CHAPTER 9

CURVED BEAMS

The theory of beam bending, presented in Chapter 7, is limited to straight beams or to beams that are mildly curved relative to their depth. However, if the ratio of the radius of curvature to depth for a beam is less than 5, the flexure formula (Eq. 7.1) is generally inadequate for describing the flexural stresses in the beam. For beams that are curved in such a manner, the theory of bending must also include consideration of the curvature. Such a theory is developed in this chapter based on mechanics of materials methods. Two important differences with respect to straight-beam bending result. First, the flexural stress distribution in a curved beam is nonlinear. Based on this result the neutral axis will not coincide with the centroidal axis of the cross section when the beam is subjected to pure bending. Second, a curved beam carries radial stresses as a consequence of the internal bending moment. These radial stresses have important design implications for thin-wall cross sections and for materials (such as wood and unidirectional composites) with relatively low tensile strength in the radial direction.

9.1 INTRODUCTION

Timoshenko and Goodier (1970) presented a solution based on the theory of elasticity for the linear elastic behavior of curved beams of rectangular cross sections for the loading shown in Figure 9.1*a*. They obtained relations for the radial stress σ_{rr}, the circumferential stress $\sigma_{\theta\theta}$, and the shear stress $\sigma_{r\theta}$ (Figure 9.1*b*). However, most curved beams do not

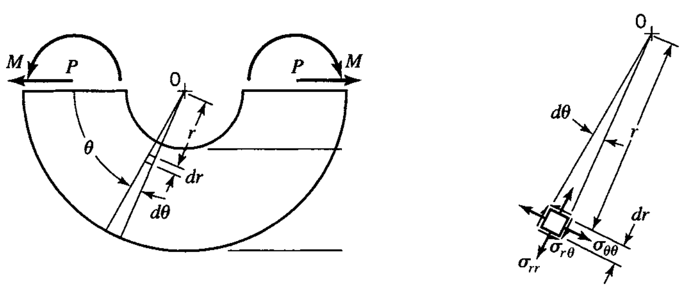


FIGURE 9.1 Rectangular section curved beam. (*a*) Curved beam loading. (*b*) Stress components.

have rectangular cross sections. Therefore, in Section 9.2 we present an approximate curved beam solution that is generally applicable to all symmetrical cross sections. This solution is based on two simplifying assumptions: 1. plane sections before loading remain plane after loading and 2. the radial stress σ_{rr} and shear stress $\sigma_{r\theta}$ are sufficiently small so that the state of stress is essentially one dimensional. The resulting formula for the circumferential stress $\sigma_{\theta\theta}$ is the curved beam formula.

9.2 CIRCUMFERENTIAL STRESSES IN A CURVED BEAM

Consider the curved beam shown in Figure 9.2*a*. The cross section of the beam has a plane of symmetry and the polar coordinates (r, θ) lie in the plane of symmetry, with origin at 0, the center of curvature of the beam. We assume that the applied loads lie in the plane of symmetry. A positive moment is defined as one that causes the radius of curvature at each section of the beam to increase in magnitude. Thus, the applied loads on the curved beams in Figures 9.1 and 9.2*a* cause positive moments. We wish to determine an approximate formula for the circumferential stress distribution $\sigma_{\theta\theta}$ on section *BC*. A free-body diagram of an element *FBCH* of the beam is shown in Figure 9.2*b*. The normal traction *N*, at the centroid of the cross section, the shear *V*, and moment M_x acting on face *FH* are shown in their positive directions. These forces must be balanced by the resultants due to the normal

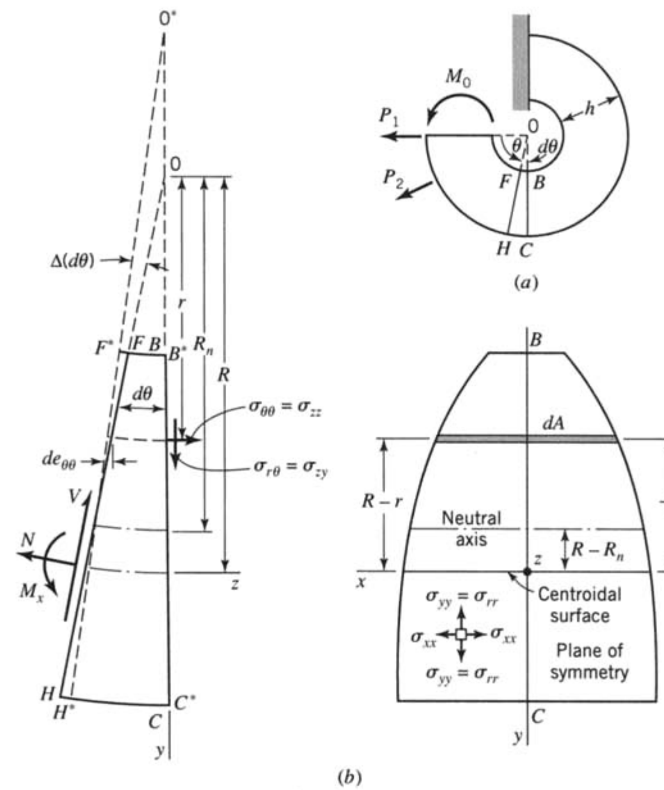


FIGURE 9.2 Curved beam.

stress $\sigma_{\theta\theta}$ and shear stress $\sigma_{r\theta}$ that act on face BC. The effect of the shear stress $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is usually small, except for curved beams with very thin webs. However, ordinarily, practical curved beams are not designed with thin webs because of the possibility of failure by excessive radial stresses (see Section 9.3). Therefore, neglecting the effect of $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is reasonable.

Let the z axis be normal to face BC (Figure 9.2b). By equilibrium of forces in the z direction and of moments about the centroidal x axis, we find

$$\sum F_z = \int \sigma_{\theta\theta}\, dA - N = 0$$

$$\sum M_x = \int \sigma_{\theta\theta}(R-r)\, dA - M_x = 0$$

or

$$N = \int \sigma_{\theta\theta}\, dA \tag{9.1}$$

$$M_x = \int \sigma_{\theta\theta}(R-r)\, dA \tag{9.2}$$

where R is the distance from the center of curvature of the curved beam to the centroid of the beam cross section and r locates the element of area dA from the center of curvature. The integrals of Eqs. 9.1 and 9.2 cannot be evaluated until $\sigma_{\theta\theta}$ is expressed in terms of r. The functional relationship between $\sigma_{\theta\theta}$ and r is obtained from the assumed geometry of deformation and stress–strain relations for the material.

The curved beam element $FBCH$ in Figure 9.2b represents the element in the undeformed state. The element $F^*B^*C^*H^*$ represents the element after it is deformed by the loads. For convenience, we have positioned the deformed element so that face B^*C^* coincides with face BC. As in the case of straight beams, we assume that planes B^*C^* and F^*H^* remain plane under the deformation. Face F^*H^* of the deformed curved beam element forms an angle $\Delta(d\theta)$ with respect to FH. Line F^*H^* intersects line FH at the neutral axis of the cross section (axis for which $\sigma_{\theta\theta} = 0$) at distance R_n from the center of curvature. The movement of the center of curvature from point 0 to point 0* is exaggerated in Figure 9.2b to illustrate the geometry changes. For infinitesimally small displacements, the movement of the center of curvature is infinitesimal. The elongation $de_{\theta\theta}$ of a typical element in the θ direction is equal to the distance between faces FH and F^*H^* and varies linearly with the distance $(R_n - r)$. However, the corresponding strain $\epsilon_{\theta\theta}$ is a nonlinear function of r, since the element length $r\, d\theta$ also varies with r. This fact distinguishes a curved beam from a straight beam. Thus, by Figure 9.2b, we obtain for the strain

$$\epsilon_{\theta\theta} = \frac{de_{\theta\theta}}{r\, d\theta} = \frac{(R_n - r)\Delta(d\theta)}{r\, d\theta} = \left(\frac{R_n}{r} - 1\right)\omega \tag{9.3}$$

where

$$\omega = \frac{\Delta(d\theta)}{d\theta} \tag{9.4}$$

It is assumed that the transverse normal stress σ_{xx} is sufficiently small so that it may be neglected. Hence, the curved beam is considered to be a problem in plane stress. Although radial stress σ_{rr} may, in certain cases, be of importance (see Section 9.3), here we neglect its effect on $\epsilon_{\theta\theta}$. Then, by Hooke's law, we find

$$\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = \frac{R_n - r}{r}E\omega = \frac{E\omega R_n}{r} - E\omega \tag{9.5}$$

Substituting Eq. 9.5 into Eqs. 9.1 and 9.2, we obtain

$$N = R_n E\omega \int \frac{dA}{r} - E\omega \int dA = R_n E\omega A_m - E\omega A \tag{9.6}$$

$$\begin{aligned} M_x &= R_n R E\omega \int \frac{dA}{r} - (R + R_n)E\omega \int dA + E\omega \int r dA \\ &= R_n R E\omega A_m - (R + R_n)E\omega A + E\omega R A = R_n E\omega (R A_m - A) \end{aligned} \tag{9.7}$$

where A is the cross-sectional area of the curved beam and A_m has the dimensions of length and is defined by the relation

$$A_m = \int \frac{dA}{r} \tag{9.8}$$

Equation 9.7 can be rewritten in the form

$$R_n E\omega = \frac{M_x}{R A_m - A} \tag{9.9}$$

Then substitution into Eq. 9.6 gives

$$E\omega = \frac{A_m M_x}{A(R A_m - A)} - \frac{N}{A} \tag{9.10}$$

The circumferential stress distribution for the curved beam is obtained by substituting Eqs. 9.9 and 9.10 into Eq. 9.5 to obtain the curved beam formula

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M_x(A - r A_m)}{A r (R A_m - A)} \tag{9.11}$$

The normal stress distribution given by Eq. 9.11 is hyperbolic in form; that is, it varies as $1/r$. For the case of a curved beam with rectangular cross section ($R/h = 0.75$) subjected to pure bending, the normal stress distribution is shown in Figure 9.3.

Since Eq. 9.11 has been based on several simplifying assumptions, it is essential that its validity be verified. Results predicted by the curved beam formula can be compared with those obtained from the elasticity solution for curved beams with rectangular sections and with those obtained from experiments on, or finite element analysis of, curved beams with other kinds of cross sections. The maximum value of circumferential stress $\sigma_{\theta\theta(\text{CB})}$

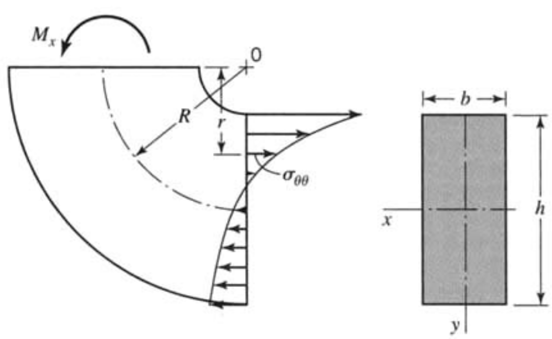


FIGURE 9.3 Circumferential stress distribution in a rectangular section curved beam ($R/h = 0.75$).

as given by the curved beam formula may be computed from Eq. 9.11 for curved beams of rectangular cross sections subjected to pure bending and shear (Figure 9.4). For rectangular cross sections, the ratios of $\sigma_{\theta\theta(\text{CB})}$ to the elasticity solution $\sigma_{\theta\theta(\text{elast})}$ are listed in Table 9.1 for pure bending (Figure 9.4*a*) and for shear loading (Figure 9.4*b*), for several values of the ratio R/h, where h denotes the beam depth (Figure 9.2*a*). The nearer this ratio is to 1, the less error in Eq. 9.11.

The curved beam formula is more accurate for pure bending than shear loading. The value of R/h is usually greater than 1.0 for curved beams, so that the error in the curved beam formula is not particularly significant. However, possible errors occur in the curved beam formula for I- and T-section curved beams. These errors are discussed in Section 9.4. Also listed in Table 9.1 are the ratios of the maximum circumferential stress $\sigma_{\theta\theta(st)}$ given by the straight-beam flexure formula (Eq. 7.1) to the value $\sigma_{\theta\theta(\text{elast})}$. The straight-beam solution is appreciably in error for small values of R/h and is in error by 7% for $R/h = 5.0$; the error is nonconservative. Generally, for curved beams with R/h greater than 5.0, the straight-beam formula may be used.

As R becomes large compared to h, the right-hand term in Eq. 9.11 reduces to $-M_x y/I_x$. The negative sign results because the sign convention for positive moments for curved beams is opposite to that for straight beams (see Eq. 7.1). To prove this reduction, note that $r = R + y$. Then the term RA_m in Eq. 9.11 may be written as

$$RA_m = \int\left(\frac{R}{R+y} + 1 - 1\right)dA = A - \int\frac{y}{R+y}\,dA \tag{a}$$

Hence, the denominator of the right-hand term in Eq. 9.11 becomes, for $R/h \to \infty$,

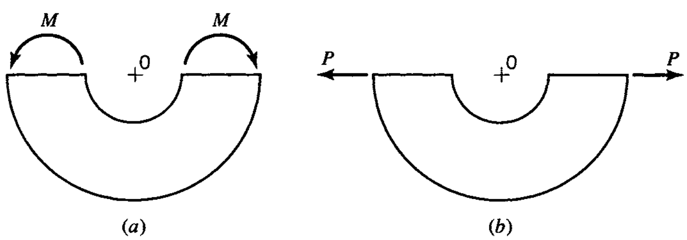


FIGURE 9.4 Types of curved beam loadings. (*a*) Pure bending. (*b*) Shear loading.

TABLE 9.1 Ratios of the Maximum Circumferential Stress in Rectangular Section Curved Beams as Computed by Elasticity Theory, the Curved Beam Formula, and the Flexure Formula

	Pure bending		Shear loading	
$\frac{R}{h}$	$\frac{\sigma_{\theta\theta\,(\text{CB})}}{\sigma_{\theta\theta\,(\text{elast})}}$	$\frac{\sigma_{\theta\theta\,(st)}}{\sigma_{\theta\theta\,(\text{elast})}}$	$\frac{\sigma_{\theta\theta\,(\text{CB})}}{\sigma_{\theta\theta\,(\text{elast})}}$	$\frac{\sigma_{\theta\theta\,(st)}}{\sigma_{\theta\theta\,(\text{elast})}}$
0.65	1.046	0.439	0.855	0.407
0.75	1.012	0.526	0.898	0.511
1.0	0.997	0.654	0.946	0.653
1.5	0.996	0.774	0.977	0.776
2.0	0.997	0.831	0.987	0.834
3.0	0.999	0.888	0.994	0.890
5.0	0.999	0.933	0.998	0.934

$$
\begin{aligned}
Ar(RA_m - A) &= -A\int\left(\frac{Ry}{R+y} + y - y\right)dA - Ay\int\frac{y}{R+y}\,dA \\
&= \frac{A}{R}\int\frac{y^2}{1+(y/R)}\,dA - A\int y\,dA - \frac{Ay}{R}\int\frac{y}{1+(y/R)}\,dA \\
&= \frac{AI_x}{R} \qquad \text{(b)}
\end{aligned}
$$

since as $R/h \to \infty$, then $y/R \to 0$, $1 + y/R \to 1$, $\int[y^2\,dA/(1 + y/R)] \to I_x$, and $\int[y\,dA/(1 + y/R)] \to \int y\,dA = 0$. The right-hand term in Eq. 9.11 then simplifies to

$$
\begin{aligned}
\frac{M_xR}{AI_x}(A - RA_m - yA_m) &= \frac{M_xR}{AI_x}\left(\int\frac{y/R}{1+(y/R)}\,dA - \frac{y}{R}\int\frac{dA}{1+(y/R)}\right) \\
&= -\frac{M_xy}{I_x} \qquad \text{(c)}
\end{aligned}
$$

The curved beam formula (Eq. 9.11) requires that A_m, defined by Eq. 9.8, be calculated for cross sections of various shapes. The number of significant digits retained in calculating A_m must be greater than that required for $\sigma_{\theta\theta}$ since RA_m approaches the value of A as R/h becomes large [see Eq. (a) above]. Explicit formulas for A, A_m, and R for several curved beam cross-sectional areas are listed in Table 9.2. Often, the cross section of a

TABLE 9.2 Expressions for A, R, and $A_m = \int\frac{dA}{r}$

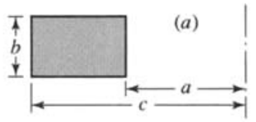
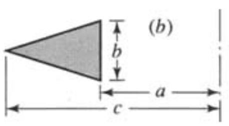
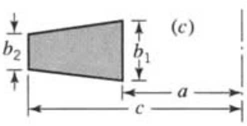
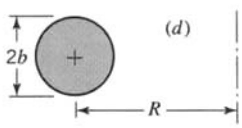
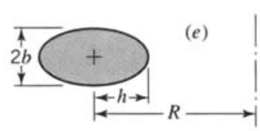
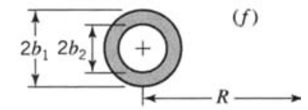
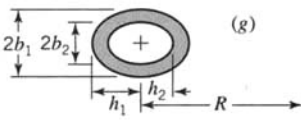
	$A = b(c-a)$; $R = \frac{a+c}{2}$ $A_m = b\ln\frac{c}{a}$
	$A = \frac{b}{2}(c-a)$; $R = \frac{2a+c}{3}$ $A_m = \frac{bc}{c-a}\ln\frac{c}{a} - b$
	$A = \frac{b_1+b_2}{2}(c-a)$; $R = \frac{a(2b_1+b_2)+c(b_1+2b_2)}{3(b_1+b_2)}$ $A_m = \frac{b_1c-b_2a}{c-a}\ln\frac{c}{a} - b_1 + b_2$
	$A = \pi b^2$ $A_m = 2\pi\left(R - \sqrt{R^2-b^2}\right)$
	$A = \pi bh$ $A_m = \frac{2\pi b}{h}\left(R - \sqrt{R^2-h^2}\right)$

TABLE 9.2 Expressions for A, R, and $A_m = \int \frac{dA}{r}$ *(continued)*



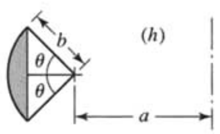
$$A = \pi\left(b_1^2 - b_2^2\right)$$

$$A_m = 2\pi\left(\sqrt{R^2 - b_2^2} - \sqrt{R^2 - b_1^2}\right)$$



$$A = \pi(b_1 h_1 - b_2 h_2)$$

$$A_m = 2\pi\left(\frac{b_1 R}{h_1} - \frac{b_2 R}{h_2} - \frac{b_1}{h_1}\sqrt{R^2 - h_1^2} + \frac{b_2}{h_2}\sqrt{R^2 - h_2^2}\right)$$



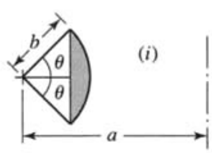
$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a + \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$

For $a > b$,

$$A_m = 2a\theta - 2b\sin\theta - \pi\sqrt{a^2 - b^2} + 2\sqrt{a^2 - b^2}\sin^{-1}\left(\frac{b + a\cos\theta}{a + b\cos\theta}\right)$$

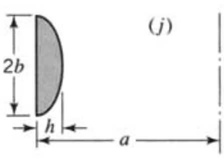
For $b > a$,

$$A_m = 2a\theta - 2b\sin\theta + 2\sqrt{b^2 - a^2}\ln\left(\frac{b + a\cos\theta + \sqrt{b^2 - a^2}\sin\theta}{a + b\cos\theta}\right)$$



$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a - \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$

$$A_m = 2a\theta + 2b\sin\theta - \pi\sqrt{a^2 - b^2} - 2\sqrt{a^2 - b^2}\sin^{-1}\left(\frac{b - a\cos\theta}{a - b\cos\theta}\right)$$



$$A = \frac{\pi bh}{2}; \quad R = a - \frac{4h}{3\pi}$$

$$A_m = 2b + \frac{\pi b}{h}\left(a - \sqrt{a^2 - h^2}\right) - \frac{2b}{h}\sqrt{a^2 - h^2}\sin^{-1}\left(\frac{h}{a}\right)$$

curved beam is composed of two or more of the fundamental areas listed in Table 9.2. The values of A, A_m, and R for the composite area are given by summation. Thus, for composite cross sections,

$$A = \sum_{i=1}^{n} A_i \tag{9.12}$$

$$A_m = \sum_{i=1}^{n} A_{mi} \tag{9.13}$$

$$R = \frac{\sum_{i=1}^{n} R_i A_i}{\sum_{i=1}^{n} A_i} \tag{9.14}$$

where n is the number of fundamental areas that form the composite area.

9.2.1 Location of Neutral Axis of Cross Section

The neutral axis of bending of the cross section is defined by the condition $\sigma_{\theta\theta} = 0$. The neutral axis is located at distance R_n from the center of curvature. The distance R_n is obtained from Eq. 9.11 with the condition that $\sigma_{\theta\theta} = 0$ on the neutral surface $r = R_n$. Thus, Eq. 9.11 yields

$$R_n = \frac{AM_x}{A_m M_x + N(A - RA_m)} \tag{9.15}$$

For pure bending, $N = 0$, and then Eq. 9.15 yields

$$R_n = \frac{A}{A_m} \tag{9.16}$$

EXAMPLE 9.1
Stress in Curved Beam Portion of a Frame

The frame shown in Figure E9.1 has a 50 mm by 50 mm square cross section. The load P is located 100 mm from the center of curvature of the curved beam portion of the frame. The radius of curvature of the inner surface of the curved beam is $a = 30$ mm. For $P = 9.50$ kN, determine the values for the maximum tensile and compressive stresses in the frame.

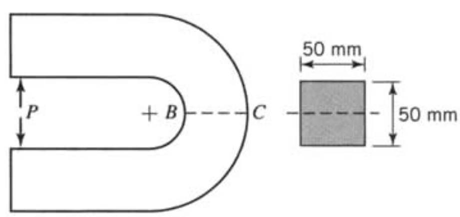


FIGURE E9.1

Solution

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. 9.11. Required values for A, A_m, and R for the curved beam are calculated using the equations in row (a) of Table 9.2. For the curved beam $a = 30$ mm and $c = 80$ mm. Therefore,

$$A = b(c-a) = 50(80-30) = 2500 \text{ mm}^2$$

$$A_m = b \ln\frac{c}{a} = 50 \ln\frac{80}{30} = 49.04$$

$$R = \frac{a+c}{2} = \frac{30+80}{2} = 55 \text{ mm}$$

Hence, the maximum tensile stress is (at point B)

$$\sigma_{\theta\theta B} = \frac{P}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} = \frac{9500}{2500} + \frac{155(9500)[2500 - 30(49.04)]}{2500(30)[55(49.04) - 2500]}$$

$$= 106.2 \text{ MPa}$$

The maximum compressive stress is (at point C)

$$\sigma_{\theta\theta C} = \frac{9500}{2500} + \frac{155(9500)[2500 - 80(49.04)]}{2500(80)[55(49.04) - 2500]} = -49.3 \text{ MPa}$$

EXAMPLE 9.2 Semicircular Aircraft Beam

In a test of a semicircular aircraft fuselage beam, the beam is subjected to an end load $P = 300$ N that acts at the centroid of the beam cross section (Figure E9.2*a*).

(a) Using Eq. 9.11, determine the normal stress $\sigma_{\theta\theta}$ that acts on the section AB as a function of radius r and angle θ, where, by Figure E9.2*a*, 1.47 m $\leq r \leq$ 1.53 m and $0 \leq \theta \leq \pi$.

(b) Determine the value of θ for which the stress $\sigma_{\theta\theta}$ is maximum.

(c) For the value of θ obtained in part (b), determine the maximum tensile and compressive stresses and their locations.

(d) Determine the maximum tensile and compressive stresses acting on the section at $\theta = \pi/2$.

(e) Compare the results obtained in parts (c) and (d) to those obtained using straight-beam theory, where $\sigma_{\theta\theta} = -My/I$.

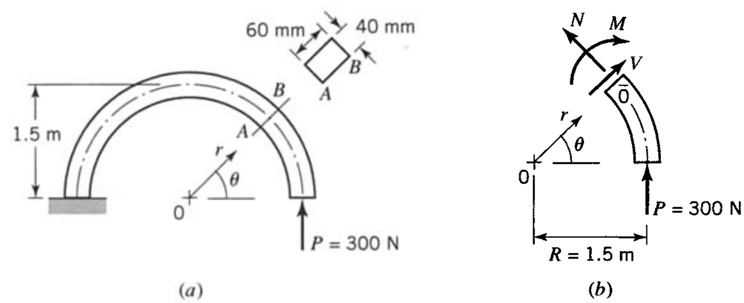


FIGURE E9.2

Solution

(a) Consider the free-body diagram of the beam segment $0 < \theta < \pi$ (Figure E9.2*b*), where N, V, and M are the normal force, the shear force, and the bending moment acting on the section at θ, respectively. By Figure E9.2*b*, we have

$$\sum F_r = V + P\sin\theta = 0$$
$$\sum F_\theta = N + P\cos\theta = 0$$
$$\circlearrowleft + \quad \sum M_{\bar{0}} = PR(1 - \cos\theta) - M = 0$$

or

$$\begin{aligned} V &= -P\sin\theta = -300\sin\theta \text{ [N]} \\ N &= -P\cos\theta = -300\cos\theta \text{ [N]} \\ M &= PR(1 - \cos\theta) = 450(1 - \cos\theta) \text{ [N} \cdot \text{m]} \end{aligned} \tag{a}$$

For the cross section, by Figure E9.2*b* and Table 9.2,

$$\begin{aligned} A &= b(c - a) = 0.04(1.53 - 1.47) = 0.0024 \text{ m}^2 \\ A_m &= b\ln\frac{c}{a} = 0.04\ln\frac{1.53}{1.47} = 0.00160021 \text{ m} \\ R &= 1.5 \text{ m} \end{aligned} \tag{b}$$

Note that the number of digits of precision shown for A_m is required in Eq. 9.11. Now, by Eqs. (a), (b), and 9.11, we have

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)}$$

$$\sigma_{\theta\theta} = -125\cos\theta + \left(\frac{14.2857 - 9.5250r}{r}\right)(1 - \cos\theta) \times 10^5 \text{ [kPa]} \tag{c}$$

(b) For maximum (or minimum) $\sigma_{\theta\theta}$,

$$\frac{d\sigma_{\theta\theta}}{d\theta} = \left[125 + \left(\frac{14.2857 - 9.5250r}{r}\right) \times 10^5\right]\sin\theta = 0$$

Hence, $\sigma_{\theta\theta}$ is minimum at $\theta = 0$ and it is maximum at $\theta = \pi$, with values given by Eq. (c).

(c) From Eq. (c), the maximum tensile and compressive stresses at $\theta = \pi$ are as follows:

For $r = 1.47$ m, the tensile stress at A is

$$\sigma_{\theta\theta} = 125 + 38{,}633 = 38{,}758 \text{ kPa} = 38.76 \text{ MPa} \tag{d}$$

For $r = 1.53$ m, the compressive stress at B is

$$\sigma_{\theta\theta} = 125 - 37{,}588 = -37{,}463 \text{ kPa} = -37.46 \text{ MPa} \tag{e}$$

(d) By Eq. (c), with $\theta = \pi/2$,

For $r = 1.47$ m, the tensile stress at A is

$$\sigma_{\theta\theta} = 0 + 19{,}316 \text{ kPa} = 19.32 \text{ MPa} \tag{f}$$

For $r = 1.53$ m, the compressive stress at B is

$$\sigma_{\theta\theta} = 0 - 18{,}794 \text{ kPa} = -18.79 \text{ MPa} \tag{g}$$

(e) Using straight-beam theory, we have

$$\sigma_{\theta\theta} = -\frac{My}{I} \tag{h}$$

where, by Figure E9.2*a*,

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.04)(0.06)^3 = 7.2 \times 10^{-7} \text{ m}^4$$

and for $\theta = \pi$,

$$M = 2PR = 2(300)(1.5) = 900 \text{ N} \cdot \text{m}$$

Hence, by Eq. (h),

$$\sigma_{\theta\theta} = -(1.25 \times 10^9)y \tag{i}$$

For $y = -0.03$ m (point A in Figure E9.2*a*), Eq. (i) yields $\sigma_{\theta\theta} = 37.50$ MPa, compared to $\sigma_{\theta\theta} = 38.76$ MPa in part (c). For $y = +0.03$ m (point B in Figure E9.2*a*), Eq. (i) yields $\sigma_{\theta\theta} = -37.50$ MPa, compared to $\sigma_{\theta\theta} = 37.46$ MPa in part (c).

For $\theta = \pi/2$,

$$M = PR = (300)(1.5) = 450 \text{ N} \cdot \text{m}$$

and then by Eq. (h),

$$\sigma_{\theta\theta} = -(6.25 \times 10^8)y \tag{j}$$

For $y = -0.03$ m (point A), Eq. (j) yields $\sigma_{\theta\theta} = 18.75$ MPa, compared to $\sigma_{\theta\theta} = 19.32$ MPa in part (d). For $y = +0.03$ m (point B), Eq. (j) yields $\sigma_{\theta\theta} = -18.75$ MPa, compared to $\sigma_{\theta\theta} = 18.79$ MPa in part (d).

EXAMPLE 9.3
Stresses in a Crane Hook

Section BC is the critically stressed section of a crane hook (Figure E9.3*a*). For a large number of manufactured crane hooks, the critical section BC can be closely approximated by a trapezoidal area with half of an ellipse at the inner radius and an arc of a circle at the outer radius. Such a section is shown in Figure E9.3*b*, which includes dimensions for the critical cross section. The crane hook is made of a ductile steel that has a yield stress of $Y = 500$ MPa. Assuming that the crane hook is designed with a factor of safety of $SF = 2.00$ against initiation of yielding, determine the maximum load P that can be carried by the crane hook.

Note: An efficient algorithm to analyze crane hooks has been developed by Wang (1985).

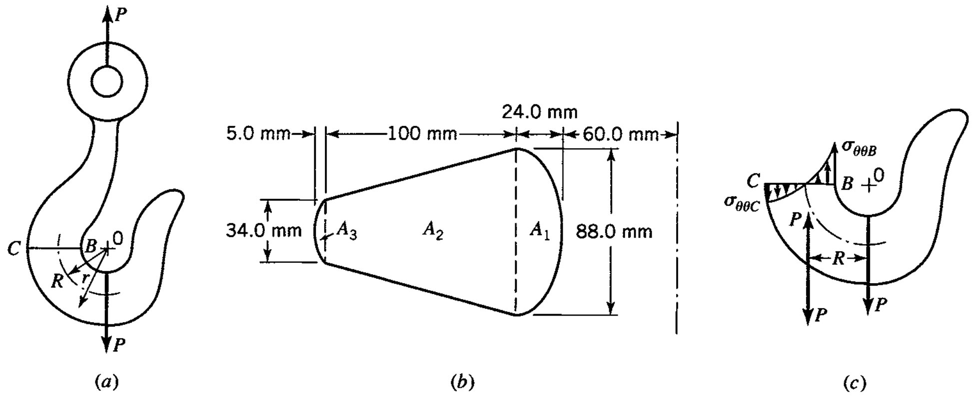


FIGURE E9.3 Crane hook.

Solution

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. 9.11. To calculate values of A, R, and A_m for the curved beam cross section, we divide the cross section into basic areas A_1, A_2, and A_3 (Figure E9.3*b*).

For area A_1, $a = 84$ mm. Substituting this dimension along with other given dimensions into Table 9.2, row (j), we find

$$A_1 = 1658.76 \text{ mm}^2, \quad R_1 = 73.81 \text{ mm}, \quad A_{m1} = 22.64 \text{ mm} \tag{a}$$

For the trapezoidal area A_2, $a = 60 + 24 = 84$ mm and $c = a + 100 = 184$ mm. Substituting these dimensions along with other given dimensions into Table 9.2, row (c), we find

$$A_2 = 6100.00 \text{ mm}^2, \quad R_2 = 126.62 \text{ mm}, \quad A_{m2} = 50.57 \text{ mm} \tag{b}$$

For area A_3, $\theta = 0.5721$ rad, $b = 31.40$ mm, and $a = 157.60$ mm. When these values are substituted into Table 9.2, row (h), we obtain

$$A_3 = 115.27 \text{ mm}^2, \quad R_3 = 186.01 \text{ mm}, \quad A_{m3} = 0.62 \text{ mm} \tag{c}$$

Substituting values of A_i, R_i, and A_{mi} from Eqs. (a)–(c) into Eqs. 9.12–9.14, we calculate

$$\begin{aligned} A &= 6100.00 + 115.27 + 1658.76 = 7874.03 \text{ mm}^2 \\ A_m &= 50.57 + 0.62 + 22.64 = 73.83 \text{ mm} \\ R &= \frac{6100.00(126.62) + 115.27(186.01) + 1658.76(73.81)}{7874.03} \\ &= 116.37 \text{ mm} \end{aligned}$$

As indicated in Figure E9.3*c*, the circumferential stress distribution $\sigma_{\theta\theta}$ is due to the normal load $N = P$ and moment $M_x = PR$. The maximum tension and compression values of $\sigma_{\theta\theta}$ occur at points B and C, respectively. For points B and C, by Figure E9.3*b*, we find

$$r_B = 60 \text{ mm}$$
$$r_C = 60 + 24 + 100 + 5 = 189 \text{ mm}$$

Substituting the required values into Eq. 9.11, we find

$$\sigma_{\theta\theta B} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 60(73.83)]}{7874.03(60)[116.37(73.83) - 7874.03]}$$
$$= 0.000127P + 0.001182P$$
$$= 0.001309P \quad \text{(tension)}$$

$$\sigma_{\theta\theta C} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 189(73.83)]}{7874.03(189)[116.37(73.83) - 7874.03]}$$
$$= 0.000127P - 0.000662P$$
$$= -0.000535P \quad \text{(compression)}$$

Since the absolute magnitude of $\sigma_{\theta\theta B}$ is greater than $\sigma_{\theta\theta C}$, initiation of yield occurs when $\sigma_{\theta\theta B}$ equals the yield stress Y. The corresponding value of the failure load (P_f) is the load at which yield occurs. Dividing the failure load $P_f = Y/(0.001309)$ by the factor of safety $SF = 2.00$, we obtain the design load P; namely,

$$P = \frac{500}{2.00(0.001309)} = 190{,}900 \text{ N}$$

EXAMPLE 9.4
Proof Test of a Crane Hook

To proof test a crane hook an engineer applies a load P to the hook through a pin (Figure E9.4*a*). Assume that the pin exerts a pressure $p \sin \theta$ [N/mm^2] at radius r_i for $0 \le \theta \le \pi$, where p is a constant. The hook has a uniform rectangular cross section of thickness t.

(a) Determine the circumferential stress $\sigma_{\theta\theta}$ as a function of P, r_o, r_i, r, and θ.

(b) For $r_i = 60$ mm, $r_o = 180$ mm, and $t = 50$ mm, determine the maximum tensile and compressive stresses on the cross section at $\theta = \pi/2$ and $\theta = \pi$ in terms of P.

(c) If the maximum allowable tensile stress is $\sigma_{\theta\theta} = 340$ MPa, what is the allowable load P for a safety factor of 2.2?

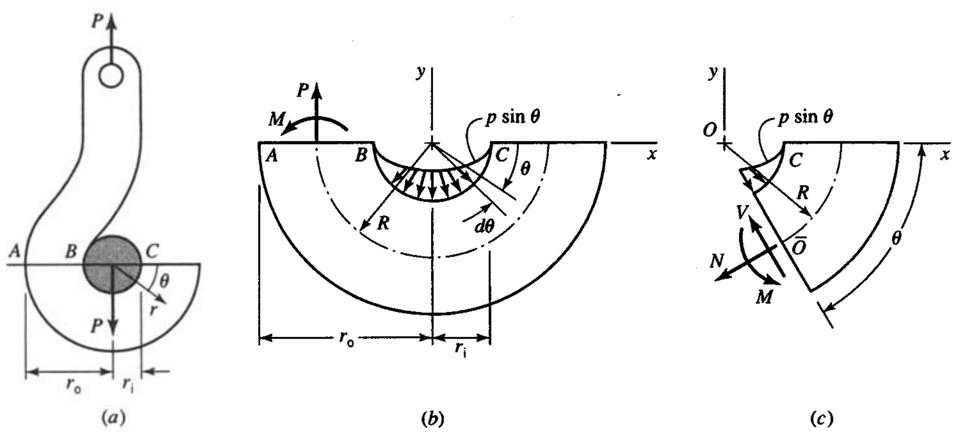


FIGURE E9.4

Solution (a) Consider the free-body diagram of the hook segment ABC (Figure E9.4b). Summing forces in the y direction, we have

$$\sum F_y = P - \int_0^{\pi} [(p\sin\phi)(r_i\, d\phi)t]\sin\phi = 0$$

or

$$p = \frac{2P}{\pi r_i t} \tag{a}$$

Next consider the free-body diagram of an element of the hook. By Figure E9.4c we have for equilibrium in the x direction

$$\sum F_x = -N\sin\theta - V\cos\theta + pr_i t\int_0^{\theta} \sin\phi\, \cos\phi\, d\phi = 0$$

or

$$N\sin\theta + V\cos\theta = \frac{1}{4}pr_i t(1 - \cos 2\theta) \tag{b}$$

For equilibrium in the y direction, we have

$$\sum F_y = -N\cos\theta + V\sin\theta - pr_i t\int_0^{\theta} \sin\phi\, \sin\phi\, d\phi = 0$$

or

$$N\cos\theta - V\sin\theta = -\frac{1}{4}pr_i t(2\theta - \sin 2\theta) \tag{c}$$

For equilibrium of moments

$$\circlearrowleft\!+ \quad \sum M_0 = M - NR = 0$$

or

$$M = NR \tag{d}$$

The solution of Eqs. (b), (c), and (d) is

$$N = \frac{1}{2}pr_i t(\sin\theta - \theta\cos\theta) \tag{e}$$

$$V = \frac{1}{2}pr_i t(\theta\sin\theta) \tag{f}$$

$$M = \frac{1}{2}pr_i Rt(\sin\theta - \theta\cos\theta) \tag{g}$$

By Eqs. (e), (g), and 9.11,

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)} \tag{h}$$

where

$$
\begin{aligned}
A &= (r_o - r_i)t \\
A_m &= t \ln\frac{r_o}{r_i} \\
R &= \frac{1}{2}(r_o + r_i)
\end{aligned}
\tag{i}
$$

and by Eqs. (a), (e), and (g),

$$
\begin{aligned}
N &= \frac{P}{\pi}(\sin\theta - \theta\cos\theta) \\
M &= \frac{PR}{\pi}(\sin\theta - \theta\cos\theta)
\end{aligned}
\tag{j}
$$

Hence, by Eqs. (h), (i), and (j),

$$
\sigma_{\theta\theta} = \frac{P(\sin\theta - \theta\cos\theta)}{\pi(r_o - r_i)t}\left\{1 + \frac{(r_o + r_i)\left(r_o - r_i - r\ln\dfrac{r_o}{r_i}\right)}{r\left[(r_o + r_i)\ln\dfrac{r_o}{r_i} - 2(r_o - r_i)\right]}\right\} \tag{k}
$$

(b) For $r_i = 60$ mm, $r_o = 180$ mm, and $t = 50$ mm, Eq. (k) yields

$$
\sigma_{\theta\theta} = P(\sin\theta - \theta\cos\theta)\left(\frac{0.06456}{r} - 0.0005372\right) \tag{l}
$$

For $\theta = \pi/2$, Eq. (l) yields

$$
\sigma_{\theta\theta} = P\left(\frac{0.06456}{r} - 0.0005372\right)
$$

For maximum tensile stress, $r = r_i = 60$ mm, at which

$$
(\sigma_{\theta\theta})_{\max} = 0.000539P \tag{m}
$$

For maximum compressive stress, $r = r_o = 180$ mm, at which

$$
(\sigma_{\theta\theta})_{\max} = -0.000179P \tag{n}
$$

For $\theta = \pi$, Eq. (l) yields

$$
\sigma_{\theta\theta} = P\left(\frac{0.2028}{r} - 0.001690\right)
$$

For maximum tensile stress, $r = r_i = 60$ mm, at which

$$
(\sigma_{\theta\theta})_{\max} = 0.00169P \tag{o}
$$

For maximum compressive stress, $r = r_o = 180$ mm, at which

$$
(\sigma_{\theta\theta})_{\max} = -0.000563P \tag{p}
$$

(c) For a maximum allowable tensile stress of 340 MPa and a safety factor of 2.2, Eq. (o) yields

$$\frac{340}{2.2} = 0.00169P$$

or the maximum allowable load is $P = 91{,}447$ N = 91.45 kN.

9.3 RADIAL STRESSES IN CURVED BEAMS

The curved beam formula for circumferential stress $\sigma_{\theta\theta}$ (Eq. 9.11) is based on the assumption that the effect of radial stress is small. This assumption is accurate for curved beams with circular, rectangular, or trapezoidal cross sections, that is, cross sections that do not possess thin webs. However, in curved beams with cross sections in the form of an H, T, or I, the webs may be so thin that the maximum radial stress in the web may exceed the maximum circumferential stress. Also, although the radial stress is usually small, it may be significant relative to radial strength, for example, when anisotropic materials, such as wood, are formed into curved beams. The beam should be designed to take such conditions into account.

To illustrate these remarks, we consider the tensile radial stress, resulting from a positive moment, that occurs in a curved beam at radius r from the center of curvature 0 of the beam (Figure 9.5*a*). Consider equilibrium of the element $BDGF$ of the beam, shown enlarged in the free-body diagram in Figure 9.5*c*. The faces BD and GF, which subtend the infinitesimal angle $d\theta$, have the area A' shown shaded in Figure 9.5*b*. The distribution of $\sigma_{\theta\theta}$ on each of these areas produces a resultant circumferential force T (Figure 9.5*c*) given by the expression

$$T = \int_a^r \sigma_{\theta\theta}\, dA \tag{9.17}$$

The components of the circumferential forces along line $0L$ are balanced by the radial stress σ_{rr} acting on the area $tr\, d\theta$, where t is the thickness of the cross section at the distance r from the center of curvature 0 (Figure 9.5*b*). Thus for equilibrium in the radial direction along $0L$,

$$\sum F_r = 0 = \sigma_{rr} tr\, d\theta - 2T\sin(d\theta/2) = (\sigma_{rr} tr - T)\, d\theta$$

since for infinitesimal angle $d\theta/2$, $\sin(d\theta/2) = d\theta/2$. Therefore, the tensile stress resulting from the positive moment is

$$\sigma_{rr} = \frac{T}{tr} \tag{9.18}$$

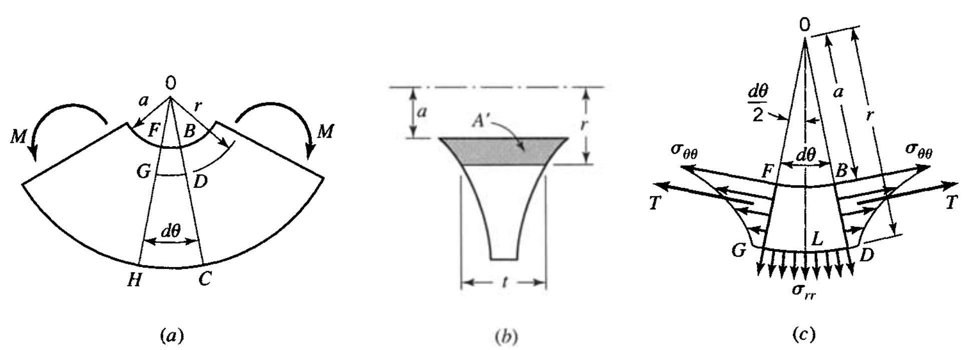


FIGURE 9.5 Radial stress in a curved beam. (*a*) Side view. (*b*) Cross-sectional shape. (*c*) Element *BDGF*.

The force T is obtained by substitution of Eq. 9.11 into Eq. 9.17. Thus,

$$\begin{aligned} T &= \frac{N}{A}\int_a^r dA + \frac{M_x}{RA_m - A}\int_a^r \frac{dA}{r} - \frac{M_x A_m}{A(RA_m - A)}\int_a^r dA \\ T &= \frac{A'}{A}N + \frac{AA'_m - A'A_m}{A(RA_m - A)}M_x \end{aligned} \tag{9.19}$$

where

$$A'_m = \int_a^r \frac{dA}{r} \quad \text{and} \quad A' = \int_a^r dA \tag{9.20}$$

Substitution of Eq. 9.19 into Eq. 9.18 yields the relation for the radial stress. For rectangular cross section curved beams subjected to shear loading (Figure 9.4*b*), a comparison of the resulting approximate solution with the elasticity solution indicates that the approximate solution is conservative. Furthermore, for such beams it remains conservative to within 6% for values of $R/h > 1.0$ even if the term involving N in Eq. 9.19 is discarded. Consequently, if we retain only the moment term in Eq. 9.19, the expression for the radial stress may be approximated by the formula

$$\sigma_{rr} = \frac{AA'_m - A'A_m}{trA(RA_m - A)}M_x \tag{9.21}$$

to within 6% of the elasticity solution for rectangular cross section curved beams subjected to shear loading (Figure 9.4*b*).

9.3.1 Curved Beams Made from Anisotropic Materials

Typically, the radial stresses developed in curved beams of *stocky* (rectangular, circular, etc.) cross sections are small enough that they can be neglected in analysis and design. However, some anisotropic materials may have low strength in the radial direction. Such materials include fiber-reinforced composites (fiberglass) and wood. For these materials, the relatively small radial stress developed in a curved beam may control the design of the beam owing to the corresponding relatively low strength of the material in the radial direction. Hence, it may be important to properly account for radial stresses in curved beams of certain materials.

EXAMPLE 9.5
Radial Stress in a T-Section

The curved beam in Figure E9.5 is subjected to a load P = 120 kN. The dimensions of section BC are also shown. Determine the circumferential stress at B and radial stress at the junction of the flange and web at section BC.

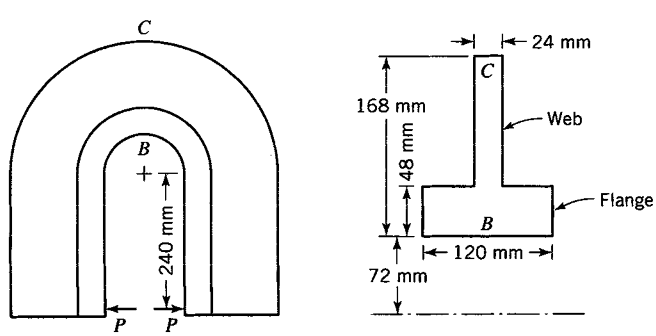


FIGURE E9.5

Solution

The magnitudes of A, A_m, and R are given by Eqs. 9.12, 9.13, and 9.14, respectively. They are

$$A = 48(120) + 120(24) = 8640 \text{ mm}^2$$

$$R = \frac{48(120)(96) + 120(24)(180)}{8640} = 124.0 \text{ mm}$$

$$A_m = 120 \ln\frac{120}{72} + 24 \ln\frac{240}{120} = 77.93 \text{ mm}$$

The circumferential stress is given by Eq. 9.11. It is

$$\sigma_{\theta\theta B} = \frac{120{,}000}{8640} + \frac{364.0(120{,}000)[8640 - 72(77.93)]}{8640(72)[124.0(77.93) - 8640]}$$
$$= 13.9 + 207.8 = 221.7 \text{ MPa}$$

The radial stress at the junction of the flange and web is given by Eq. 9.21, with r = 120 mm and t = 24 mm. Magnitudes of A' and A'_m are

$$A' = 48(120) = 5760 \text{ mm}^2$$

$$A'_m = 120 \ln\frac{120}{72} = 61.30 \text{ mm}$$

Substitution of these values into Eq. 9.21, which neglects the effect of N, gives

$$\sigma_{rr} = \frac{364.0(120{,}000)[8640(61.30) - 5760(77.93)]}{24(120)(8640)[124.0(77.93) - 8640]} = 138.5 \text{ MPa}$$

Hence, the magnitude of this radial stress is appreciably less than the maximum circumferential stress ($|\sigma_{\theta\theta B}| > |\sigma_{\theta\theta C}|$) and may not be of concern for the design engineer.

EXAMPLE 9.6
Radial Stress in an I-Section

The curved section of the frame of a press is subjected to a positive moment M_0 = 96 kN • m and a shear load P = 120 kN (Figure E9.6*a*). The dimensions of section BC are shown in Figure E9.6*b*. Determine the circumferential stress $\sigma_{\theta\theta}$ at point B and the radial stress σ_{rr} at points B' and C' of section BC. Include the effects of traction N.

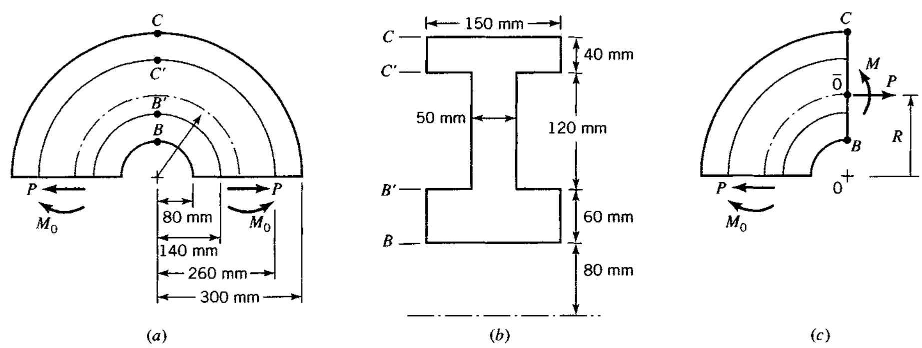


FIGURE E9.6

Solution

The magnitudes of A, A_m, and R are given by Eqs. 9.12, 9.13, and 9.14. They are

$$A = 150(60) + 50(120) + 150(40) = 21{,}000 \text{ mm}^2$$

$$A_m = 150 \ln\frac{140}{80} + 50 \ln\frac{260}{140} + 150 \ln\frac{300}{260} = 136.360 \text{ mm} \qquad \text{(a)}$$

$$R = \frac{150(60)110 + 50(120)200 + 150(40)280}{21{,}000} = 184.286 \text{ mm}$$

By Figure E9.6*c*,

$$\circlearrowleft\!+\ \sum M_{\bar{0}} = M - M_0 - PR = 0$$

or

$$M = 96{,}000{,}000 + 120{,}000(184.286) = 118.1 \times 10^6 \text{ N} \bullet \text{mm}$$

Then, by Eq. 9.11 with $r = 80$ mm, the circumferential stress at b is

$$\begin{aligned}\sigma_{\theta\theta} &= \frac{120{,}000}{21{,}000} + \frac{118{,}100{,}000\,[21{,}000 - 80(136.360)]}{21{,}000(80)[184.286(136.360) - 21{,}000]}\\ &= 5.71 + 171.80 = 177.51 \text{ MPa}\end{aligned}$$

To find the radial stress σ_{rr} at the junction of the flange and web (point B'), we require the geometric terms A' and A'_m. By Eq. 9.20,

$$\begin{aligned}A' &= 150(60) = 9000 \text{ mm}^2\\ A'_m &= \int_{80}^{140} 150\frac{dr}{r} = 150\ln\frac{140}{80} = 83.94 \text{ mm}\end{aligned} \tag{b}$$

With the values in Eq. (b), $r = 140$ mm, and $t = 50$ mm, Eqs. 9.18 and 9.19 yield

$$\begin{aligned}\sigma_{rr} &= \frac{A'}{A}\frac{N}{tr} + \frac{AA'_m - A'A_m}{trA(RA_m - A)}M\\ &= \frac{9000}{21{,}000}\frac{120{,}000}{50(140)} + \frac{[21{,}000(83.94) - 9000(136.360)]}{50(140)(21{,}000)[184.286(136.360) - 21{,}000]}(118.1\times10^6)\\ &= 7.347 + 104.189 = 111.54 \text{ MPa}\end{aligned}$$

Here we see that the effect of N represents $(7.347/111.54) \times 100\% = 6.59\%$ of the total σ_{rr} at B'.

Similarly, for the radial stress at point C', where $r = 260$ mm and $t = 50$ mm, the geometric terms A' and A'_m are

$$\begin{aligned}A' &= 150(60) + 50(120) = 15{,}000 \text{ mm}^2\\ A'_m &= \int_{80}^{140} 150\frac{dr}{r} + \int_{140}^{260} 50\frac{dr}{r} = 114.89 \text{ mm}\end{aligned} \tag{c}$$

Then, by Eqs. 9.18, 9.19, and (c), we have

$$\begin{aligned}\sigma_{rr} &= \frac{15{,}000}{21{,}000}\frac{120{,}000}{50(260)} + \frac{[21{,}000(114.89) - 15{,}000(136.360)]}{50(260)(21{,}000)[184.286(136.360) - 21{,}000]}(118.1\times10^6)\\ &= 6.59 + 38.48 = 45.07 \text{ MPa}\end{aligned}$$

At C', the effect of N represents 14.6% of the total radial stress. In either case (point B' or C'), σ_{rr} is considerably less than $\sigma_{\theta\theta} = 177.54$ MPa.

EXAMPLE 9.7
Radial Stress in Glulam Beam

A glued laminated timber (glulam) beam is used in a roof system. The beam has a simple span of 15 m and the middle half of the beam is curved with a mean radius of 10 m. The beam depth and width are both constant: $d = 0.800$ m and $b = 0.130$ m. Dead load is 2400 N/m and snow load is 4800 N/m. The geometry of the beam and assumed loading are shown in Figure E9.7.

(a) Determine the maximum circumferential and radial stresses in the beam.

(b) Compare the maximum circumferential stress to that obtained from the straight-beam flexure formula.

(c) Compare the maximum circumferential and radial stresses to the allowable stress limits for Douglas fir: $\sigma_{\theta\theta(\text{allow})} = 15.8$ MPa, $\sigma_{rr(\text{allow})} = 0.119$ MPa (AITC, 1994).

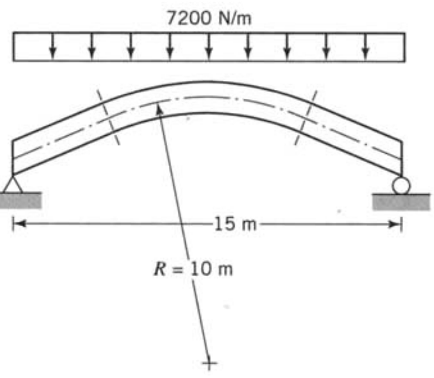


FIGURE E9.7

Solution

(a) The maximum bending moment occurs at midspan and has magnitude $M_x = wl^2/8 = 202{,}500$ N • m. Circumferential stress $\sigma_{\theta\theta}$ is calculated using Eq. 9.11. For the curved beam described,

$$a = R - \frac{d}{2} = 9.6 \text{ m}$$

$$c = R + \frac{d}{2} = 10.4 \text{ m}$$

$$A = 0.13 \times 0.80 = 0.104 \text{ m}^2$$

$$A_m = 0.13 \ln\frac{10.4}{9.6} = 0.0104056$$

The maximum circumferential stress occurs at the inner edge of the beam $r = a$. It is

$$\sigma_{\theta\theta(\max)} = \frac{M_x(A - aA_m)}{Aa(RA_m - A)} = \frac{202{,}500[0.104 - 9.6(0.0104056)]}{0.104(9.6)[10.0(0.0104056) - 0.104]} = 15.0 \text{ MPa}$$

The maximum radial stress $\sigma_{rr(\max)}$ is calculated using Eq. 9.21. However, the location at which $\sigma_{rr(\max)}$ occurs is unknown. Thus, we must maximize σ_{rr} with respect to r. For a rectangular cross section, the quantities in Eq. 9.21 are

$$\begin{aligned}
t &= b = \text{width of cross section} \\
d &= c - a = \text{depth of cross section} \\
A &= bd \\
A' &= b(r - a) \\
A_m &= b \ln\frac{c}{a} \\
A'_m &= b \ln\frac{r}{a}
\end{aligned}$$

Substitution of these expressions into Eq. 9.21 gives

$$\sigma_{rr} = \frac{M_x}{b}\left[\frac{d\ln\left(\frac{r}{a}\right) - (r-a)\ln\left(\frac{c}{a}\right)}{rd\left[R\ln\left(\frac{c}{a}\right) - d\right]}\right] \tag{a}$$

Maximizing σ_{rr} with respect to r, we find that $\sigma_{rr(\max)}$ occurs at

$$r = ae^{\left(1 - \frac{a}{d}\ln\frac{c}{a}\right)} \tag{b}$$

We evaluate Eq. (b) for the particular cross section of this example to obtain $r = 9.987$ m. At that location, the radial stress is, by Eq. (a),

$$\sigma_{rr(\max)} = \frac{202{,}500}{0.13}\left[\frac{0.80\ln\left(\frac{9.987}{9.6}\right) - \left[(9.987 - 9.6)\ln\left(\frac{10.4}{9.6}\right)\right]}{9.987(0.80)\left[10.0\ln\left(\frac{10.4}{9.6}\right) - 0.80\right]}\right] \tag{c}$$

$$= 0.292 \text{ MPa}$$

An approximate formula for computing radial stress in curved beams of rectangular cross section is (AITC, 1994, p. 227)

$$\sigma_{rr} = \frac{3M}{2Rbd} \tag{d}$$

Using this expression, we determine the radial stress to be $\sigma_{rr} = 0.292$ MPa. The approximation of Eq. (d) is quite accurate in this case! In fact, for rectangular curved beams with $R/d > 3$, the error in Eq. (d) is less than 3%. However, as R/d becomes small, the error grows substantially and Eq. (d) is nonconservative.

(b) Using the curved beam formula, Eq. 9.11, we obtain the maximum circumferential stress as $\sigma_{\theta\theta(\max)} = 15.0$ MPa. Using the straight-beam flexure formula, Eq. 7.1, with $I_x = bd^3/12 = 0.005547\ \text{m}^4$, we obtain $\sigma_{\theta\theta} = 202{,}500(0.40)/0.005547 = 14.6$ MPa. Thus, the straight-beam flexure formula is within 3% of the curved beam formula. One would generally consider the flexure formula adequate for this case, in which $R/d = 12.5$.

(c) The maximum circumferential stress is just within its limiting value; the beam is understressed just 5%. However, the maximum radial stress is 245% over its limit. It would be necessary to modify beam geometry or add mechanical reinforcement to make this design acceptable.

9.4 CORRECTION OF CIRCUMFERENTIAL STRESSES IN CURVED BEAMS HAVING I, T, OR SIMILAR CROSS SECTIONS

If the curved beam formula is used to calculate circumferential stresses in curved beams having thin flanges, the computed stresses are considerably in error and the error is nonconservative. The error arises because the radial forces developed in the curved beam