Probability

6.1 Introduction

Probability and statistics are concerned with events which occur by chance. Examples include occurrence of accidents, errors of measurements, production of defective and non defective items from a production line. In each case we may have some knowledge of the likelihood of various possible results, but we cannot predict with any certainty the outcome of any particu- lar trial. Probability and statistics are used throughout engineering. Chemical engineers use probability and statis- tics to assess experimental data and control and improve chemical processes. It is essential for today's engineer to master these tools.

6.2 The basic concepts of probability.

A-Random Experiments: a measurement process that produces quantifiable results (e.g. throwing two dice, measuring heights of people, recording proton-proton collisions)

Example 6.1
If we toss a die, the result of the experiment is that it will come up with one of the numbers in the set {1, 2, 3, 4, 5, 6}.

B-Sample Spaces (S): the set of all possible outcomes from an experiment

Example 6.2
If we toss a die, then one sample space is given by {6,5,4,3,2,1} while another is {even, odd}. It is clear, however, that the latter would not be adequate to determine, for example, whether an outcome is divisible by 3.

Example 6.3 : Tossing a coin: **S** = {H,T}
Example 6.4 : Number of customers in a queue: **S** = {0,1,2,...}

The number of all possible outcomes may be

1- If a sample space has a finite number of points, it is called ***a finite sample space***.
2- If it has as many points as there are natural numbers 1, 2, 3, …. , it is called ***a countable infinite sample space***.
3- If it has as many points as there are in some interval on the ***x*** axis, such as $\mathbf{0 \le x \le 1}$, it is called a ***noncountable infinite sample space***.

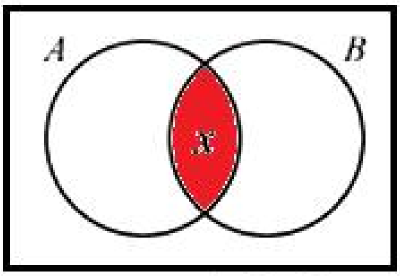
A sample space that is ***finite*** or ***countable finite*** is often called ***a discrete sample space***, while one that is ***noncountable infinite*** is called a ***nondiscrete*** sample space.

C-Events (E).

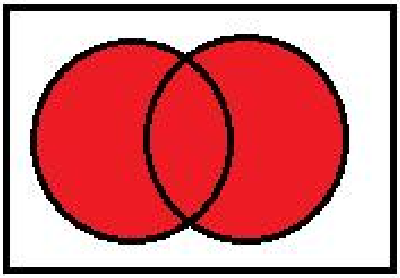
An event is any subset of a sample set (including the empty set, and the whole set) Two events that have no outcome in common are called mutually exclusive events.

Combination of events

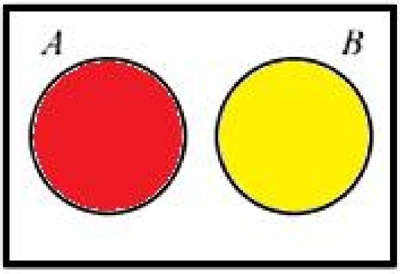
- Union "A or B": $A \cup B = \{\omega \in \mathbf{S} \mid \omega \in A \text{ or } \omega \in B\}$
- Intersection "A and B": $A \cap B = \{\omega \in \mathbf{S} \mid \omega \in A \text{ and } \omega \in B\}$
- Complement "not A": $A^c = \{\omega \in \mathbf{S} \mid \omega \notin A\}$
- Events A and B are disjoint if – $A \cap B = \emptyset$



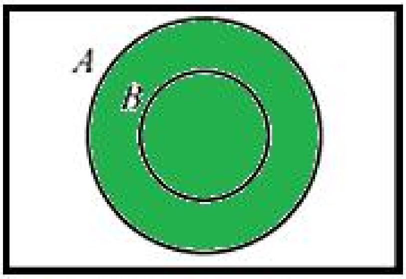
Intersection $A \cap B = x$



Union $A \cup B$



Disjoint $A \cap B = \emptyset$



Subset $B \in A$

Example 6.5 : S =(3,4,2,8,9,10,27,23,14)

A=(2,4,8)   B=(3,4,8,27)
$A \cup B$=(2,3,4,8,27)
$A \cap B$=(4,8)
$A - B$=(2)

6.3 The Probability

In any random experiment there is always uncertainty as to whether a particular event will or will not occur. As a measure of the chance, or probability, with which we can expect the event to occur, it is convenient to assign a number between **0** and **1**. For example, the probability is 1/4 , we would say that there is a 25% chance it will occur and a 75% chance that it will not occur. Equivalently, we can say that the odds against occurrence are 75% to 25%, or 3 to 1.

Example

260 bolts are examined as they are produced. Five of them are found to be defective.

On the basis of this information, estimate the probability that a bolt will be defective.

Answer: The probability of a defective bolt is approximately equal to the relative frequency, which is 5 / 260 = 0.019 or 1.9%.

Example

Two fair coins are tossed. What is the probability of getting one heads and one tails?.

Answer: For a fair or unbiased coin, for each toss of each coin

Pr [heads] = Pr [tails] =1/2

Theory of Probability

Definitions

Suppose an event (E) can happen in (h) ways out of a total of (n) possible equally likely ways. Then the probability of the occurrence of the event (E) is denoted by:-

$$P(E)=\frac{h}{n}=\frac{\text{favorable outcomes}}{\text{total outcomes}}$$

Also, the probability of non-occurrence of the event (E) is denoted by:-

$$q(E)=P(not\ E)=\frac{n-h}{n}=1-\frac{h}{n}=1-P(E)$$

Therefore;

$$P(E)+q(E)=1$$

For example, let the event (E) be the numbers (3) or (4) turn up in a single toss of die. So, there are 6 equally likely ways can happened in the single toss of the die, these are from number 1 to 6 and two ways out of them are favorable (3.4). Then the probability of occurrence of event (E) is:-

$P(E)=\frac{2}{6}=\frac{1}{3}$ and the probability of non-occurrence event (E)

$$P(not E)=1-\frac{1}{3}=\frac{2}{3}$$

Random Experiment:-

It is the experiment in which there is no way to estimate its outcome results (unknown and random outcomes).

Sample Space (S):-

It is the set of all possible outcomes of any random experiment.

Equally Likely Outcomes:-

If (for a random experiment) no outcome is any more likely to occur than any other, the outcomes are said to be equally likely.

Mutually Exclusive Events:-

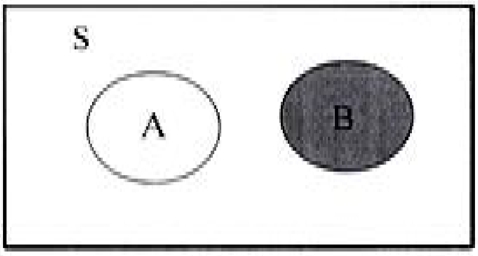
When no two outcomes of an experiment can occur at the same time the outcomes are said to be mutually exclusive.

<u>Some Laws of Probability Relationships</u>

Multiplication Law:-

If A and B are independent events and the occurrence of one does not effect on the occurrence of the other, then the probability of both events to occur is equal to the product of the probability each one.

$$P(A \text{ and } B) = P(A) * P(B) = P(AB)$$



Also, for more than two events (E_1, E_2, E_3, ..., E_n), the probability of all events to occur is :-

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{............and } E_n) = P(E_1) * P(E_2) * \text{..........} * P(E_n) = \prod_{i=1}^{n} P(E_i)$$

Example:

A box contains 6 red balls, 4 white balls, and 5 black balls. If three balls are drawn successively (with replacement), find the probability that they are drawn in order red, white, and black.

Solution:-

Let

R : event of get a red ball in the first drawn
W : event of get a white ball in the second drawn
B : event of get a black ball in the third drawn

If each ball is replaced that is mean the total number of balls did not change and the events R, W, and B are independent events so :-

P(RWB) = P(R) * P(W) * P(B) = (6/15) * (4/15) * (5/15) = 8/225

Addition Law:-

If A and B are independent and mutually exclusive events (i. e. can not occur together). Then the probability of A or B to occur equals to the sum of their individual probabilities:-

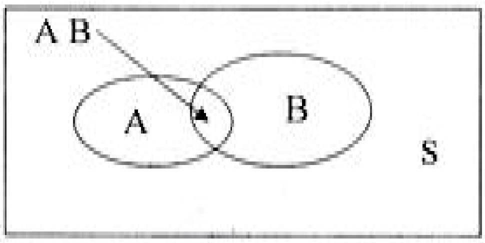
$$P(A\, or\, B) = P(A) + P(B)$$

And for more than two events:-

$$P(E_1\, or\, E_2\, or \ldots\ldots or\, E_n) = \sum_{i=1}^{n} P(A_i)$$

If A and B are independent events and are not mutually exclusive, then the probability of at least one of these events to occur (A or B or Both):-

$$P(A\ and/or\ B) = P(A) + P(B) - P(AB)$$



The generalization of this law for more than two events is obtained by adding all odd number combinations and subtracting all even number combinations. For example, the law for three events A, B, and C is:-

$$P(A\ and/or\ B\ and/or\ C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Example:

Find the probability of a number (4) turning up at least one in two tosses of die.

Solution:-

Let

E_1 : obtaining (4) at first toss.

E_2 : obtaining (4) at second toss.

E_1+E_2 : event number (4) on first toss or second toss or both tosses.

Since E_1 and E_2 are not mutually exclusive, then:-

$$P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_1E_2)$$

Also, E1 and E2 are independent then

$$P(E_1 E_2) = P(E_1) * P(E_2)$$

$P(E_1) = 1/6$ $\quad\quad$ $P(E_2) = 1/6$

Then

$$P(E_1 + E_2) = (1/6) + (1/6) - (1/6) * (1/6)$$
$$= 11/36$$

Conditional Probability

If A and B are two events, the probability that B occur given that A has occurred is denoted by P(B/A) or P(B given A) and is called conditional probability of (B) given that (A) has occurred so (A) and (B) are dependent events.

The probability of a compound events A and B is equal to the probability of one event times the conditional probability of the other event under the condition that the first event has occurred.

$$P(AB) = P(A) * P(B/A)$$

Example:-

Three balls are drawn successively (without replacement) from a box contains 5 red balls and 6 black balls. What is the probability that the one draw in order red, black and red balls?

Solution:-

Let R_1 : event that get red ball in the first draw

B : event that get black ball in the second draw

R_3 : event that get red ball in the third draw

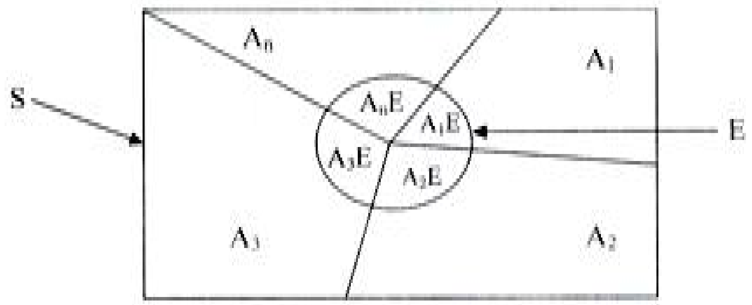
If each ball is not replaced then, R_1, B, and R_3 are dependent events and:-

$$P(R_1 B R_3) = P(R_1) * P(B/R_1) * P(R_3/R_1 B)$$
$$= \frac{5}{5+6} * \frac{6}{4+6} * \frac{4}{4+5} = 0.1212$$

Total Probability Theorem

Suppose that $A_1, A_2, A_3, \ldots\ldots, A_n$ are mutually exclusive events, and be a partition of a sample space S. For any event (E) within the sample space we have:-

$$P(E) = P(A_1) * P(E/A_1) + P(A_2) * P(E/A_2) + \ldots\ldots\ldots\ldots + P(A_n) * P(E/A_n)$$



Example:
Box A contains 5 white balls and 7 red balls, and box B contains 6 white balls and 4 red balls. A box is chosen at random, and then a ball is drawn from it. Find the probability that the ball is red.

Solution:-
Let A : event box (A) is chosen.
B : event box (B) is chosen.
R : event (red) ball drawn.

P(R) = P(A)* P(R/A) + P(B)*P(R/B)
P(A) = P(B) =1/2
P(R) = (1/2)(7/12) + (1/2)(4/10)
= 0.4916

Example:
Three similar boxes, the first contains 8 white balls and 2 red balls, the second contains 6 white balls and 6 red balls, and the third box contains 3 white balls and 9 red balls. A box is chosen at random, and then a ball is drawn from it. Find the probability that the ball is white.

Solution:-
Let
B_1 : event that the ball from box 1.
B_2 : event that the ball from box 2.
B_3 : event that the ball from box 3.
W : event that get white ball.

$P(W) = P(B_1)*P(W/B_1) + P(B_2)*P(W/B_2) + P(B_3)*P(W/B_3)$
= (1/3)(8/10) + (1/3)(6/12) + (1/3)(3/12)
= 0.5167

Bays Theorem of Inverse Probability

If an event A is dependent on one several mutually exclusive events B_1, B_2, B_3, B_4, ……, B_n. If the event has occurred, then the probability that event B_i has also occurred is defined by Bays theorem as:-

$$P(B_i / A) = \frac{P(B_i) * P(A / B_i)}{P(A)}$$

where:

$$P(A) = P(B_1) * P(A/B_1) + P(B_2) * P(A/B_2) + \ldots\ldots + P(B_n) * P(A/B_n)$$

Example:

Box A contains 3 red balls and 5 white balls. Box B contains 2 red balls and 1 white. Box C contains 2 red balls and 3 white. A box is chosen randomly and then a ball is drawn form it. If the ball is red, find the probability that this ball was drawn from box A.

Solution:-

Let A: event for chosen box A

R : event for drawn red ball

$$P(A/R) = \frac{P(A) * P(R/A)}{P(R)}$$

$$P(R) = P(A) * P(R/A) + P(B) * P(R/B) + P(C) * P(R/C)$$

$$= (1/3)(3/8) + (1/3)(2/3) + (1/3)(2/5) = 0.4805$$

$$P(A/R) = \frac{(1/3) * (3/8)}{0.4805} = 0.26$$

Permutations

An arrangement of a set of n different objects is called the permutations of the objects. The arrangement of any $r \leq n$ of these objects in a given order is called a permutation of n objects taken r at a time and is denoted by P_r^n or $P(r, n)$.

The following notes must be taken into account:-

1. The number of permutations of n different objects taken all at a time is (n!)
2. The number of permutations of n different objects taken r at a time is:-

$$P_r^n = n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

where $0 \leq r \leq n$

3. If repetition is allowed, then the total number of arrangements of n different objects in r places is (n^r).
4. The number of arrangements of n objects such that r_1 of them from one kind, r_2 of them from second kind, r_k of them from kth kind, is denoted by:-

$$P^n_{r1,r2,r3,\ldots,rk} = \frac{n!}{(r_1!)(r_2!)(r_3!)\ldots\ldots\ldots\ldots(r_k!)}$$

where $r_1 + r_2 + r_3 + \ldots\ldots + r_k = n$

5. The total number of arrangement of n objects around a circle is (n-1)! .

Example:
How many 3-digit numbers can be formed from the digits, 2, 3, 4, 5, 6, 7, if (a) repetition is not allowed, (b) repetition is allowed.

Solution:-

(a) $n = 6$ $\quad r = 3$ $\quad P_3^6 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$

(b) $P_r^n = n^r = 6^3 = 216$

Example:
It is required to seat 5 men and 4 women in a raw so that the women occupy the even places. How many such arrangements are available?

Solution:-
The men can be seated in P_5^5 ways and the women in P_4^4 ways. Each arrangement of the men may be associated with each arrangement of the women so:

The required number of arrangements is:-

$$P_5^5 * P_4^4 = 5! * 4! = 120 * 24 = 2880$$

Combinations

A combination of n different objects taken r at a time is a selection of r out of n objects with no attention given to the order of arrangements. For example, the combinations of the letters a, b, c, d taken 3 at time are:-

abc abd acd bcd

The combinations of n objects taken r at a time is denoted by C_r^n and is given by:-

$$C_r^n = \frac{n!}{r! * (n-r)!} = \frac{P_r^n}{r!}$$

Example:

Out of 5 civil engineers and 7 mechanical engineers, a committee consisting of 2 civil engineers and 3 mechanical engineers is required. In how any ways can this be done if:-

a) any one of the civil engineers or mechanical engineer can be included.
b) One particular mechanical engineer must be in the committee.
c) Two particular civil engineers cannot be in the committee.

Solution:-

a) 2 civil engineers out of 5 can be selected C_2^5 ways and 3 mechanical engineers out of 7 can be selected in C_3^7 ways

$$Total\ Number\ of\ Selection = C_2^5 * C_3^7 = 250$$

b) 2 civil engineers out of 5 can be selected C_2^5 and 2 additional mechanical engineers out of 6 in C_2^6 ways

$$Total\ Number\ of\ Selection = C_2^5 * C_2^6 = 150$$

c) 2 civil engineers out of 3 can be selected C_2^3 ways and 3 mechanical engineers out of 7 can be selected in C_3^7 ways

$$Total\ Number\ of\ Selection = C_2^3 * C_3^7 = 105$$

Example:

A box contains 8 red balls, 3 white balls, and 9 blue balls. If 3 balls are drawn successively (without replacement), determine the probability that (a) all 3 balls to be red. (b) 2 are red and 1 white. (c) at least 1 is white.

Solution:-

a) $$P(R_1, R_2, R_3) = \frac{number\ of\ combination\ of\ 3\ out\ of\ 8\ red\ balls}{number\ of\ combination\ of\ 3\ out\ of\ 20\ balls} = \frac{C_3^8}{C_3^{20}} = \frac{14}{285}$$

b) $$P(2R, W) = \frac{C_2^8 * C_1^3}{C_3^{20}} = \frac{7}{95}$$

c) $$P(not\ W) = \frac{C_3^{17}}{C_3^{20}} = \frac{34}{57} \qquad \therefore P(at\ least\ 1\ ball\ is\ white) = 1 - \frac{34}{57} = \frac{23}{57}$$

Geometric Probability

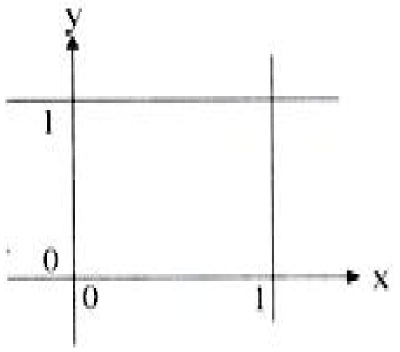
It is a method by which the probability of an event can be calculated by using the geometric areas. Each one of the events in the sample space is representing by an area.

xample:

wo numbers are chosen which their values between 0 and 1 randomly. What is the robability that the sum of their squares is greater than 1.

olution:-

1. Represent each number by a coordinate axis and then determine the sample space.



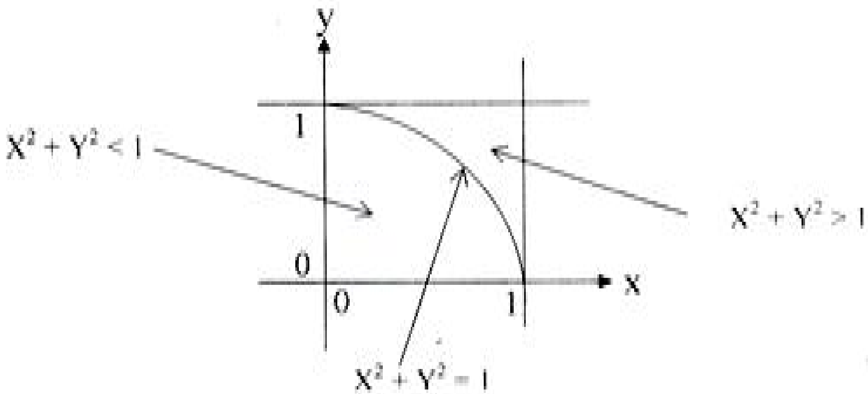
2. Derive a mathematical expression for the required event. So ,

   $X^2 + Y^2 > 1$ (this is the required event)

3. Change the above expression to a mathematical equation.

   $X^2 + Y^2 = 1$ (a circle equation with r = 1)

4. Now, draw the above equation in the sample space.



2) when the first rod have a capacity greater than the second, that's mean.

$x > y$ (mathematical expression for the event)
so,
$x = y$ (equation of a line)

at $y = 12$ ⟶ $x = 12$
at $x = 15$ ⟶ $y = 15$

$P(x > y) = A_1 / A$
$A_1 = 0.5 \times 3 \times 3 = 4.5$ (triangle area)
$A = 8 \times 5 = 40$

$P(x > y) = 4.5/40 = 0.1125$