CHAPTER 7

BENDING OF STRAIGHT BEAMS

In this chapter, we assume that there is a plane in which the forces that act on a beam lie. This plane is called the *plane of loads*. In addition, we assume that the plane of loads passes through a point (the *shear center*) in the beam cross section, so that there is no twisting (torsion) of the beam; that is, the resulting forces that act on any cross section of the beam consist only of bending moments and shear forces. The concept of shear center is studied in Chapter 8. For now, though, we simply assume that the net torque is zero. We introduce the concepts of symmetrical and nonsymmetrical bending of straight beams and the plane of loads in Section 7.1. In Section 7.2, we develop formulas for stresses in beams subjected to nonsymmetrical bending. In Section 7.3, deflections of beams are computed. In Section 7.4, the effect of an inclined load relative to a principal plane is examined. Finally, in Section 7.5, a method is presented for computing fully plastic loads for cross sections in nonsymmetrical bending.

7.1 FUNDAMENTALS OF BEAM BENDING

7.1.1 Centroidal Coordinate Axes

The straight cantilever beam shown in Figure 7.1 has a cross section of arbitrary shape. It is subjected to pure bending by the end couple $\mathbf{M_0}$. Let the origin 0 of the coordinate system (x, y, z) be chosen at the centroid of the beam cross section at the left end of the beam, with the z axis directed along the centroidal axis of the beam, and the (x, y) axes taken in the plane of the cross section. Generally, the orientation of the (x, y) axis is arbitrary. However, we often choose the (x, y) axes so that the moments of inertia of the cross section I_x, I_y, and I_{xy} are easily calculated, or we may take them to be principal axes of the cross section (see Appendix B).

The bending moment that acts at the left end of the beam (Figure 7.1a) is represented by the vector $\mathbf{M_0}$ directed perpendicular to a plane that forms an angle ϕ taken positive when measured counterclockwise from the x–z plane as viewed from the positive z axis. This plane is called the *plane of load* or the *plane of loads*. Consider now a cross section of the beam at distance z from the left end. The free-body diagram of the part of the beam to the left of this section is shown in Figure 7.1b. For equilibrium of this part of the beam, a moment $\mathbf{M}$, equal in magnitude but opposite in sense to $\mathbf{M_0}$, must act at section z. For the case shown ($\pi/2 \le \phi \le \pi$), the (x, y) components (M_x, M_y) of $\mathbf{M}$ are related to the signed magnitude M of $\mathbf{M}$ by the relations $M_x = M \sin \phi$, $M_y = -M \cos \phi$. Since $\pi/2 \le \phi \le \pi$, $\sin \phi$ is positive and $\cos \phi$ is negative. Since (M_x, M_y) are positive (Figure 7.1b), the sign of M is positive.

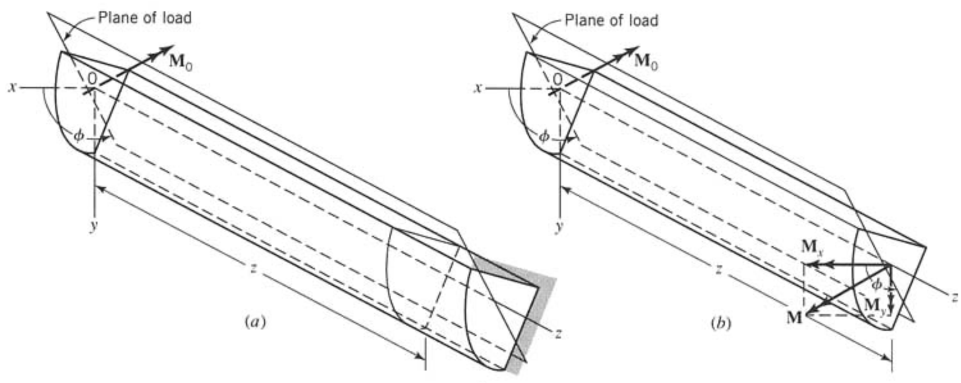

FIGURE 7.1 Cantilever beam with an arbitrary cross section subjected to pure bending.

7.1.2 Shear Loading of a Beam and Shear Center Defined

Let the beam shown in Figure 7.2*a* be subjected to a concentrated force **P** that lies in the end plane ($z = 0$) of the beam cross section. The vector representing **P** lies in the plane of the load that forms angle ϕ, taken positive when measured counterclockwise from the x–z plane as viewed from the positive z axis. Consider a cross section of the beam at distance z from the left end. The free-body diagram of the part of the beam to the left of this section is shown in Figure 7.2*b*. For equilibrium of this part of the beam, a moment **M**, with components $\mathbf{M}_x$ and $\mathbf{M}_y$, shear components $\mathbf{V}_x$ and $\mathbf{V}_y$, and in general, a twisting moment **T** (with vector directed along the positive z axis) must act on the section at z. However, if the line of action of force **P** passes through a certain point C (the shear center) in the cross section, then $\mathbf{T} = 0$. In this discussion, we assume that the line of action of **P** passes through the shear center. Hence, **T** is not shown in Figure 7.2*b*. Note that in Figure 7.2*b*, the force **P** requires $\mathbf{V}_x$, $\mathbf{V}_y$ to be positive [directed along positive (x, y) axes, respectively]. The component $\mathbf{M}_x$ is also directed along the positive x axis. However, since $\phi < \pi/2$, $\mathbf{M}_y$ is negative (directed along the negative y axis).

There is a particular axial line in the beam called the *bending axis of the beam*, which is parallel to the centroidal axis of the beam (the line that passes through the cen-

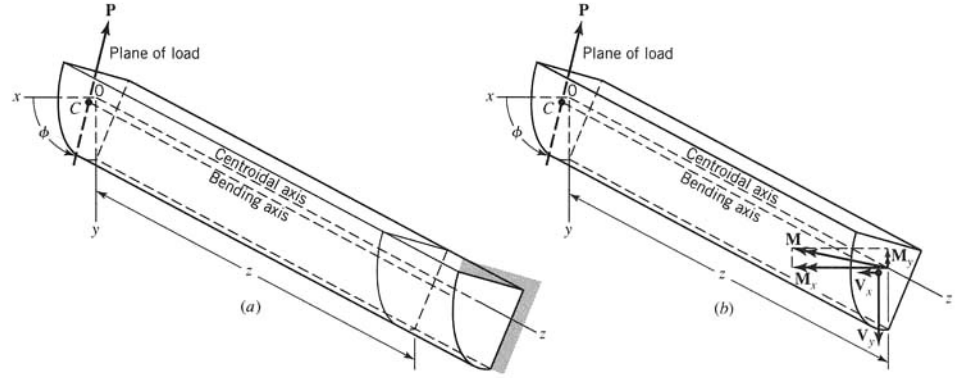

FIGURE 7.2 Cantilever beam with an arbitrary cross section subjected to shear loading.

troids of all of the cross sections of the beam). Except for special cases, the bending axis does not coincide with the centroidal axis (Figure 7.2).

The intersection of the bending axis with any cross section of the beam locates a point C in that cross section called the *shear center* of the cross section (see Section 8.1). Thus, the bending axis passes through the shear centers of all the cross sections of the beam.

In Section 7.2, formulas are derived for the normal stress component σ_{zz} that acts on the cross section at z in terms of the bending moment components $(\mathbf{M}_x, \mathbf{M}_y)$. Also, one may derive formulas for the shear stress components (τ_{zx}, τ_{zy}) resulting from the shear forces $(\mathbf{V}_x, \mathbf{V}_y)$. However, if the length L of the beam is large compared to the maximum cross-section dimension D, such that $L/D > 5$, the maximum shear stress is small compared to the maximum normal stress. In this chapter we ignore the shear stresses resulting from $(\mathbf{V}_x, \mathbf{V}_y)$; that is, we consider beams for which $L/D > 5$.

For bending of a beam by a concentrated force and for which the shear stresses are negligible, the line of action of the force must pass through the shear center of a cross section of the beam; otherwise, the beam will be subjected to both bending and torsion (twist). Thus, for the theory of pure bending of beams, we assume that the shear stresses resulting from concentrated loads are negligible and that the lines of action of concentrated forces that act on the beam pass through the shear center of a beam cross section. If the cross section of a beam has either an axis of symmetry or an axis of antisymmetry, the shear center C is located on that axis (Figure 7.3). If the cross section has two or more axes of symmetry or antisymmetry, the shear center is located at the intersection of the axes (Figures 7.3*a* and 7.3*d*). For a general cross section (Figure 7.1) or for a relatively thick, solid cross section (Figure 7.3*c*), the determination of the location of the shear center requires advanced computational methods (Boresi and Chong, 2000).

In this chapter, unless the shear center is located by intersecting axes of symmetry or antisymmetry, its location is approximated. The reader should have a better understanding of such approximations after studying Chapter 8.

7.1.3 Symmetrical Bending

In Appendix B, it is shown that every beam cross section has principal axes (X, Y). With respect to principal axes (X, Y), the product of inertia of the cross section is zero; $I_{XY} = 0$. The principal axes (X, Y) for the cross section of the cantilever beam of Figure 7.1 are shown in Figure 7.4. For convenience, axes (X, Y) are also shown at a section of the beam at distance z from the left end of the beam. At the left end, let the beam be subjected to a couple $\mathbf{M}_0$ with sense in the negative X direction and a force $\mathbf{P}$ through the shear center C with sense in the negative Y direction (Figure 7.4*a*). These loads are reacted by a bending

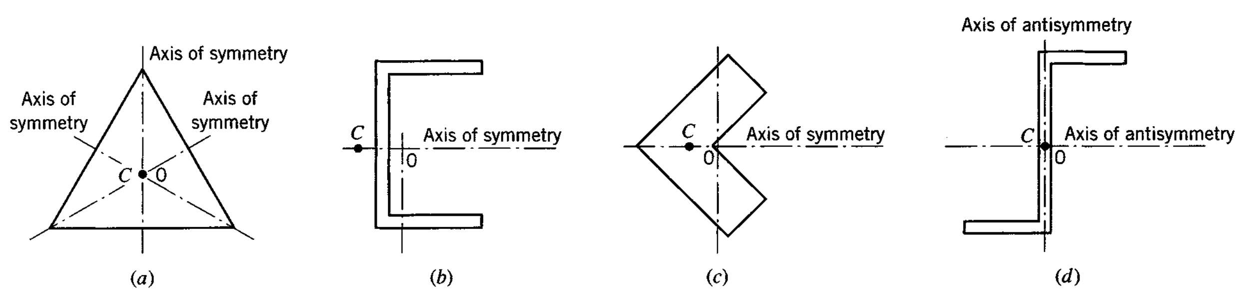

FIGURE 7.3 (*a*) Equilateral triangle section. (*b*) Open channel section. (*c*) Angle section. (*d*) Z-section.

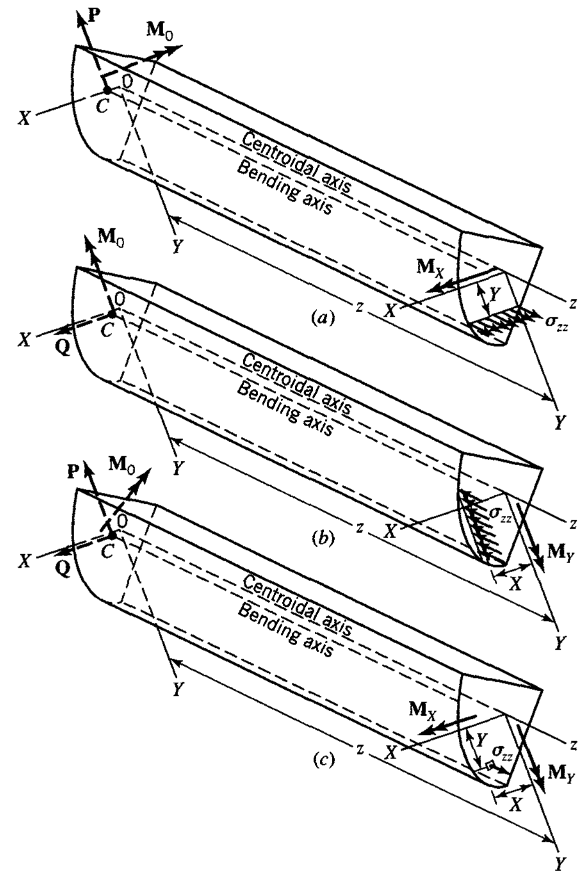

FIGURE 7.4 Cantilever beam with an arbitrary cross section.

moment $\mathbf{M} = \mathbf{M}_X$ at the cut section with sense in the positive X direction. By Bernoulli's beam theory (Boresi and Chong, 2000), the stress σ_{zz} normal to the cross section is given by the flexure formula

$$\sigma_{zz} = \frac{M_X Y}{I_X} \tag{7.1}$$

where M_X is positive since $\mathbf{M}_X$ is in the positive X sense, Y is the distance from the principal axis X to the point in the cross section at which σ_{zz} acts, and I_X is the principal moment of inertia of the cross-sectional area relative to the X axis. Equation 7.1 shows that σ_{zz} is zero for $Y = 0$ (along the X axis). Consequently, the X axis is called the *neutral axis of bending of the cross section*, that is, the axis for which $\sigma_{zz} = 0$. We take M_X as positive when the sense of the vector representing $\mathbf{M}_X$ is in the positive X direction. If $\mathbf{M}_X$ is directed in the negative X sense, M_X is negative. Since M_X is related to σ_{zz} by Eq. 7.1, σ_{zz} is a tensile stress for positive values of Y and a compressive stress for negative values of Y. In addition to causing a bending moment component M_X, load P produces a positive shear V_Y at the cut section. It is assumed that the maximum shear stress τ_{ZY} resulting from V_Y is small compared to the maximum value of σ_{zz}. Hence, since this chapter treats bending effects only, we neglect shear stresses in this chapter.

Likewise, if a load **Q** (applied at the shear center *C*) directed along the positive *X* axis and a moment $\mathbf{M}_0$ directed along the negative *Y* axis are applied to the left end of the beam (Figure 7.4*b*), they are reacted by a bending moment $\mathbf{M} = \mathbf{M}_Y$ directed along the positive *Y* axis. The normal stress distribution σ_{zz}, resulting from $\mathbf{M}_Y$, is also given by the flexure formula. Thus,

$$\sigma_{zz} = -\frac{M_Y X}{I_Y} \tag{7.2}$$

where M_Y is positive since $\mathbf{M}_Y$ is in the positive *Y* sense, *X* is the distance from the principal axis *Y* to the point in the cross section at which σ_{zz} acts, and I_Y is the principal moment of inertia of the cross-sectional area relative to the *Y* axis. The negative sign arises from the fact that a positive M_Y produces compressive stresses on the positive side of the *X* axis. Now for $X = 0$ (along the *Y* axis), $\sigma_{zz} = 0$. Hence, in this case, the *Y* axis is the *neutral axis of bending of the cross section*, that is, the axis for which $\sigma_{zz} = 0$. In either case (Eq. 7.1 or 7.2), the beam is subjected to *symmetrical bending*. (Bending occurs about a neutral axis in the cross section that coincides with the corresponding principal axis.)

As a simple case, consider a straight cantilever beam of constant rectangular cross section (Figure 7.5*a*) subjected to the end couple $\mathbf{M}_0$ directed along the negative *X* axis. Axes (*X*, *Y*) are principal axes of the cross section, and axis *z* is the centroidal axis of the beam. In this case, the shear center and the centroid of the cross section coincide, and hence the bending axis coincides with the centroidal axis *z*. Consider now a cross section of the beam at distance *z* from the left end. The free-body diagram of the part of the beam to the left of this section is shown in Figure 7.5*b*. For equilibrium of this part, a moment $\mathbf{M}_X$ equal in magnitude to $\mathbf{M}_0$, but opposite in sense, acts at section *z*. That is, $\mathbf{M}_X$ is directed in the positive *X* sense. Hence, the stress σ_{zz} normal to the cross section is (see Eq. 7.1)

$$\sigma_{zz} = \frac{M_X Y}{I_X}$$

The magnitude $\sigma_{\max}$ of the maximum flexural stress occurs at $Y = \pm h/2$. For positive M_X, σ_{zz} is tensile at $Y = h/2$ and compressive at $Y = -h/2$. Also, $I_X = bh^3/12$ for the rectangular cross section. Hence, for a constant rectangular cross section

$$\sigma_{\max} = \frac{6|M_X|}{bh^2}$$

where $|M_X|$ denotes the absolute value of M_X. More generally for a beam of general cross section subjected to symmetrical bending, the flexural stress increases linearly from zero at

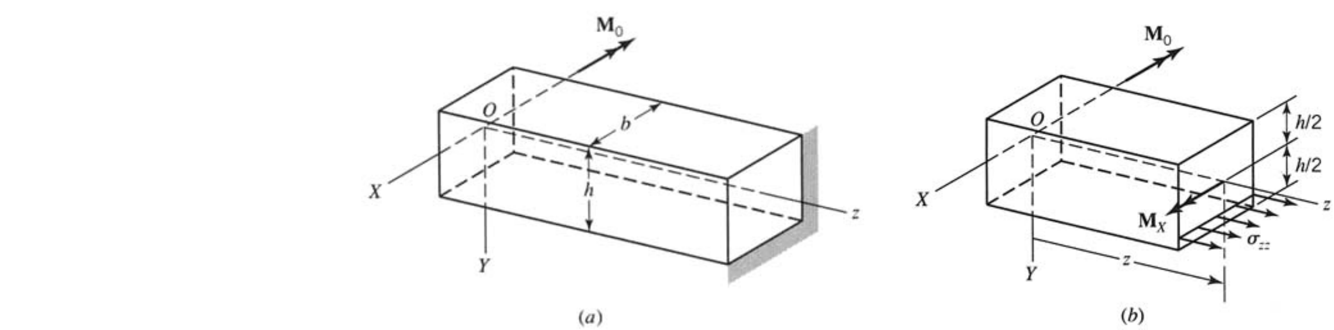

FIGURE 7.5 Cantilever beam with rectangular cross section.

the neutral axis ($Y = 0$) to an absolute maximum at either the top or bottom of the beam, whichever is farther from the neutral axis. For example, let c_1 be the distance from the neutral axis to the bottom of the beam and c_2 be the distance to the top of the beam. Then,

$$\sigma_{max} = \frac{|M_X| c_{max}}{I_X} = \frac{|M_X|}{S_X} \tag{7.3}$$

where c_{max} is the larger of c_1 and c_2, and $S_X = I_X/c_{max}$. The factor S_X is called the *elastic section modulus*. For a beam of constant cross section, S_X is a constant. The elastic section moduli for many manufactured cross sections are listed in handbooks for the convenience of design engineers. See also Appendix C.

Equation 7.3 indicates that σ_{max} is inversely proportional to S_X. For example, the five beam cross sections shown in Figure 7.6 all have a cross-sectional area of 10,400 mm^2. The magnitude of S_X increases from 149×10^3 mm^3 for the solid circular cross section to 1.46×10^6 mm^3 for the S460 × 81.4 I-beam. For a given cross section shape, the magnitude of S_X increases with depth of cross section and as a larger portion of the area is moved away from the neutral axis. However, the depth of the beam cross section (and the web thickness of a standard I-beam) is limited to prevent local buckling of the cross section elements (see Chapter 12).

7.1.4 Nonsymmetrical Bending

In Figure 7.4*c*, the beam is subjected to moment $\mathbf{M}_0$ with components in the negative directions of both the X and Y axes as well as concentrated forces **P** and **Q** acting through the shear center C. These loads result in a bending moment **M** at the cut section with positive projections (M_X, M_Y). For this loading, the stress σ_{zz} normal to the cross section may be obtained by the superposition of Eqs. 7.1 and 7.2. Thus,

$$\sigma_{zz} = \frac{M_X Y}{I_X} - \frac{M_Y X}{I_Y} \tag{7.4}$$

In this case, the moment $\mathbf{M} = (M_X, M_Y)$ is not parallel to either of the principal axes (X, Y). Hence the bending of the beam occurs about an axis that is not parallel to either the X or Y axis. When the axis of bending does not coincide with a principal axis direction, the bending of the beam is said to be *nonsymmetrical*. The determination of the *neutral axis of bending of the cross section* for nonsymmetrical bending is discussed in Section 7.2.

7.1.5 Plane of Loads: Symmetrical and Nonsymmetrical Loading

Often, a beam is loaded by forces that lie in a plane that coincides with a plane of symmetry of the beam, as depicted in Figure 7.7. In this figure, the y axis is an axis of symmetry for the cross section; it is a principal axis. Hence, if axes (x, y) are principal axes for the cross section, the beams in Figures 7.7*a* and *b* undergo symmetrical bending, that is, bending about a principal axis of a cross section, since the moment vector in Figure 7.7*a* and the force vectors in Figure 7.7*b* are parallel to principal axes. We further observe that since the shear center lies on the y axis, the plane of the load contains the axis of bending of the beam.

Consider next two beams with cross sections shown in Figure 7.8. Since a rectangular cross section (Figure 7.8*a*) has two axes of symmetry that pass through its centroid 0, the shear center C coincides with the centroid 0. Let the intersection of the plane of the loads and the plane of the cross section be denoted by line L–L, which forms angle

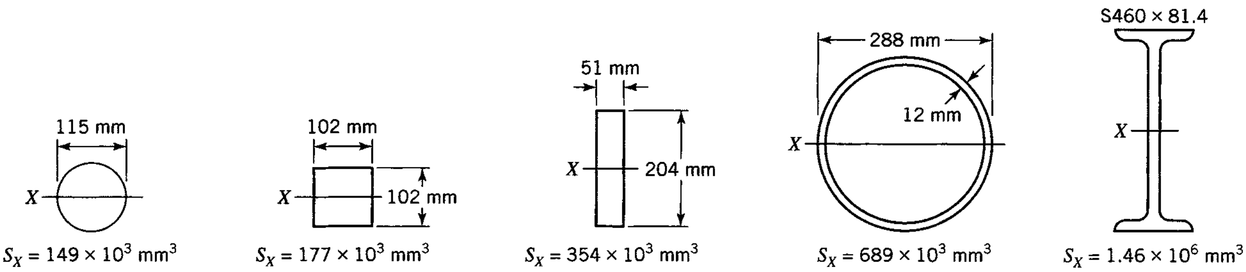

FIGURE 7.6 Cross sections with the same area but different values of S_X.

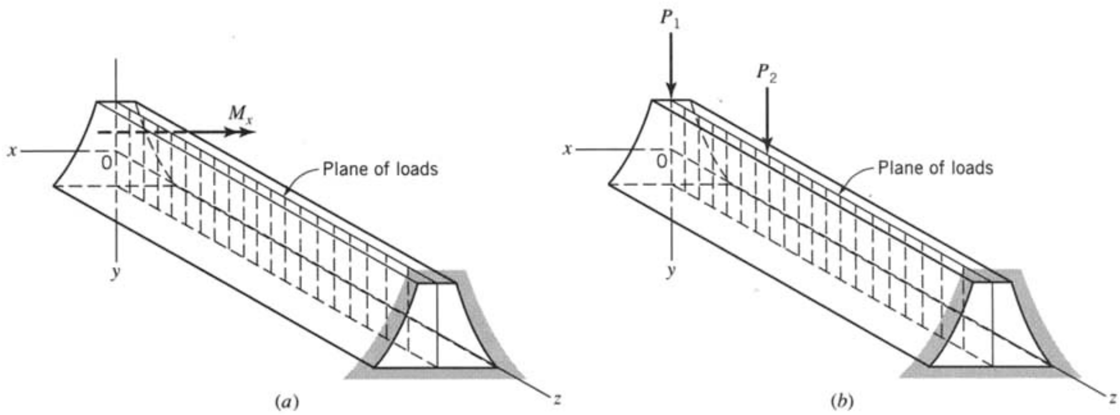

FIGURE 7.7 Symmetrical bending: Plane of loads coincident with the plane of symmetry of the beam. (*a*) Couple loads. (*b*) Lateral loads.

ϕ measured counterclockwise from the x–z plane and passes through the shear center C. Since the plane of loads contains point C, the bending axis of the rectangular beam lies in the plane of the loads. If the angle ϕ equals 0 or $\pi/2$, the rectangular beam will undergo symmetrical bending. For other values of ϕ, the beam undergoes nonsymmetrical bending, that is, bending for which the neutral axis of bending of the cross section does not coincide with either of the principal axes X–Y.

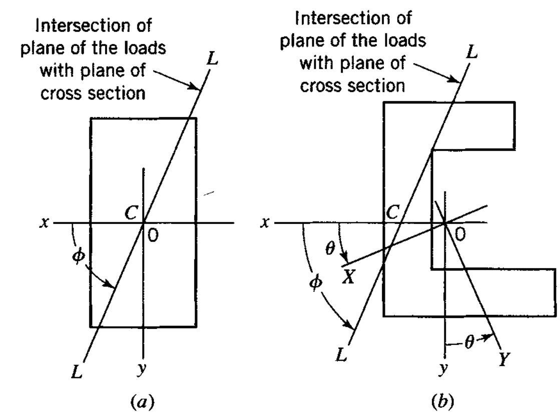

FIGURE 7.8 Nonsymmetrically loaded beams. (*a*) Rectangular cross section. (*b*) Channel cross section.

In the case of a general channel section (Figure 7.8*b*), the principal axes X–Y are located by a rotation through angle θ (positive θ is taken counterclockwise) from the x–y axes as shown. The value of θ is determined by Eq. B.12 in Appendix B. Although the plane of loads contains the shear center C (and hence the bending axis of the beam), it is not parallel to either of the principal planes X–z or Y–z. Hence, in general, the channel beam (Figure 7.8*b*) undergoes nonsymmetrical bending. However, for the two special cases, $\phi = \theta$ or $\phi = \theta + \pi/2$, the channel beam does undergo symmetrical bending.

EXAMPLE 7.1 *Cantilever Beam Subjected to Uniform Load*

A cantilever beam (Figure E7.1*a*) has a design requirement that its depth h be twice its width. It is made of structural steel (E = 200 GPa and Y = 250 MPa). The design of the beam is based on a factor of safety SF = 1.9 for failure by general yielding when the beam is subjected to a uniform load w = 1.0 kN/m. Determine the depth of the beam's cross section.

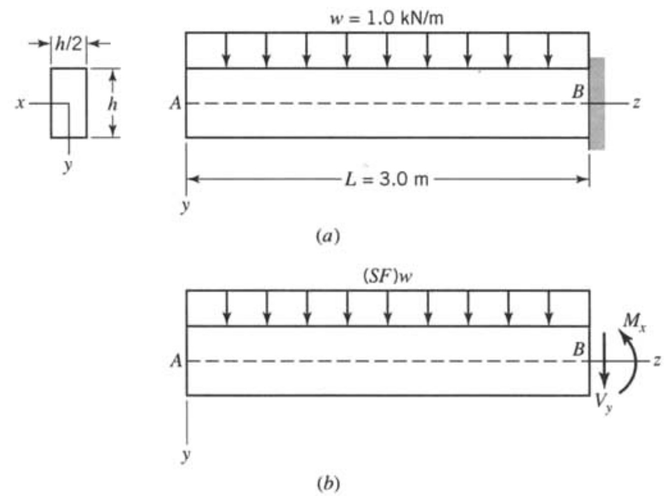

FIGURE E7.1

Solution

To account for the safety factor, we multiply w by 1.9. Yielding will be initiated at the location where the bending moment is maximum. By Figure E7.1*a*, the maximum bending moment occurs at the wall. The free-body diagram of the beam to the left of the wall is shown in Figure E7.1*b*. Taking moments about the horizontal axis at section B, we have

$$\sum M_B = M_x + \frac{(SF)wL^2}{2} = 0$$

or

$$M_x = -8550 \text{ N} \cdot \text{m}$$

The negative sign for M_x indicates that the bottom of the beam is in compression and the top is in tension. By Eq. 7.3, the maximum magnitude of flexural stress is

$$\sigma_{\max} = \frac{|M_x| c_{\max}}{I_x} = \frac{8550(h/2)}{h^4/24} = \frac{102{,}600}{h^3}$$

At yielding, $\sigma_{\max} = Y = 250 \times 10^6$. Hence,

$$h = 0.0743 \text{ m} = 74.3 \text{ mm}$$

EXAMPLE 7.2
Symmetrically Loaded T-Section Beam

A simple beam is subjected to loads of 1.5 kN and 4.5 kN, as shown in Figure E7.2*a*. The cross section of the beam is shown in Figure E7.2*b*.

(a) Determine the values for the maximum tensile and the maximum normal stresses σ_{zz} for the section at midspan of the beam.

(b) Sketch the distribution of σ_{zz} for the section at midspan.

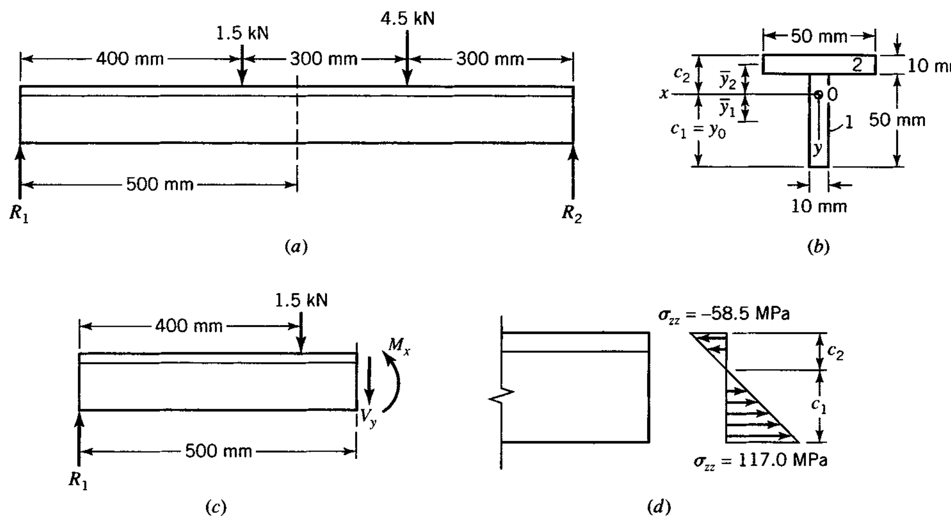

FIGURE E7.2

Solution

(a) To locate the centroid of the cross section, we consider the first area moments of two rectangles 1 and 2 in Figure E7.2*b*, relative to the bottom of the T-section. Since the total area is $A = 1000\ \text{mm}^2$ and the areas of the rectangles are $A_1 = A_2 = 500\ \text{mm}^2$,

$$Ay_0 = A_1y_1 + A_2y_2$$

where y_1 and y_2 are the distances from the centroids of rectangles 1 and 2 from the bottom of the beam. Hence, substituting known values for area and distance into this equation, we obtain

$$y_0 = c_1 = 40.0\ \text{mm}$$

and therefore at the top of the beam (Figure E7.2*b*)

$$c_2 = 20\ \text{mm}$$

The moment of inertia about the centroidal axis, found by the parallel axis theorem and Figure E7.2*b*, is

$$\begin{aligned} I_x &= \frac{1}{12}b_1h_1^3 + A_1\bar{y}_1^2 + \frac{1}{12}b_2h_2^3 + A_2\bar{y}_2^2 \\ &= 333{,}300\ \text{mm}^4 = 3.333 \times 10^{-7}\ \text{m}^4 \end{aligned}$$

where $\bar{y}_1$ and $\bar{y}_2$ are centroidal distances of A_1 and A_2 from the x axis, respectively. The free-body diagram of the beam segment to the left of midspan is shown in Figure E7.2*c*. The equations of equilibrium for this portion of the beam yield

$$R_1 = 2250\ \text{N}, \quad V_y = 750\ \text{N}, \quad M_x = 975\ \text{N} \cdot \text{m}$$

The maximum tensile stress occurs at the bottom of the beam, at a distance c_1 from the centroidal axis:

$$\sigma_{zz} = \frac{M_x c_1}{I_x} = 117.0 \text{ MPa}$$

The maximum compressive stress occurs at the top of the beam, at a distance $y = -c_2 - 20$ mm from the centroidal axis:

$$\sigma_{zz} = \frac{M_x(-c_2)}{I_x} = -58.5 \text{ MPa}$$

(b) The flexural stress σ_{zz} varies linearly from $y = 40$ to $y = -20$ mm over the cross section at midspan and is shown in Figure E7.2*d*.

7.2 BENDING STRESSES IN BEAMS SUBJECTED TO NONSYMMETRICAL BENDING

Let a cutting plane be passed through a straight cantilever beam at section z. The free-body diagram of the beam to the left of the cut is shown in Figure 7.9*a*. The beam has constant cross section of arbitrary shape. The origin 0 of the coordinate axes is chosen at the centroid of the beam cross section at the left end of the beam, with the z axis taken parallel to the beam. The left end of the beam is subjected to a bending couple $\mathbf{M}_0$ that is equilibrated by bending moment $\mathbf{M}$ acting on the cross section at z, with positive components $(\mathbf{M}_x, \mathbf{M}_y)$ as shown. The bending moment $\mathbf{M} = (\mathbf{M}_x, \mathbf{M}_y)$ is the resultant of the forces due to the normal stress σ_{zz} acting on the section (Figure 7.9*b*). For convenience, we show (x, y) axes at the cross section z. It is assumed that the (x, y) axes are not principal axes for the cross section. In this article, we derive the load–stress formula that relates the normal stress σ_{zz} acting on the cross section to the components $(\mathbf{M}_x, \mathbf{M}_y)$.

The derivation of load–stress and load–deformation relations for the beam requires that equilibrium equations, compatibility conditions, and stress–strain relations be satisfied for the beam along with specified boundary conditions for the beam.

7.2.1 Equations of Equilibrium

Application of the equations of equilibrium to the free body in Figure 7.9*b* yields (since there is no net resultant force in the z direction)

$$\begin{aligned} 0 &= \int \sigma_{zz}\, dA \\ M_x &= \int y\sigma_{zz}\, dA \\ M_y &= -\int x\sigma_{zz}\, dA \end{aligned} \qquad (7.5)$$

where dA denotes an element of area in the cross section and the integration is performed over the area A of the cross section. The other three equilibrium equations are satisfied identically, since σ_{zz} is the only nonzero stress component. To evaluate the integrals in Eq. 7.5, it is necessary that the functional relation between σ_{zz} and (x, y) be known. The determination of σ_{zz} as a function of (x, y) is achieved by considering the geometry of deformation and the stress–strain relations.

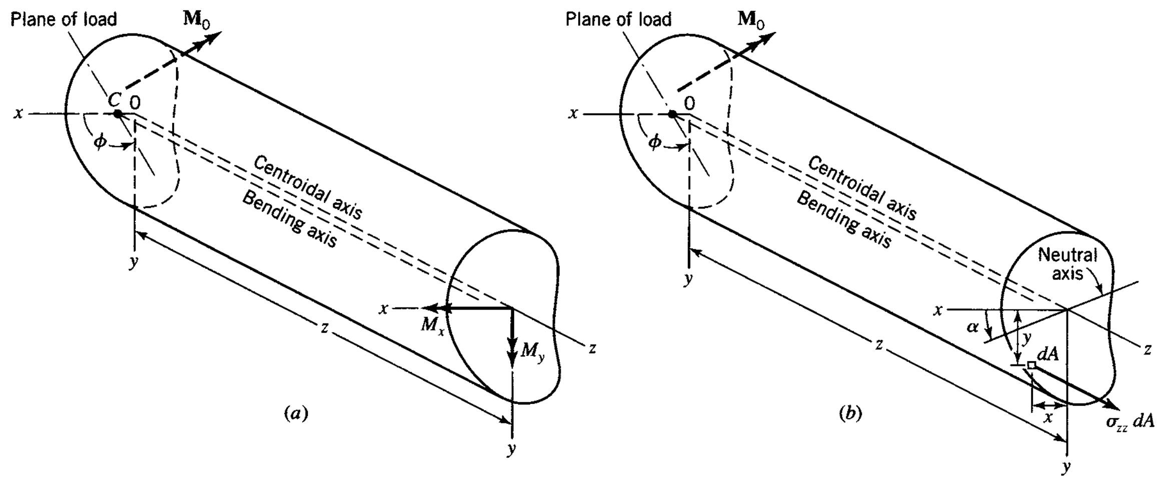

FIGURE 7.9 Pure bending of a nonsymmetrically loaded cantilever beam.

7.2.2 Geometry of Deformation

We assume that plane sections of an unloaded beam remain plane after the beam is subjected to pure bending. Consider two plane cross sections perpendicular to the bending axis of an unloaded beam such that the centroids of the two sections are separated by a distance Δz. These two planes are parallel since the beam is straight. These planes rotate with respect to each other when moments M_x and M_y are applied. Hence, the extension e_{zz} of longitudinal fibers of the beam between the two planes can be represented as a linear function of (x, y), namely,

$$e_{zz} = a'' + b''x + c''y \tag{a}$$

where a'', b'', and c'' are constants. Since the beam is initially straight, all fibers have the same initial length Δz so that the strain ϵ_{zz} can be obtained by dividing Eq. (a) by Δz. Thus,

$$\epsilon_{zz} = a' + b'x + c'y \tag{7.6}$$

where $\epsilon_{zz} = e_{zz}/\Delta z$, $a' = a''/\Delta z$, $b' = b''/\Delta z$, and $c' = c''/\Delta z$.

7.2.3 Stress–Strain Relations

According to the theory of pure bending of straight beams, the only nonzero stress component in the beam is σ_{zz}. For linearly elastic conditions, Hooke's law states

$$\sigma_{zz} = E\epsilon_{zz} \tag{7.7}$$

Eliminating ϵ_{zz} between Eqs. 7.6 and 7.7, we obtain

$$\sigma_{zz} = a + bx + cy \tag{7.8}$$

where $a = Ea'$, $b = Eb'$, and $c = Ec'$.

7.2.4 Load–Stress Relation for Nonsymmetrical Bending

Substitution of Eq. 7.8 into Eqs. 7.5 yields

$$
\begin{aligned}
0 &= \int (a + bx + cy)\,dA = a\int dA + b\int x\,dA + c\int y\,dA \\
M_x &= \int (ay + bxy + cy^2)\,dA = a\int y\,dA + b\int xy\,dA + c\int y^2 dA \\
M_y &= -\int (ax + bx^2 + cxy)\,dA = -a\int x\,dA - b\int x^2 dA - c\int xy\,dA
\end{aligned}
\tag{7.9}
$$

Since the z axis passes through the centroid of each cross section of the end beam, $\int x\,dA = \int y\,dA = 0$. The other integrals in Eqs. 7.9 are defined in Appendix B. Equations 7.9 simplify to

$$
\begin{aligned}
0 &= aA \\
M_x &= bI_{xy} + cI_x \\
M_y &= -bI_y - cI_{xy}
\end{aligned}
\tag{7.10}
$$

where I_x and I_y are the centroidal moments of inertia of the beam cross section with respect to the x and y axes, respectively, and I_{xy} is the centroidal product of inertia of the beam cross section. Solving Eqs. 7.10 for the constants a, b, and c, we obtain

$$
\begin{aligned}
a &= 0 \quad (\text{because } A \neq 0) \\
b &= -\frac{M_y I_x + M_x I_{xy}}{I_x I_y - I_{xy}^2} \\
c &= \frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2}
\end{aligned}
\tag{7.11}
$$

Substitution of Eqs. 7.11 into Eq. 7.8 gives the normal stress distribution σ_{zz} on a given cross section of a beam subjected to nonsymmetrical bending in the form

$$
\sigma_{zz} = -\left(\frac{M_y I_x + M_x I_{xy}}{I_x I_y - I_{xy}^2}\right)x + \left(\frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2}\right)y
\tag{7.12}
$$

Equation 7.12 is not the most convenient form for the determination of the maximum value of the flexural stress σ_{zz}. Also, Eq. 7.12 does not lend itself readily to visualization of the bending behavior of the beam. A more convenient and a more visually meaningful form follows.

7.2.5 Neutral Axis

Before the location of the points of maximum tensile and compressive stresses in the cross section are determined, it is useful to locate the neutral axis. For this purpose, it is desirable to express the neutral axis orientation in terms of the angle ϕ between the plane of the loads and the x–z plane, where ϕ is measured positive counterclockwise (Figure 7.8). The magnitude of ϕ is generally in the neighborhood of $\pi/2$. The bending moments M_x and M_y can be written in terms of ϕ as follows:

$$
\begin{aligned}
M_x &= M\sin\phi \\
M_y &= -M\cos\phi
\end{aligned}
\tag{7.13}
$$

in which M is the signed magnitude of moment **M** at the cut section. The sign of M is positive if the x projection of the vector M is positive; it is negative if the x projection of **M** is negative. Because the (x, y) axes are chosen for the convenience of the one making the calculations, they are chosen so that the magnitude of M_x is not zero. Therefore, by Eqs. 7.13,

$$\cot\phi = -\frac{M_y}{M_x} \tag{7.14}$$

The neutral axis of the cross section of a beam subjected to nonsymmetrical bending is defined as the axis in the cross section for which $\sigma_{zz} = 0$. Thus, by Eq. 7.12, the equation of the neutral axis of the cross section is

$$y = \left(\frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}}\right) x = x \tan\alpha \tag{7.15}$$

where α is the angle between the neutral axis of bending and the x axis; α is measured positive counterclockwise (Figure 7.9), and

$$\tan\alpha = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}} \tag{7.16}$$

Since $x = y = 0$ satisfies Eq. 7.15, the neutral axis passes through the centroid of the section. The right side of Eq. 7.16 can be expressed in terms of the angle ϕ by using Eq. 7.14. Thus,

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} \tag{7.17}$$

7.2.6 More Convenient Form for the Flexure Stress σ_{zz}

Elimination of M_y between Eqs. 7.12 and 7.16 results in a more convenient form for the normal stress distribution σ_{zz} for beams subjected to nonsymmetrical bending, namely,

$$\sigma_{zz} = \frac{M_x(y - x\tan\alpha)}{I_x - I_{xy}\tan\alpha} \tag{7.18}$$

where tan α is given by Eq. 7.17. Once the neutral axis is located on the cross sections at angle α as indicated in Figure 7.9*b*, points in the cross section where the tensile and compressive flexure stresses are maxima are easily determined. The coordinates of these points can be substituted into Eq. 7.18 to determine the magnitudes of these stresses. If M_x is zero, Eq. 7.12 may be used instead of Eq. 7.18 to determine magnitudes of these stresses, or axes (x, y) may be rotated by $\pi/2$ to obtain new reference axes (x', y').

Note: Equations 7.17 and 7.18 have been derived by assuming that the beam is subjected to pure bending. These equations are exact for pure bending. Although they are not exact for beams subjected to transverse shear loads, often the equations are assumed to be valid for such beams. The error in this assumption is usually small, particularly if the beam has a length of at least five times its maximum cross-sectional dimension.

In the derivation of Eqs. 7.17 and 7.18, the (x, y) axes are any convenient set of orthogonal axes that have an origin at the centroid of the cross-sectional area. The equations are valid if (x, y) are principal axes; in this case, $I_{xy} = 0$. If the axes are principal axes and $\phi = \pi/2$, Eq. 7.17 indicates that $\alpha = 0$ and Eq. 7.18 reduces to Eq. 7.1.

For convenience in deriving Eqs, 7.17 and 7.18, the origin for the (x, y, z) coordinate axes was chosen (see Figure 7.9*b*) at the end of the free body opposite from the cut section with the positive z axis toward the cut section. The equations are equally valid if the origin is taken at the cut section with the positive z axis toward the opposite end of the free body. If ϕ_2 is the magnitude of ϕ for the second choice of axes and ϕ_1 is the magnitude of ϕ for the first choice of axes, then $\phi_2 = \pi - \phi_1$.

EXAMPLE 7.3 Channel Section Beam

The cantilever beam in Figure E7.3*a* has a channel section as shown in Figure E7.3*b*. A concentrated load $P = 12.0$ kN lies in the plane making an angle $\phi = \pi/3$ with the x–z plane. Load P lies in the plane of the cross section of the free end of the beam and passes through shear center C; in Chapter 8 we find that the shear center lies on the y axis as shown. Locate points of maximum tensile and compressive stresses in the beam and determine the stress magnitudes.

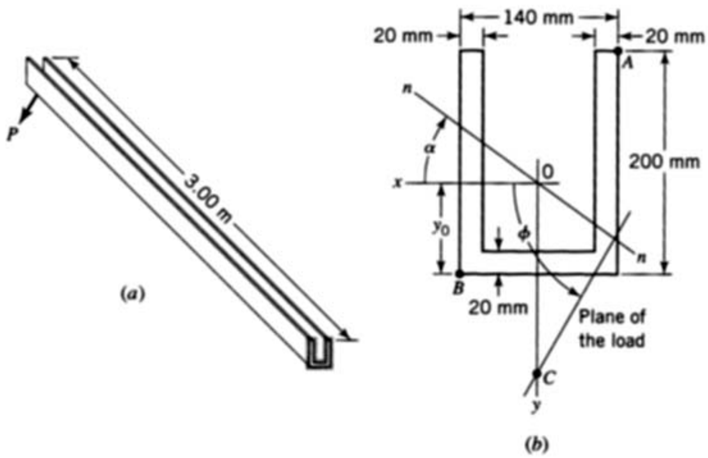

FIGURE E7.3

Solution

Several properties of the cross-sectional area are needed (see Appendix B):

$$A = 10{,}000 \text{ mm}^2, \quad I_x = 39.69 \times 10^6 \text{ mm}^4$$

$$y_0 = 82.0 \text{ mm}, \quad I_y = 30.73 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 0$$

The orientation of the neutral axis for the beam, given by Eq. 7.17, is

$$\tan\alpha = -\frac{I_x}{I_y}\cot\phi = -\frac{39{,}690{,}000}{30{,}730{,}000}(0.5774) = -0.7457$$

$$\alpha = -0.6407 \text{ rad}$$

The negative sign indicates that the neutral axis n–n, which passes through the centroid ($x = y = 0$), is located clockwise 0.6407 rad from the x axis (Figure E7.3*b*). The maximum tensile stress occurs at point A, whereas the maximum compressive stress occurs at point B. These stresses are given by Eq. 7.18 after M_x has been determined. From Figure E7.3*a*

$$M = -3.00P = -36.0 \text{ kN} \bullet \text{m}$$

$$M_x = M\sin\phi = -31.18 \text{ kN} \bullet \text{m}$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha}$$

$$= \frac{-31{,}180{,}000[-118-(-70)(-0.7457)]}{39{,}690{,}000}$$

$$= 133.7 \text{ MPa}$$

$$\sigma_B = \frac{-31{,}180{,}000[82-70(-0.7457)]}{39{,}690{,}000} = -105.4 \text{ MPa}$$

EXAMPLE 7.4 *Angle-Beam*

Plates are welded together to form the 120 mm by 80 mm by 10 mm angle-section beam shown in Figure E7.4*a*. The beam is subjected to a concentrated load $P = 4.00$ kN as shown. The load P lies in the plane making an angle $\phi = 2\pi/3$ with the x–z plane. Load P passes through shear center C which is located at the intersection of the two legs of the angle section. Determine the maximum tensile and compressive bending stresses at the section of the beam where the load is applied.

(a) Solve the problem using the load–stress relations derived for nonsymmetrical bending.

(b) Solve the problem using Eq. 7.4.

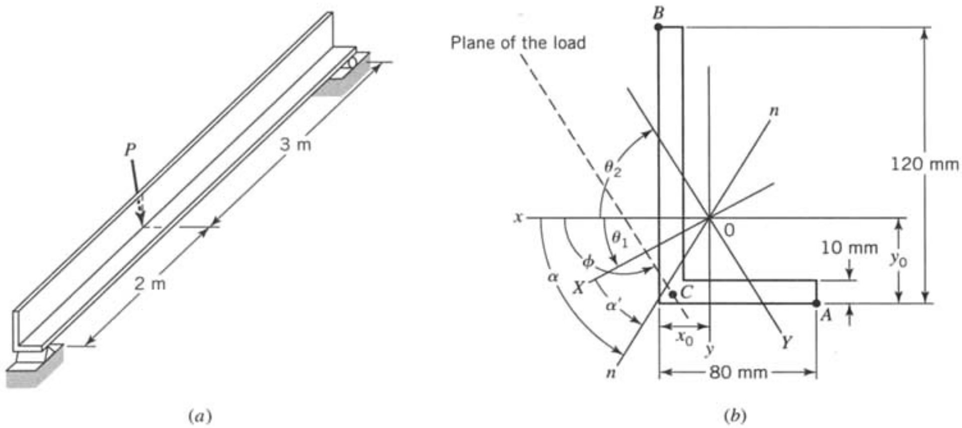

FIGURE E7.4

Solution

(a) Several properties of the cross-sectional area are needed (see Appendix B):

$$A = 1900 \text{ mm}^2, \quad I_x = 2.783 \times 10^6 \text{ mm}^4$$

$$x_0 = 19.74 \text{ mm}, \quad I_y = 1.003 \times 10^6 \text{ mm}^4$$

$$y_0 = 39.74 \text{ mm}, \quad I_{xy} = -0.973 \times 10^6 \text{ mm}^4$$

The orientation of the neutral axis for the beam is given by Eq. 7.17. Thus,

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi}$$

$$= \frac{-0.973 \times 10^6 - 2.783 \times 10^6(-0.5774)}{1.003 \times 10^6 - (-0.973 \times 10^6)(-0.5774)} = 1.436$$

$$\alpha = 0.9626 \text{ rad}$$

The positive sign indicates that the neutral axis n–n, which passes through the centroid ($x = y = 0$), is located counterclockwise 0.9628 rad from the x axis (Figure E7.4b). The maximum tensile stress occurs at point A, whereas the maximum compressive stress occurs at point B. These stresses are given by Eq. 7.18 after M_x has been determined. From Figure E7.4a

$$M = 1.2P = 4.80 \text{ kN} \cdot \text{m}$$

$$M_x = M\sin\phi = 4.80 \times 10^3(0.8660) = 4.157 \text{ kN} \cdot \text{m}$$

$$\begin{aligned}\sigma_A &= \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha}\\ &= \frac{4.157 \times 10^6[39.74 - (-60.26)(1.4363)]}{2.783 \times 10^6 - (-0.973 \times 10^6)(1.4363)}\\ &= 125.6 \text{ MPa}\end{aligned}$$

$$\begin{aligned}\sigma_B &= \frac{4.157 \times 10^6[-80.26 - 19.74(1.4363)]}{2.783 \times 10^6 - (-0.973 \times 10^6)(1.4363)}\\ &= -108.0 \text{ MPa}\end{aligned}$$

(b) To solve the problem using Eq. 7.4, it is necessary that the principal axes for the cross section be determined. The two values of the angle θ between the x axis and the principal axes are given by Eq. B.12. Thus, we obtain

$$\tan 2\theta = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-0.973 \times 10^6)}{2.783 \times 10^6 - 1.003 \times 10^6} = 1.0933$$

$$\theta_1 = 0.4150 \text{ rad} \quad (\theta_2 = -1.156 \text{ rad})$$

The principal X and Y axes are shown in Figure E7.2b. Thus (see Eq. B.10 in Appendix B)

$$I_X = I_x\cos^2\theta_1 + I_y\sin^2\theta_1 - 2I_{xy}\sin\theta_1\cos\theta_1 = 3.212 \times 10^6 \text{ mm}^4$$

$$I_Y = I_x + I_y - I_X = 0.574 \times 10^6 \text{ mm}^4$$

Note that now angle ϕ is measured from the X axis and not from the x axis as for part (a). Hence,

$$\phi = \frac{2\pi}{3} - \theta_1 = 1.6794 \text{ rad}$$

Angle α', which determines the orientation of the neutral axis, is now measured from the X axis (Figure E7.4b), and is given by Eq. 7.17. Hence, we find

$$\tan\alpha' = -\frac{I_X\cot\phi}{I_Y} = -\frac{3.212 \times 10^6(-0.1090)}{0.574 \times 10^6} = 0.6098$$

$$\alpha' = 0.5476 \text{ rad}$$

which gives the same orientation for the neutral axis as for part (a), that is,

$$\begin{aligned}\alpha &= \alpha' + \theta_1\\ &= 0.5476 + 0.4150\\ &= 0.9626 \text{ rad}\end{aligned}$$

To use Eq. 7.4 relative to axes (X, Y), the X and Y coordinates of points A and B are needed. They are (Eq. B.9)