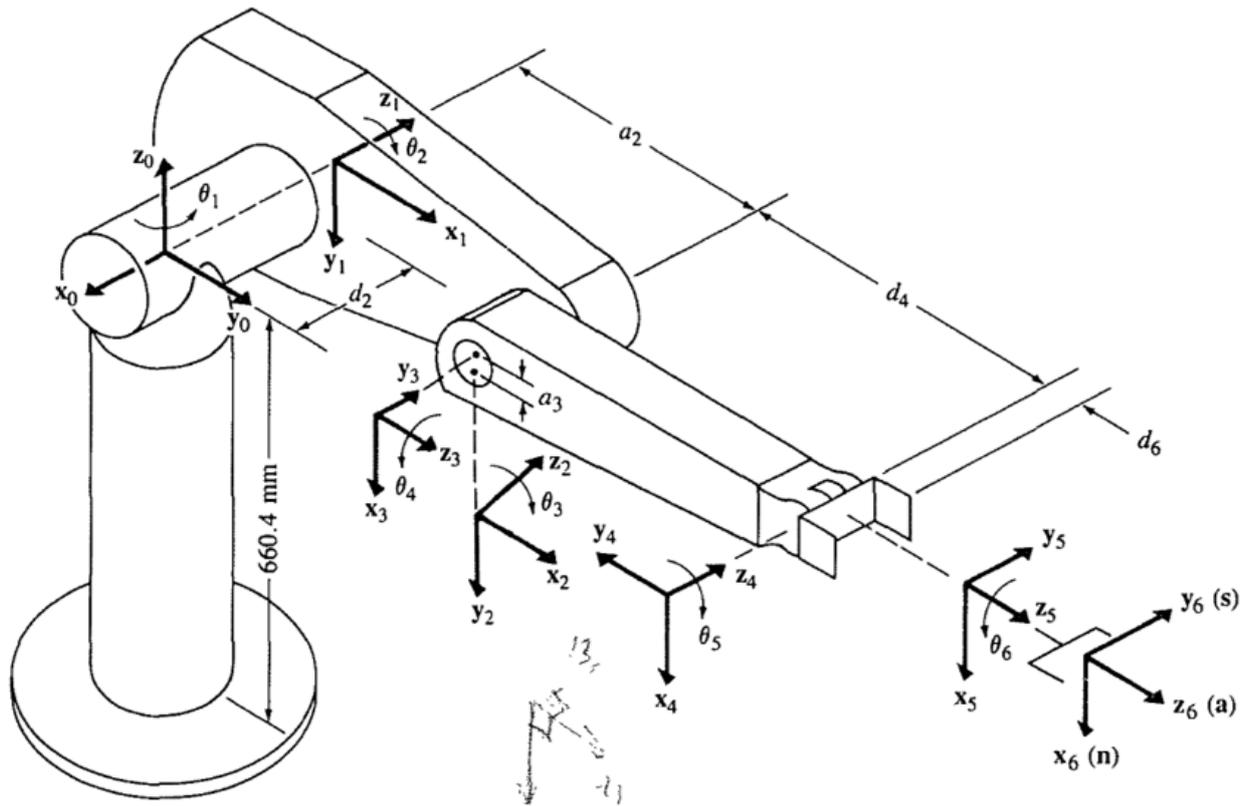
PUMA robot arm link coordinate parameters					
Joint i	θ_i	α_i	a_i	d_i	Joint range
1	90	−90	0	0	−160 to +160
2	0	0	431.8 mm	149.09 mm	−225 to 45
3	90	90	−20.32 mm	0	−45 to 225
4	0	−90	0	433.07 mm	−110 to 170
5	0	90	0	0	−100 to 100
6	0	0	0	56.25 mm	−266 to 266

Fig. 2.5: Establishing link coordinate systems for a PUMA robot.
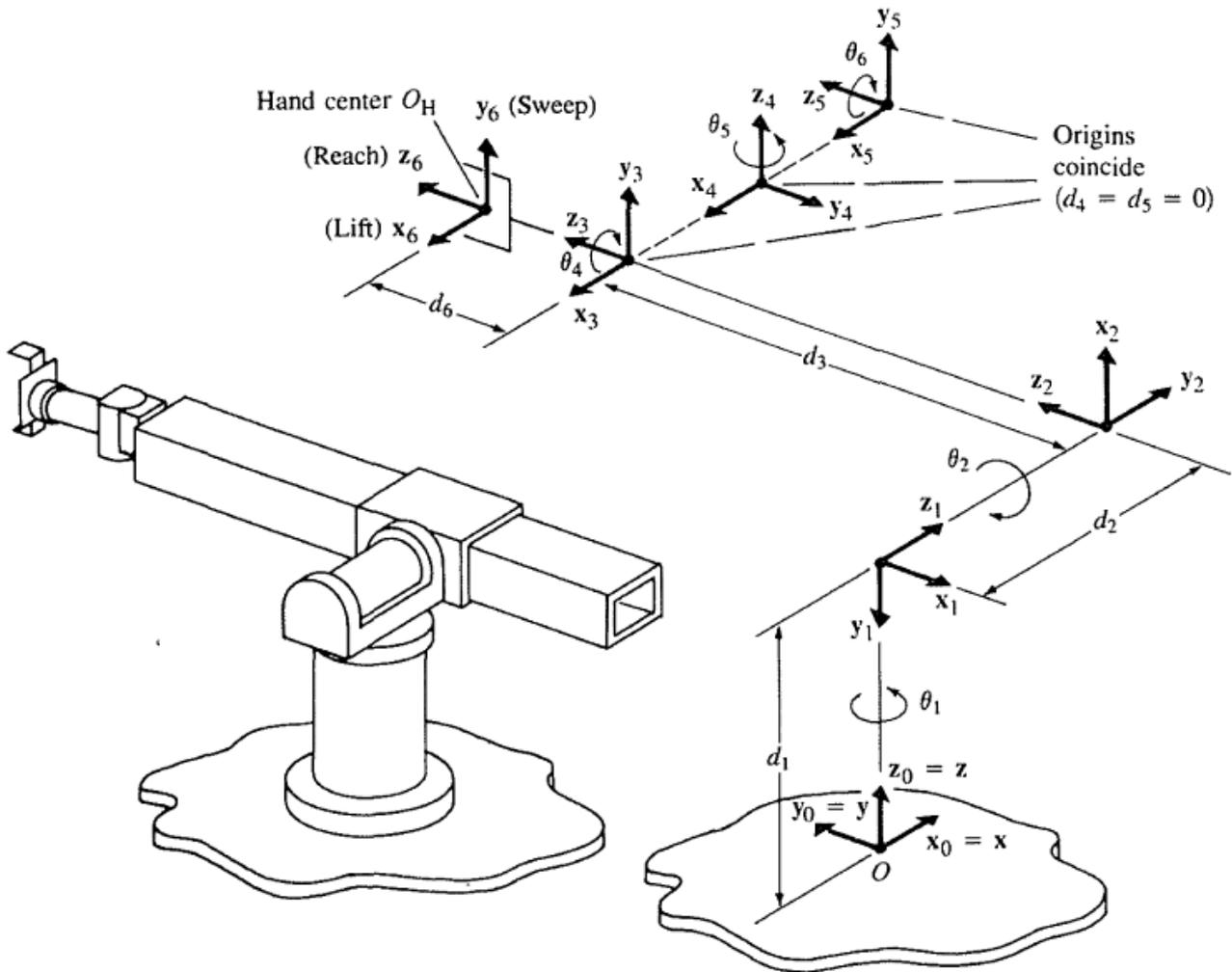
Stanford robot arm link coordinate parameters				
Joint i	θ_i	α_i	a_i	d_i
I	$\theta_1 = -90$	-90	0	d_1
2	$\theta_2 = -90$	90	0	d_2
3	-90	0	0	d_3
4	$\theta_4 = 0$	-90	0	0
5	$\theta_5 = 0$	90	0	0
6	$\theta_6 = 0$	0	0	d_6

Fig. 2.6: Establishing link coordinate systems for a Stanford robot.

Once the D-H coordinate system has been established for each link, a homogeneous transformation matrix can easily be developed relating the ith coordinate frame to the (i-1)th coordinate frame. Looking at Fig. 2.5, it is obvious that a point r_i expressed in the ith coordinate system may be expressed in the (i-1)th coordinate system as r_{i-1} by performing the following successive transformations:

1. Rotate about the z_{i-1} axis an angle of θ_i to align the x_{i-1} axis with the x_i axis (x_{i-1} axis is parallel to x_i and pointing in the same direction).

2. Translate along the z_{i-1} axis a distance of d_i to bring the x_{i-1} and x_i axes into coincidence.

3. Translate along the x_i axis a distance of a_i to bring the two origins as well as the x-axis into coincidence.

4. Rotate about the x_i axis an angle of α_i to bring the two coordinate systems into coincidence.

Each of these four operations can be expressed by a basic homogeneous rotation-translation matrix and the product of these four basic homogeneous transformation matrices yields a composite homogeneous transformation matrix, $^{i-1}A_i$, known as the D-H transformation matrix for adjacent coordinate frames, i and i - 1. Thus,

$$^{i-1}A_i = T_{z,d}\ T_{z,\theta}\ T_{x,a}\ T_{x,\alpha}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (23)$$

Using Eq. (22), the inverse of this transformation can be found to be

$$[^{i-1}A_i]^{-1} = {}^{i}A_{i-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i\sin\theta_i & \cos\alpha_i\cos\theta_i & \sin\alpha_i & -d_i\sin\alpha_i \\ \sin\alpha_i\sin\theta_i & -\sin\alpha_i\cos\theta_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (24)$$

where α_i, a_i, d_i are constants while θ_i is the joint variable for a revolute joint.

For a prismatic joint, the joint variable is d_i, while α_i, a_i, and θ_i are constants. In this case, $^{i-1}A_i$ becomes

$$^{i-1}A_i = T_{z,\theta}\ T_{z,d}\ T_{x,\alpha} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & 0 \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (25)$$

$$[^{i-1}\mathbf{A}_i]^{-1} = {}^i\mathbf{A}_{i-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & 0 \\ -\cos\alpha_i\sin\theta_i & \cos\alpha_i\cos\theta_i & \sin\alpha_i & -d_i\sin\alpha_i \\ \cos\alpha_i\sin\theta_i & -\sin\alpha_i\cos\theta_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(26)

Using the $^{i-1}\mathbf{A}_i$ matrix, one can relate a point p_i at rest in link i, and expressed in homogeneous coordinates with respect to coordinate system i, to the coordinate system i - 1 established at link i - 1 by

$$P_{i-1} = {}^{i-1}\mathbf{A}_i \ p_i \qquad\qquad (27)$$

where $p_{i-1} = (x_{i-1}, y_{i-1}, z_{i-1}, 1)^T$ and $p_i = (x_i, y_i, z_i, 1)^T$.

The six $^{i-1}\mathbf{A}_i$ transformation matrices for the six-axis PUMA robot arm have been found on the basis of the coordinate systems established in Fig. 2.5. These $^{i-1}\mathbf{A}_i$ matrices are listed in Fig. 2.7.

$$^{i-1}\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$^0\mathbf{A}_1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^1\mathbf{A}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2\mathbf{A}_3 = \begin{bmatrix} C_3 & 0 & S_3 & a_3 C_3 \\ S_3 & 0 & -C_3 & a_3 S_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^3\mathbf{A}_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^4\mathbf{A}_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^5\mathbf{A}_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_1 \equiv {}^0\mathbf{A}_1 \, {}^1\mathbf{A}_2 \, {}^2\mathbf{A}_3 = \begin{bmatrix} C_1 C_{23} & -S_1 & C_1 S_{23} & a_2 C_1 C_2 + a_3 C_1 C_{23} - d_2 S_1 \\ S_1 C_{23} & C_1 & S_1 S_{23} & a_2 S_1 C_2 + a_3 S_1 C_{23} + d_2 C_1 \\ -S_{23} & 0 & C_{23} & -a_2 S_2 - a_3 S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_2 \equiv {}^3\mathbf{A}_4 \, {}^4\mathbf{A}_5 \, {}^5\mathbf{A}_6 = \begin{bmatrix} C_4 C_5 C_6 - S_4 S_6 & -C_4 C_5 S_6 - S_4 C_6 & C_4 S_5 & d_6 C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 & -S_4 C_5 S_6 + C_4 C_6 & S_4 S_5 & d_6 S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 & d_6 C_5 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $C_i \equiv \cos\theta_i$; $S_i \equiv \sin\theta_i$; $C_{ij} \equiv \cos(\theta_i + \theta_j)$; $S_{ij} \equiv \sin(\theta_i + \theta_j)$.

Fig. 2.7: PUMA link coordinate transformation matrices.

The homogeneous matrix ^0T_i which specifies the location of the ith coordinate frame with respect to the base coordinate system is the chain product of successive coordinate transformation matrices of $^{i-1}A_i$, and is expressed as

$$^0T_i = ^0A_1 \, ^1A_2 \ldots ^{i-1}A_i = \prod_{j=1}^{i} {}^{j-1}A_j, \qquad \text{for } i = 1, 2, \ldots, n$$

$$= \begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ^0\mathbf{R}_i & ^0\mathbf{p}_i \\ 0 & 1 \end{bmatrix}$$

(28)

Where

$[x_i, y_i, z_i]$ = orientation matrix of the ith coordinate system established at link i with respect to the base coordinate system. It is the upper left 3 x 3 partitioned matrix of ^0T_i.

p_i = position vector which points from the origin of the base coordinate system to the origin of the ith coordinate system. It is the upper right 3 X 1 partitioned matrix of ^0T_i.

Specifically, for i = 6, we obtain the T matrix, T = $^0A_6$, which specifies the position and orientation of the endpoint of the manipulator with respect to the base coordinate system. This T matrix is used so frequently in robot arm kinematics that it is called the "arm matrix." Consider the T matrix to be of the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{x}_6 & \mathbf{y}_6 & \mathbf{z}_6 & \mathbf{p}_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ^0\mathbf{R}_6 & ^0\mathbf{p}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(29)

where (see Fig. 2.8)

$\mathbf{n}$ = normal vector of the hand. Assuming a parallel jaw hand, it is orthogonal to the fingers of the robot arm.

$\mathbf{s}$ = sliding vector of the hand. It is pointing in the direction of the finger motion as the gripper opens and closes.

$\mathbf{a}$ = approach vector of the hand. It is pointing in the direction normal to the palm of the hand (i.e., normal to the tool mounting plate of the arm).

$\mathbf{p}$ = position vector of the hand. It points from the origin of the base coordinate system to the origin of the hand coordinate system, which is usually located at the centre point of the fully closed fingers.
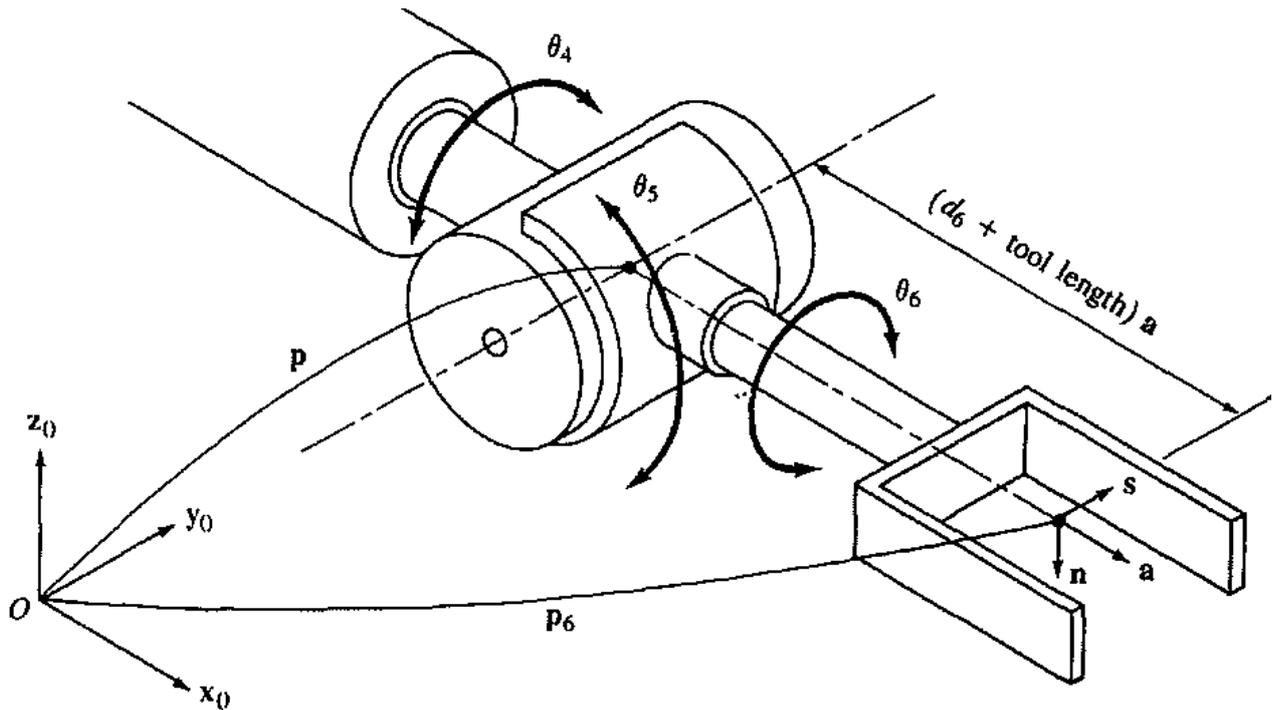
Fig. 2.8: Hand coordinate system and [n, s, a].

If the manipulator is related to a reference coordinate frame by a transformation B and has a tool attached to its last joint's mounting plate described by H, then the endpoint of the tool can be related to the reference coordinate frame by multiplying the matrices B, $^{o}T_6$, and H together as

$$^{ref}T_{Tool} = B \ ^{0}T_6 \ H \qquad\qquad (30)$$

Note that $H = {}^{6}A_{tool}$ and $B = {}^{ref}A_0$.

The direct kinematics solution of a six-link manipulator is, therefore, simply a matter of calculating $T = {}^{o}A_6$ by chain multiplying the six $^{i-1}A_i$ matrices and evaluating each element in the T matrix.

A method that has both fast computation and flexibility is to "hand" multiply the first three $^{i-1}A_i$ matrices together to form $T_1 = {}^{o}A_1 \ ^{1}A_2 \ ^{2}A_3$ and also the last three $^{i-1}A_i$ matrices together to form $T_2 = {}^{3}A_4 \ ^{4}A_5 \ ^{5}A_6$, which is a fairly straightforward task. Then, we express the elements of T_1 and T_2 out in a computer program explicitly and let the computer multiply them together to form the resultant arm matrix $T = T_1 T_2$.

For a PUMA 560 series robot, T_1 is found from Fig. 2.7 to be $^{0}A_3$

$$T_1 = {}^{0}A_3 = {}^{0}A_1 \ ^{1}A_2 \ ^{2}A_3$$

$$= \begin{bmatrix} C_1 C_{23} & -S_1 & C_1 S_{23} & a_2 C_1 C_2 + a_3 C_1 C_{23} - d_2 S_1 \\ S_1 C_{23} & C_1 & S_1 S_{23} & a_2 S_1 C_2 + a_3 S_1 C_{23} + d_2 C_1 \\ -S_{23} & 0 & C_{23} & -a_2 S_2 - a_3 S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (31)$$

and the T_2 matrix is found to be

$$T_2 = {}^{3}A_6 = {}^{3}A_4 \ ^{4}A_5 \ ^{5}A_6$$

$$= \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & d_6C_4S_5 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & d_6S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 & d_6C_5 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (32)$$

where $C_{ij} = \cos(\theta_i + \theta_j)$ and $S_{ij} = \sin(\theta_i + \theta_j)$.

The arm matrix T for the PUMA robot arm is found to be

$$\mathbf{T} = \mathbf{T_1T_2} = {}^0\mathbf{A_1}\,{}^1\mathbf{A_2}{}^2\mathbf{A_3}{}^3\mathbf{A_4}{}^4\mathbf{A_5}{}^5\mathbf{A_6} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (33)$$

Where

$$n_x = C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - S_1(S_4C_5C_6 + C_4S_6)$$

$$n_y = S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + C_1(S_4C_5C_6 + C_4S_6)$$

$$n_z = -S_{23}[C_4C_5C_6 - S_4S_6] - C_{23}S_5C_6 \qquad (34)$$

$$s_x = C_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6)$$

$$s_y = S_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6)$$

$$s_z = S_{23}(C_4C_5S_6 + S_4C_6) + C_{23}S_5S_6 \qquad (35)$$

$$a_x = C_1(C_{23}C_4S_5 + S_{23}C_5) - S_1S_4S_5$$

$$a_y = S_1(C_{23}C_4S_5 + S_{23}C_5) + C_1S_4S_5$$

$$a_z = -S_{23}C_4S_5 + C_{23}C_5 \qquad (36)$$

$$p_x = C_1[d_6(C_{23}C_4S_5 + S_{23}C_5) + S_{23}d_4 + a_3C_{23} + a_2C_2] - S_1(d_6S_4S_5 + d_2)$$

$$p_y = S_1[d_6(C_{23}C_4S_5 + S_{23}C_5) + S_{23}d_4 + a_3C_{23} + a_2C_2] + C_1(d_6S_4S_5 + d_2)$$

$$p_z = d_6(C_{23}C_5 - S_{23}C_4S_5) + C_{23}d_4 - a_3S_{23} - a_2S_2 \qquad (37)$$

As a check, if $\theta_1 = 90°$, $\theta_2 = 0°$, $\theta_3 = 90°$, $\theta_4 = 0°$, $\theta_5 = 0°$, $\theta_6 = 0°$, then the T matrix is

$$\mathbf{T} = \begin{bmatrix} 0 & -1 & 0 & -149.09 \\ 0 & 0 & 1 & 921.12 \\ -1 & 0 & 0 & 20.32 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which agrees with the coordinate systems established in Fig. 2.5.

Example 2.6: A robot work station has been set up with a TV camera (see the figure). The camera can see the origin of the base coordinate system where a six joint robot is attached. It can also see the center of an object (assumed to be a cube) to be manipulated by the robot. If a local coordinate system has been established at the center of the cube, this object as seen by the camera can be represented by a homogeneous transformation matrix T_1. If the origin of the base coordinate system as seen by the camera can also be expressed by a homogeneous transformation matrix T_2 and

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the position of the centre of the cube with respect to the base coordinate system?

(b) Assume that the cube is within the arm's reach. What is the orientation matrix [n, s, a] if you want the gripper (or finger) of the hand to be aligned with the y axis of the object and at the same time pick up the object from the top?
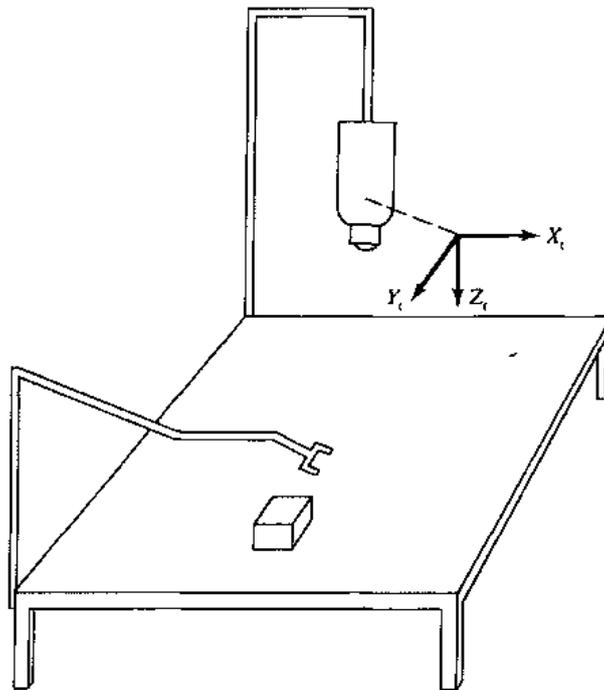
Solution:

$$^{camera}T_{cube} \equiv T_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^{camera}T_{base} \equiv T_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find basercube, we use the "chain product" rule:

$$^{base}T_{cube} = {}^{base}T_{camera} \, {}^{camera}T_{cube} = (T_2)^{-1} \, T_1$$

Using Eq. (22) to invert the T_2 matrix, we obtain the resultant transformation matrix:

$$^{base}\mathbf{T}_{cube} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 11 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the cube is at location $(11, 10, 1)^T$ from the base coordinate system.

Its x, y, and z axes are parallel to the -y, x, and z axes of the base coordinate system, respectively. To find [n, s, a], we make use of

$$^0\mathbf{T}_6 = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $p = (11, 10, 1)^T$ from the above solution. From the above figure, we want the approach vector a to align with the negative direction of the OZ axis of the base coordinate system [i.e., $a = (0, 0, -1)^T$]; the s vector can be aligned in either direction of the y axis of $^{base}T_{cabe}$ [i.e., $s = (\pm 1, 0, 0)^T$]; and the n vector can be obtained from the cross product of s and a:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s_x & s_y & s_z \\ a_x & a_y & a_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \pm 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$$

Therefore, the orientation matrix [n, s, a] is found to be

$$[\mathbf{n}, \mathbf{s}, \mathbf{a}] = \begin{bmatrix} 0 & 1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2.5 Inverse kinematics:

In order to control the position and orientation of the end-effector of a robot to reach its object, the inverse kinematics solution is more important. In other words, given the position and orientation of the end-effector of a six-axis robot arm as $°T_6$ and its joint and link parameters, we would like to find the corresponding joint angles $q = (q_1, q_2, q_3, q_4, q_5, q_6)^T$, T of the robot so that the end-effector can be positioned as desired.

We shall show the basic concept of the inverse transform technique by applying it to solve for the Euler angles. Since the 3 x 3 rotation matrix can be expressed in terms of the Euler angles (ϕ, θ, ψ), and given

$$
\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{u,\theta}\,\mathbf{R}_{w,\psi}
$$

$$
= \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -C\phi S\psi - S\phi C\theta C\psi & S\phi S\theta \\ S\phi C\psi + C\phi C\theta S\psi & -S\phi S\psi + C\phi C\theta C\psi & -C\phi S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix}
$$

We would like to find the corresponding value of ϕ, θ, ψ. Equating the elements of the above matrix equation, we have:

$$n_x = C\phi C\psi - S\phi C\theta S\psi$$

$$n_y = S\phi C\psi + C\phi C\theta S\psi$$

$$n_z = S\theta S\psi$$

$$s_x = -C\phi S\psi - S\phi C\theta C\psi$$

$$s_y = -S\phi S\psi + C\phi C\theta C\psi$$

$$s_z = S\theta C\psi$$

$$a_x = S\phi S\theta$$

$$a_y = -C\phi S\theta$$

$$a_z = C\theta$$

A solution to the above nine equations is:

$$\theta = \cos^{-1}(a_z) \tag{38}$$

$$\psi = \cos^{-1}\left[\frac{s_z}{S\theta}\right] \tag{39}$$

$$\phi = \cos^{-1}\left[\frac{-a_y}{S\theta}\right] \tag{40}$$

The above solution is inconsistent and ill conditioned because:

1. The arc cosine function does not behave well as its accuracy in determining the angle is dependent on the angle. That is, cos (θ) = cos $(-\theta)$.

2. When sin (0) approaches zero, that is, $\theta \approx 0°$ or $\theta \approx \pm180°$, Eq. (39) and (40) give inaccurate solutions or are undefined.

We must, therefore, find a more consistent approach to determining the Euler angle solution and a more consistent arc trigonometric function in evaluating the angle solution. In order to evaluate θ for

-π ≤ θ ≤ π, an arc tangent function, atan2 (y, x), which returns tan^{-1}(y/x) adjusted to the proper quadrant will be used. It is defined as:

$$\theta = \text{atan2}\,(y, x) = \begin{cases} 0° \leqslant \theta \leqslant 90° & \text{for } +x \text{ and } +y \\ 90° \leqslant \theta \leqslant 180° & \text{for } -x \text{ and } +y \\ -180° \leqslant \theta \leqslant -90° & \text{for } -x \text{ and } -y \\ -90° \leqslant \theta \leqslant 0° & \text{for } +x \text{ and } -y \end{cases}$$

Premultiplying the above matrix equation by $R_{z,\phi}^{-1}$, we have one unknown on the LHS and two unknowns (φ) on the RHS of the matrix equation, thus we have

$$\begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} C\phi n_x + S\phi n_y & C\phi s_x + S\phi s_y & C\phi a_x + S\phi a_y \\ -S\phi n_x + C\phi n_y & -S\phi s_x + C\phi s_y & -S\phi a_x + C\phi a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} C\psi & -S\psi & 0 \\ C\theta S\psi & C\theta C\psi & -S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix} \tag{41}$$

Equating the (1, 3) elements of both matrices in Eq. (41), we have:

$$C\phi a_x + S\phi a_y = 0$$

Which gives

$$\phi = \tan^{-1} \left(\frac{a_x}{-a_y} \right) = \text{atan2}\,(a_x, -a_y)$$

Equating the (1, 1) and (1, 2) elements of the both matrices, we have:

$$C\psi = C\phi n_x + S\phi n_y$$

$$S\psi = -C\phi s_x - S\phi s_y$$

which lead to the solution for ψ,

$$\psi = \tan^{-1} \left(\frac{S\psi}{C\psi} \right) = \tan^{-1} \left(\frac{-C\phi s_x - S\phi s_y}{C\phi n_x + S\phi n_y} \right)$$

$$= \text{atan2}\,(-C\phi s_x - S\phi s_y, C\phi n_x + S\phi n_y)$$

Equating the (2, 3) and (3, 3) elements of the both matrices, we have:

$$S\theta = S\phi a_x - C\phi a_y$$
$$C\theta = a_z$$

which gives us the solution for θ,

$$\theta = \tan^{-1}\left[\frac{S\theta}{C\theta}\right] = \tan^{-1}\left[\frac{S\phi a_x - C\phi a_y}{a_z}\right] = \text{atan2}\left(S\phi a_x - C\phi a_y, a_z\right)$$

H.W: Solve the above matrix equation for ϕ, θ, ψ by postmultiplying the above matrix equation by its inverse transform $R_{w,\psi}^{-1}$?

Example 2.7: Let us apply this inverse transform technique to solve the Euler angles for a PUMA robot arm (OAT solution of a PUMA robot). PUMA robots use the symbols O, A, T to indicate the Euler angles and their definitions are given as follows (with reference to Fig. 2.9):

O (orientation) is the angle formed from the yo axis to the projection of the tool a axis on the XY plane about the zo axis.

A (altitude) is the angle formed from the XY plane to the tool a axis about the s axis of the tool.

T (tool) is the angle formed from the XY plane to the tool s axis about the a axis of the tool.

Initially the tool coordinate system (or the hand coordinate system) is aligned with the base coordinate system of the robot as shown in Fig. 2.10.

Solution:

When O = A = T = 0 °, the hand points in the negative y_o axis with the fingers in a horizontal plane, and the s axis is pointing to the positive x_o axis transform that describes the orientation of the hand coordinate system (n, s, a) with respect to the base coordinate system (xo, yo, zo) is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

From the definition of the OAT angles and the initial alignment matrix, the relationship between the hand transform and the OAT angle is given by

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \mathbf{R}_{z,O} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \mathbf{R}_{s,A}\, \mathbf{R}_{a,T}$$

$$= \begin{bmatrix} CO & -SO & 0 \\ SO & CO & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} CA & 0 & SA \\ 0 & 1 & 0 \\ -SA & 0 & CA \end{bmatrix} \begin{bmatrix} CT & -ST & 0 \\ ST & CT & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Postmultiplying the above matrix equation by the inverse transform of $R_{a,T}$,

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} \begin{bmatrix} CT & ST & 0 \\ -ST & CT & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} CO & -SO & 0 \\ SO & CO & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} CA & 0 & SA \\ 0 & 1 & 0 \\ -SA & 0 & CA \end{bmatrix}$$

and multiplying the matrices out, we have:

$$\begin{bmatrix} n_x CT - s_x ST & n_x ST + s_x CT & a_x \\ n_y CT - s_y ST & n_y ST + s_y CT & a_y \\ n_z CT - s_z ST & n_z ST + s_z CT & a_z \end{bmatrix} = \begin{bmatrix} -SOSA & CO & SOCA \\ COSA & SO & -COCA \\ -CA & 0 & -SA \end{bmatrix}$$

Equating the (3, 2) elements of the above matrix equation, we have:

$$n_z ST + s_z CT = 0$$

which gives the solution of T,

$$T = \tan^{-1}\left(\frac{s_z}{-n_z}\right) = \text{atan2}(s_z, -n_z)$$

Equating the (3, 1) and (3, 3) elements of the both matrices, we have:

$$SA = -a_z$$

$$CA = -n_z CT + s_z ST$$

then the above equations give

$$A = \tan^{-1}\left(\frac{-a_z}{-n_z CT + s_z ST}\right) = \text{atan2}(-a_z, -n_z CT + s_z ST)$$

Equating the (1, 2) and (2, 2) elements of the both matrices, we have:

$$CO = n_x ST + s_x CT$$

$$SO = n_y ST + s_y CT$$

which give the solution of O,

$$O = \tan^{-1}\left(\frac{n_y ST + s_y CT}{n_x ST + s_x CT}\right)$$
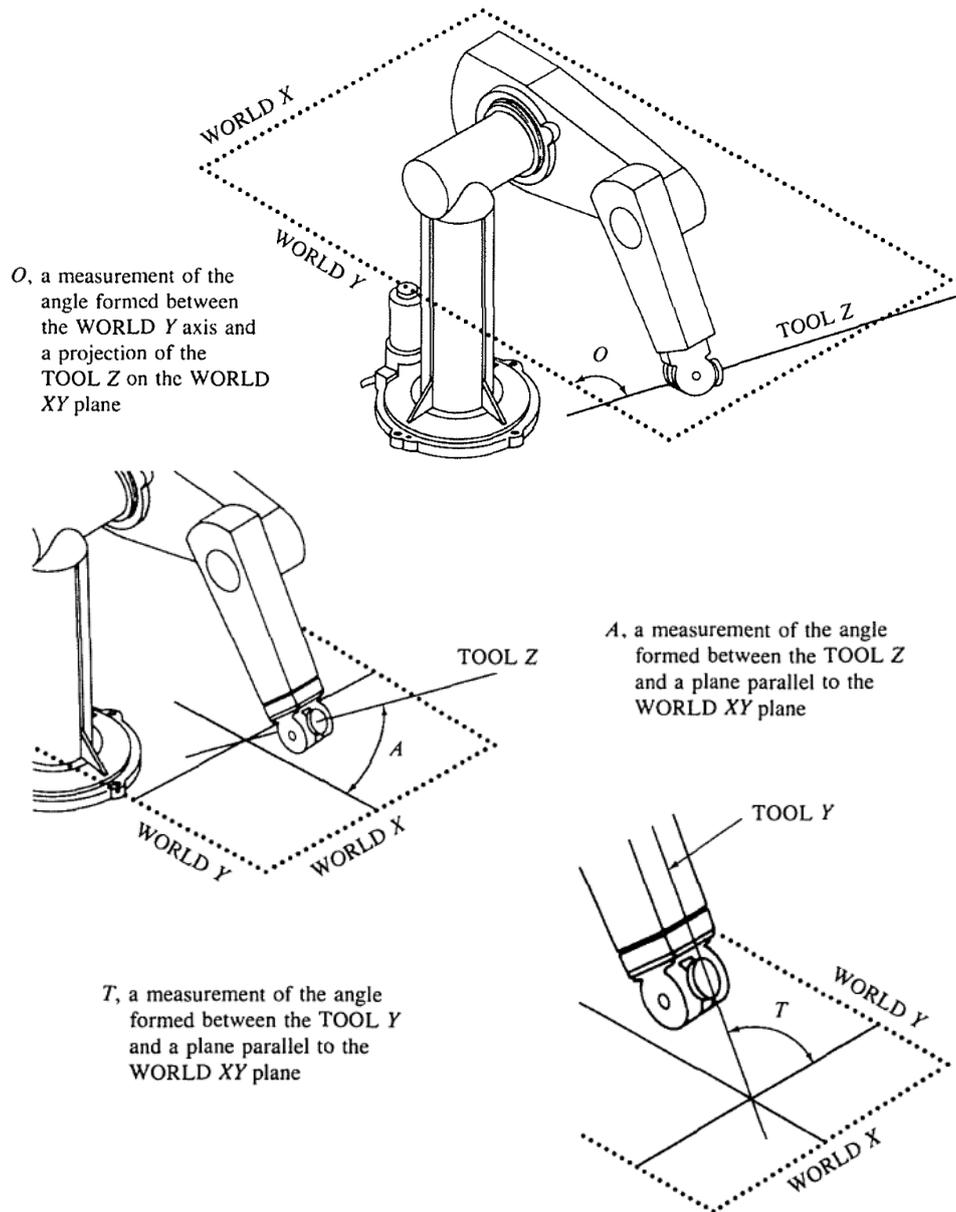
$$= \text{atan2}(n_y ST + s_y CT, n_x ST + s_x CT)$$

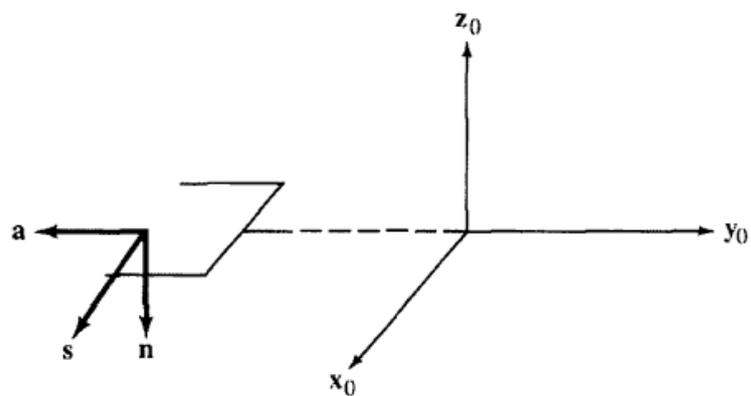*Fig. 2.9: Definition of Euler angles 0, A, and T. (Taken from PUMA robot manual 398H.)*

Fig. 2.10: Initial alignment of tool coordinate system.