(3) $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dy$ (4) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dx\,dy$

4.8 TRIPLE INTEGRAL (PHYSICAL SIGNIFICANCE)

The triple integral is defined in a manner entirely analogous to the definition of the double integral.

Let $F(x, y, z)$ be a function of three independent variables x, y, z defined at every point in a region of space V bounded by the surface S. Divided V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$ and let (x_r, y_r, z_r) be any point inside the rth sub division δV_r. Then, the limit of the sum

$$\sum_{r=1}^{n} F(x_r, y_r, z_r)\delta v_r\,, \qquad \ldots(1)$$

if exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the 'triple integral' of $R(x, y, z)$ over the region V, and is denoted by

$$\iiint F(x, y, z)\,dV \qquad \ldots(2)$$

In order to express triple integral in the 'integrated' form, V is considered to be sub-divided by planes parallel to the three coordinate planes. The volume V may then be considered as the sum of a number of vertical columns extending from the lower surface say, $z = f_1(x, y)$ to the upper surface say, $z = f_2(x, y)$ with base as the elementary areas δA_r over a region R in the xy-plance when all the columns in V are taken.

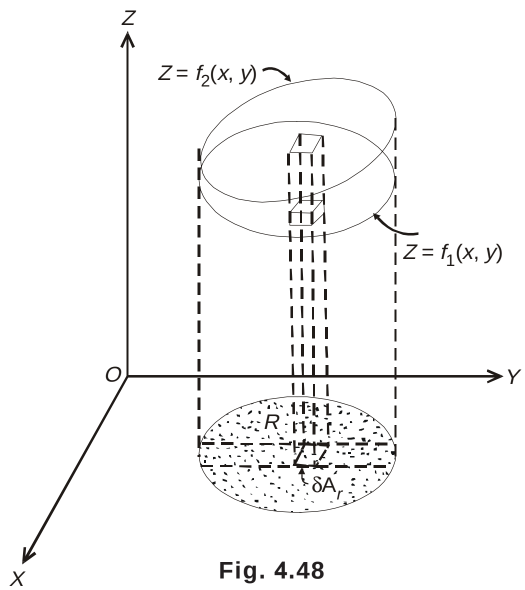

Fig. 4.48

On summing up the elementary cuboids in the same vertical columns first and then taking the sum for all the columns in V, it becomes

$$\sum_r\left[\sum_r F(x_r, y_r, z_r)\delta z\right]\delta A_r \qquad \ldots(3)$$

with the pt. (x_r, y_r, z_r) in the rth cuboid over the element δA_r.

When δA_r and δz tend to zero, we can write (3) as

$$\int_R\left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\,dz\right]dA$$

Note: An ellipsoid, a rectangular parallelopiped and a tetrahedron are regular three dimensional regions.

4.9. EVALUATION OF TRIPLE INTEGRALS

For evaluation purpose, $\iiint_V F(x, y, z)\,dV$ ...(1)

is expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(x, y, z)\, dz\, dy\, dx \qquad \ldots(2)$$

where in the order of integration depends upon the limits.

If the limits z_1 and z_2 be the functions of (x, y); y_1 and y_2 be the functions of x and x_1, x_2 be constant, then

$$I = \int_{x=a}^{x=b}\left(\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\right)dy\right)dx \qquad \ldots(3)$$

which shows that the first $F(x, y, z)$ is integrated with respect to z keeping x and y constant between the limits $z = f_1(x, y)$ to $z = f_2(x, y)$. The resultant which is a function of x, y is integrated with respect to y keeping x constant between the limits $y = f_1(x)$ to $y = f_2(x)$. Finally, the integrand is evaluated with respect to x between the limits $x = a$ to $x = b$.

Note: This order can accordingly be changed depending upon the comfort of integration.

Example 36: Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$.

Solution: On integrating first with respect to z, keeping x and y constants, we get

$$I = \int_0^a\int_0^x \left[e^{(x+y)+z}\right]_0^{(x+y)} dy\, dx, \qquad \text{[Here } (x + y) = a, \text{ (say), like some constant]}$$

$$= \int_0^a\int_0^x \left[e^{(x+y)+(x+y)} - e^{(x+y)+0}\right] dy dx$$

$$= \int_0^a\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right] dy dx$$

$$= \int_0^a \left[\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\right]_0^x dx, \quad \text{(Integrating with respect to } y\text{, keeping } x \text{ constant)}$$

$$= \int_0^a \left[\left(\frac{e^{4x}}{2} - \frac{e^{2x}}{1}\right) - \left(\frac{e^{2x}}{2} - \frac{e^x}{1}\right)\right] dx$$

On integrating with respect to x,

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + \frac{e^x}{1}\right]_0^a$$

$$= \left(\frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^a\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$$

$$\Rightarrow \qquad I = \left(\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}\right).$$

Example 37: Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\, dr\, d\theta\, dz$.

Solution: On integrating with respect to z first keeping r and θ constants, we get

$$I = \int_0^{\pi/2}\int_0^{a\sin\theta} (z)_0^{\frac{a^2-r^2}{a}}\, r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\int_0^{a\sin\theta}\left(a^2 - r^2\right) r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(a^2\frac{r^2}{2} - \frac{r^4}{4}\right)_0^{a\sin\theta} d\theta, \quad \text{(On integrating with respect to r)}$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(\frac{a^2\cdot a^2\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right)d\theta$$

$$= \frac{a^3}{4}\int_0^{\frac{\pi}{2}}\left[2\sin^2\theta - \sin^4\theta\right]d\theta$$

$$= \frac{a^3}{4}\left[2\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right],$$

$$\int_0^{\pi/2}\sin^p x\,dx = \frac{(p-1)\cdot(p-3)\ldots}{(p)\cdot(p-2)\ldots}\times\left(\frac{\pi}{2};\text{only if } p \text{ is even}\right)$$

$$\therefore \qquad I = \frac{a^3}{4}\left[\frac{\pi}{2}\left(1 - \frac{3}{8}\right)\right] = \frac{5\pi a^3}{64}$$

Example 38: Evaluate $\int_1^e\int_0^{\log y}\int_1^{e^x} \log z\,dz\,dy\,dx$.

Solution: $\int_1^e\int_0^{\log y}\left(\int_1^{e^x}\log z\,dz\right)dx\,dy$

[Here $z = f(x, y)$ with $z_1 = 1$ and $z_2 = e^{x + 0y}$

$$= \int_1^e\int_0^{\log y}\left(\int_1^{e^x}\underset{\text{Ist fun.}}{\log z}\cdot\underset{\text{IInd fun.}}{1}\right)dz\,dx\,dy$$

$$= \int_1^e\int_0^{\log y}\left[\log z\times z - \int z\frac{1}{z}dz\right]_1^{e^x} dx\,dy$$

$$= \int_1^e\int_0^{\log y}\left[\left(e^x\log e^x - 1\cdot\log 1\right) - (z)_1^{e^x}\right]dx\,dy$$

$$= \int_1^e \left(\int_0^{\log y} \left[xe^x - \left(e^x - 1 \right) \right] dx \right) dy$$

$$= \int_1^e \int_0^{\log y} \left(\left[(x-1)e^x + 1 \right] dx \right) dy$$

$$= \int_1^e \left[xe^x - 2e^x + x \right]_0^{\log y} dy$$

$$= \int_1^e \left[\underset{\substack{\text{I} \\ \text{function}}}{\left(y+1 \right)} \cdot \underset{\substack{\text{II} \\ \text{function}}}{\log y} + 2\left(1-y \right) \right] dy$$

On integrating by parts,

$$I = \left[\log y \times \left(\frac{y^2}{2} + y \right) \Bigg|_1^e - \int_1^e \frac{1}{y} \cdot \left(\frac{y^2}{2} + y \right) dy + \left(2y - \frac{2y^2}{2} \right) \Bigg|_1^e \right]$$

$$= \left[(\log e)\left(\frac{e^2}{2} + e \right) - \log 1 \cdot \left(\frac{1}{2} + 1 \right) - \int_1^e \left(\frac{y}{2} + 1 \right) dy + \left(2e - e^2 \right) - \left(2 - 1 \right) \right]$$

$$= \left[\frac{e^2}{2} + e - \left(\frac{y^2}{4} + y \right) \Bigg|_1^e + 2e - e^2 - 1 \right]$$

$$= \left[\frac{e^2}{2} + e - \frac{e^2}{4} - e + \frac{1}{4} + 1 + 2e - e^2 - 1 \right]$$

$$= \left[\frac{1}{4}\left(1 + 8e - 3e^2 \right) \right].$$

Example 39: Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} \left(x + y + z \right) dx\, dy\, dz.$

Solution: Integrating first with respect to y, keeping x and z constant,

$$I = \int_{-1}^1 \int_0^z \left(\left[xy + \frac{y^2}{2} + yz \right]_{x-z}^{x+z} \right) dx\, dz$$

$$= \int_{-1}^1 \left(\int_0^2 \left(4zx + 2z^2 \right) dx \right) dz$$

$$= \int_{-1}^1 \left[4z \frac{x^2}{2} + 2 \cdot z^2 \cdot x \right]_0^z dz$$

$$= \int_{-1}^1 \left[4z \cdot \frac{z^2}{2} + 2z^2 \cdot z \right] dz$$

$$= 4\int_{-1}^1 z^3 dz = 4 \frac{z^4}{4} \Bigg|_{-1}^1 = 0$$

ASSIGNMENT 6

Evaluate the following integrals:

(1) $\int_0^1\int_0^2\int_1^2 x^2yz\,dxdydz$ (2) $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\left(x^2+y^2+z^2\right)dxdydz$

(3) $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$ (4) $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx$

4.10 VOLUME AS A DOUBLE INTEGRAL

(Geometrical Interpretation of the Double Integral)

One of the most obvious use of double integral is the determination of volume of solids *viz*. 'volume between two surfaces'.

If $f(x, y)$ is a continuous and single valued function defined over the region R in the xy-plane with $z = f(x, y)$ as the equation of the surface. Let Γ be the closed curve which encloses R. Clearly, the surface R (*viz*. $z = f(x, y)$) is the orthogonal projection of S(*viz* $z = F(x, y)$) in the xy-plane.

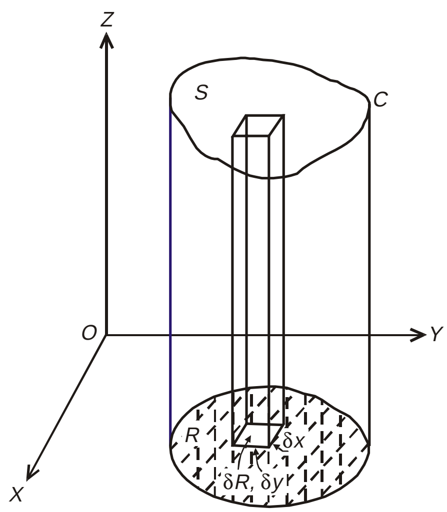

Fig. 4.49

Divided R into elementary rectangles of area $\delta x\delta y$ by drawing lines parallel to the axis of x and y. On each of these rectangles errect prisms having their lengths parallel to the z-axis. The volume of each such prism is $z\delta x\,\delta y$.

(Division of R is performed with the lines $x = x_i$ (i = 1, 2, …, m) and $y = y_j$(j = 1, 2, …, n). Through each line $x = x_i$, pass a plane parallel to yz-plane, and through each line $y = y_j$, pass a plance parallel to xz-plane. The rectangle ΔR_{ij} whose area is $\Delta A_{ij} = \Delta x_i\ \Delta y_j$ will be the base of a rectangle prism of height $f(x_{ij}, h_{ij})$, whose volume is approximately equal to the volume between the surface and the xy-plane $x = x_i - 1$, $x = x_i$; $y = y_i - 1$ $y = y_i$. Then $\sum_{\substack{i=1\\j=1}}^{n} f\left(\xi_{ij},\eta_{ij}\right)\Delta x_i\cdot\Delta y_j$ gives an approximate value for volume V of the prism of the cylinder enclosed between $z = f(x, y)$ and the xy-plane.

The volume V is the limit of the sum of each elementary volume $z\ \delta x\delta y$.

$$\therefore \qquad V = \underset{\substack{\delta x\to 0\\ \delta y\to 0}}{Lt}\sum\sum z\,\delta x\,\delta y = \iint_R z\,dx\,dy = \iint_R f\left(x,y\right)dA$$

Note: In cyllidrical co-ordinates, the equation of the surface becomes $z = f(r, \theta)$, elementary area $dA = r\ dr\ d\theta$ and volume $= \iint_R f\left(r,\theta\right)r\,dr\,d\theta$

Problems on Volume of a Solid with the Help of Double Integral

Example 40: Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes. **[Burdwan, 2003]**

Solution: Given, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \;\Rightarrow\; z=f(x,y)=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ …(1)

If $f(x, y)$ is a continuous and single valued function over the region R (see Fig. 5. 50) in the xy plane, then $z = f(x, y)$ is the equation of the surface. Let C be the closed curve that is the boundary of R. Using R as a base, construct a cylinder having elements parallel to the z-axis. This cylinder intersects $z = f(x, y)$ in a curve Γ, whose projection on the xy-plane is C.

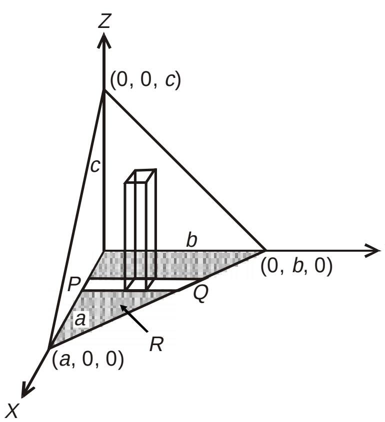

Fig. 4.50

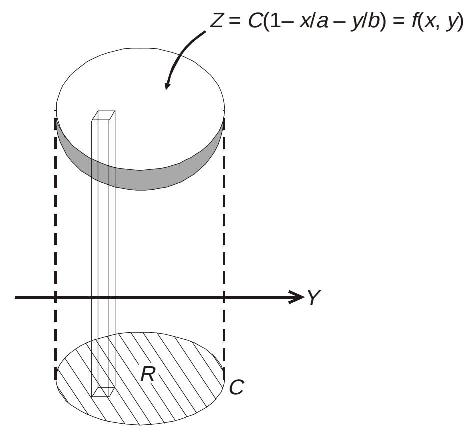

Fig. 4.51

The equation of the surface under which the region whose volume is required, may be written in the form (1) *i.e.*, $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.

Hence the volume of the region $=\iint_R a\,dA=\iint_R c\left(1-\frac{x}{a}-\frac{y}{b}\right)dx\,dy$

The equation of the inter-section of the given surface with xy-plane is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots(2)$$

If the prisms are summed first in the y-direction they will be summed from $y = 0$ to the line $y=b\left(1-\frac{x}{a}\right)$

Therefore, $$V=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a\left(y-\frac{xy}{a}-\frac{y^2}{2b}\right)\Bigg|_0^{b(1-x/a)}dx$$

$$= c\int_0^a b\left(\frac{1}{2} - \frac{x}{a} + \frac{x^2}{2a^2}\right)dx$$

$$= cb\left[\frac{x}{2} - \frac{x^2}{2a} + \frac{x^3}{6a^2}\right]_0^a$$

$$= bc\left[\frac{a}{2} - \frac{a^2}{2a} + \frac{a^3}{6a^2}\right] = \frac{abc}{6}.$$

Example 41: Prove that the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{128a^2}{15}$.

Solution: Let V be required volume which is enclosed by the cylinder $x^2 + y^2 = 2ax$ and the paraboloid $z^2 = 2ax$.

Only half of the volume is shown in Fig 5.52.

Now, it is evident from that $z = \sqrt{2ax}$ is to be evaluated over the circle $x^2 + y^2 = 2ax$(with centre at $(a, 0)$ and radius a.

Here y varies from $-\sqrt{2ax - x^2}$ to $\sqrt{2ax - x^2}$ on the circle $x^2 + y^2 = 2ax$ and finally x varies from $x = 0$ to $x = 2a$

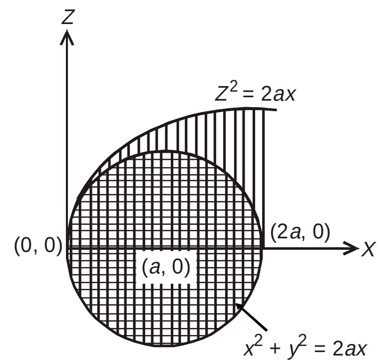

Fig. 4.52

$$\therefore \qquad V = 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}[z]\,dx\,dy \quad \text{as } z = f(x, y)$$

$$= 2\int_0^{2a}\left(2\cdot\int_0^{\sqrt{2ax-x^2}}\sqrt{2ax}\right)dy\,dx$$

$$= 4\int_0^{2a}\sqrt{2ax}\left(\int_0^{\sqrt{2ax-x^2}}dy\right)dx$$

$$= 4\int_0^{2a}\sqrt{2ax}\,|y|_0^{\sqrt{2ax-x^2}}\,dx = 4\int_0^{2a}\sqrt{2ax}\sqrt{2ax - x^2}\,dx$$

$$= 4\sqrt{2a}\int_0^{2a}x\sqrt{2a - x}\,dx$$

Let $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$. Further, for $x = 0,\ \theta = 0$

$x = 2a, \theta = \frac{\pi}{2}.$

$$\therefore \qquad V = 4\sqrt{2a}\int_0^{\pi/2}2a\sin^2\theta\sqrt{2a}\cos\theta\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$= 64\,a^3 \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots} \cdot 1, \;\; p = 3, \;\; q = 2$$

$$= 64\,a^3 \frac{(3-1)1}{5 \cdot 3} = \frac{128\,a^3}{15} \cdot$$

Problems based on Volume as a Double Integral in Cylindrical Coordinates

Example 42: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution: In cartesian co-ordinates, the section of the given hyperboloid $x^2 + y^2 - z^2 = 1$ in the *xy* plane ($z = 0$) is the circle $x^2 + y^2 = 1$, where as at the top and at the bottom end (along the *z*-axis *i.e.*, $z = \pm\sqrt{3}$) it shares common boundary with the circle $x^2 + y^2 = 4$ (Fig. 5.53 and 5.54).

Here we need to calculate the volume bounded by the two bodies (*i.e.*, the volume of shaded portion of the geometry).

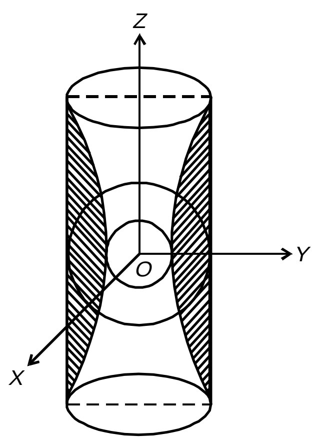

Fig. 4.53

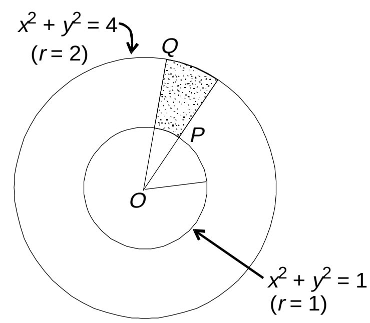

Fig. 4.54

(Best example of this geometry is a *solid damroo* in a *concentric long hollow drum*.)

In cylindrical polar coordinates, we see that here r varies from $r = 1$ to $r = 2$ and θ varies from 0 to 2π.

$$\therefore \qquad V = 2\left[\iint z\,dx\,dy\right] = 2\left[\iint f(r,\theta)\,r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{2\pi}\int_1^2 \sqrt{r^2-1}\,r\,dr\,d\theta\right] \qquad (\because \;\; x^2 + y^2 - z^2 - 1 \Rightarrow z = \sqrt{x^2+y^2-1}\,)$$

$$= 2\int_0^{2\pi}\left(\int_1^2 \frac{1}{3}\,d\left(r^2-1\right)^{\frac{3}{2}}\right)d\theta$$

$$= 2\int_0^{2\pi} \left. \frac{(r^2 - 1)^{\frac{3}{2}}}{3} \right|_1^2 d\theta$$

$$= 2\sqrt{3}\int_0^{2\pi} d\theta = 4\pi\sqrt{3}.$$

Example 43: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. **[KUK, 2000; MDU, 2002; Cochin, 2005; SVTU, 2007]**

Solution: From Fig. 5.55, it is very clear that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy-plane.

To cover the shaded portion, x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ and y varies from – 2 to 2.

Hence the desired volume,

$$V = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2}(4-y)\left(\int_0^{\sqrt{4-y^2}} dx\right)dy$$

$$= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2}\left[4\sqrt{4-y^2} - y\sqrt{4-y^2}\right]dy$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy - 0$$

(The second term vanishes as the integrand is an odd function)

$$= 8\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2} = 16\pi.$$

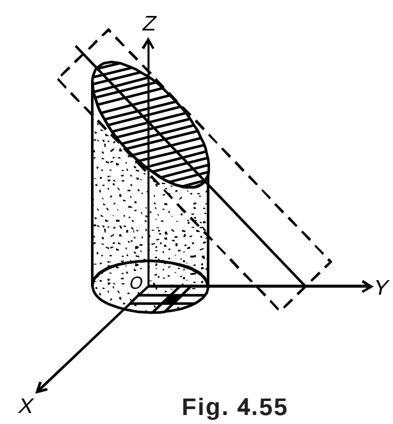

Fig. 4.55

ASSIGNMENT 7

1. Find the volume enclosed by the coordinate planes and the portion of the plane $lx + my + nz = 1$ lying in the first quadrant.
2. Obtain the volume bounded by the surface $z = c\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$ and the quadrant of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   [**Hint:** Use elliptic polar coordinates $x = ar\cos\theta$, $y = br\sin\theta$]

4.11 VOLUME AS A TRIPLE INTEGRAL

Divide the given solid by planes parallel to the coordinate plane into rectangular parallelopiped of elementary volume $\delta x \delta y \delta z$.

Then the total volume V is the limit of the sum of all elementary volume i.e.,

$$V = \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{Lt} \sum\sum\sum \delta x\, \delta y\, \delta z = \iiint dx\, dy\, dx$$

Problems based on Volume as a Triple Integral in cartesian Coordinate System

Example 44: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution: The sections of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ are the circles $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in xy and xz plane respectively.

Here in the picture, one-eighth part of the required volume (covered in the 1st octant) is shown.

Clearly, in the common region, z varies from 0 to $\sqrt{a^2 - x^2}$ *i.e.*, $\sqrt{a^2 - 1x^2 - 0y^2}$, and x and y vary on the circle $x^2 + y^2 = a^2$.

The required volume

$$\therefore \quad V = 8\int_0^a \int_{y_1=0}^{y_2=\sqrt{a^2-x^2}} \int_{z_1=0}^{z_2=\sqrt{a^2-x^2-0y^2}} dz\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(z\Big|_0^{\sqrt{a^2-x^2}} \right) dy\, dx$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy \right) dx$$

$$= 8\int_0^a \left(\sqrt{a^2 - x^2} \right) \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx$$

$$= 8\int_0^a \sqrt{a^2 - x^2} \left(\sqrt{a^2 - x^2} - 0 \right) dx$$

$$= 8\int_0^a \left(a^2 - x^2 \right) dx = 8\left[\left(a^2 x - \frac{x^3}{3} \right) \Big|_0^a \right]$$

$$= 8\left(a^3 - \frac{a^3}{3} \right) = \frac{16a^3}{3}.$$

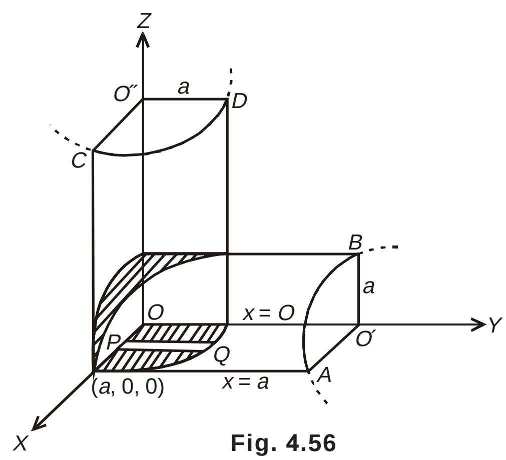

Fig. 4.56

Example 45: Find the volume bounded by the *xy* plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.

Solution: Let $V(x, y, z)$ be the desired volume enclosed laterally by the cylinder $x^2 + y^2 = 1$ (in the *xy*-plane) and on the top, by the plane $x + y + z = 3$ (= a say).

Clearly, the limits of z are from 0 (on the *xy*-plane) to $z = (3 - x - y)$ and x and y vary on the circle $x^2 + y^2 = 1$

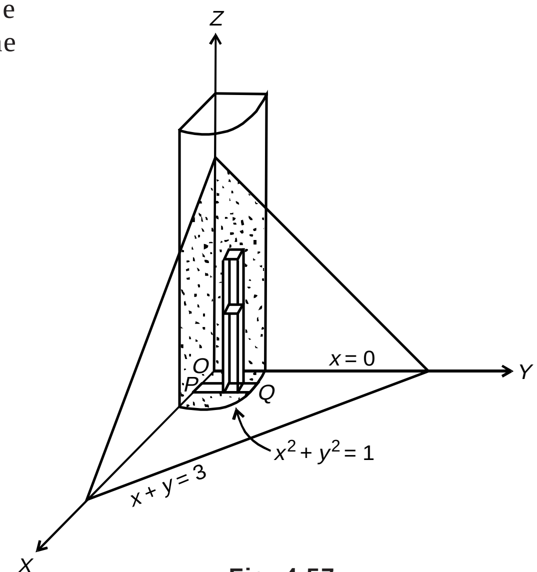

Fig. 4.57

$$\therefore \qquad V(x,y,z) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{1\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(z\Big|_0^{(3-x-y)}\right)dy\,dx$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3-x-y)\,dy\right)dx$$

$$= \int_{-1}^{1}\left[3y - xy - \frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$\Rightarrow \qquad I = \int_{-1}^{1}\left(6\times\sqrt{1-x^2} - 2x\sqrt{1-x^2}\right)dx$$

On taking $x = \sin\theta$, we get $dx = d\theta$; $\left.\begin{array}{l}\text{For } x = -1, \theta = -\dfrac{\pi}{2}\\ \text{For } x = 1, \quad \theta = \dfrac{\pi}{2}\end{array}\right\}$

Thus,

$$V = \int_{-\pi/2}^{\pi/2}\left(6\sqrt{1-\sin^2\theta} - 2\sin\theta\sqrt{1-\sin^2\theta}\right)\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(6\cos^2\theta - 2\sin\theta\cos^2\theta\right)d\theta$$

$$= 6\times 2\underbrace{\int_{0}^{\pi/2}\cos^2\theta\,d\theta}_{\text{Ist}} - 2\underbrace{\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta}_{\text{IInd}}$$

$$= 12\frac{(2-1)}{2}\cdot\frac{\pi}{2} + 2\frac{\cos^3\theta}{3}\Bigg|_{-\pi/2}^{\pi/2} = 3\pi + \frac{2}{3}\times 0 = 3\pi$$

Using $\displaystyle\int_0^{\pi/2}\cos^p\theta\,d\theta = \frac{(p-1)(p-3)\ldots}{p(p-2)\ldots}\times\left(\frac{\pi}{2}, \text{only if } p \text{ is even}\right)$ and

$$\int f'(x) f^n(x) dx = \frac{f^{n+1}(x)}{n+1}$$ for Ist and IInd integral respectively

Example 46: Find the volume bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Considering the symmetry, the desired volume is 8 times the volume of the ellipsoid into the positive octant.

The ellipsoid cuts the XOY plane in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $z = 0$.

Therefore, the required volume lies between the ellipsoid

$$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

and the plane XOY (*i.e.*, $z = 0$) and is bounded on the sides by the planes $x = 0$ and $y = 0$

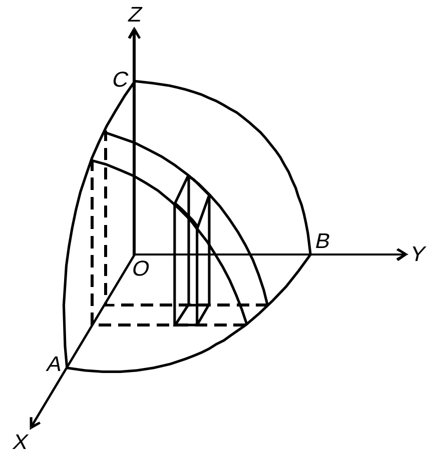

Fig. 4.58

Hence,
$$V = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a \left(\int_0^{\alpha} \frac{c}{b}\sqrt{\alpha^2-y^2}\,dy\right)dx \qquad \left(\text{taking } \sqrt{\left(1-\frac{x^2}{a^2}\right)} = \frac{\alpha}{b}\right)$$

$$V = 8\frac{c}{b}\int_0^a \left[\frac{y\sqrt{\alpha^2-y^2}}{2}+\frac{\alpha^2}{2}\sin^{-1}\frac{y}{\alpha}\right]_0^{\alpha} dx$$

$$\left(\text{Using formula } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\tan^{-1}\frac{x}{a}\right)$$

$$= 8\frac{c}{b}\int_0^a \left[0+\frac{\alpha^2}{2}\sin^{-1}1\right]dx$$

$$= \frac{4c}{b}\int_0^a \frac{\pi}{2}\alpha^2 dx = \frac{2\pi c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx, \quad \alpha = b\sqrt{1-\frac{x^2}{b^2}}$$

$$= 2\pi bc\left[x-\frac{1}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{4}{3}\pi abc.$$

Example 47: Evaluate the integral $\iiint \frac{dxdydz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ taken throughout the volume of the sphere.

Solution: Here for the given sphere $x^2 + y^2 + z^2 = a^2$, any of the three variables x, y, z can be expressed in term of the other two, say $z = \pm\sqrt{a^2 - x^2 - y^2}$.

In the xy-plane, the projection of the sphere is the circle $x^2 + y^2 = a^2$.

Thus,
$$I = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz}{\sqrt{\alpha^2 - z^2}} \right) dy \right) dx, \ \alpha^2 = (a^2 - x^2 - y^2)$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}\frac{z}{\alpha} \right)_0^{\alpha} dy \right) dx$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}1 - \sin^{-1}0 \right) dy \right) dx$$

$$= 8\frac{\pi}{2}\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx = 4\pi\int_0^a \left(y\Big|_0^{\sqrt{a^2-x^2}} \right) dx$$

$$= 4\pi\int_0^a \sqrt{a^2 - x^2}\,dx = 4\pi\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= 4\pi\left[0 + \frac{a^2}{2}\frac{\pi}{2} \right] \quad I = \pi^2 a^2.$$

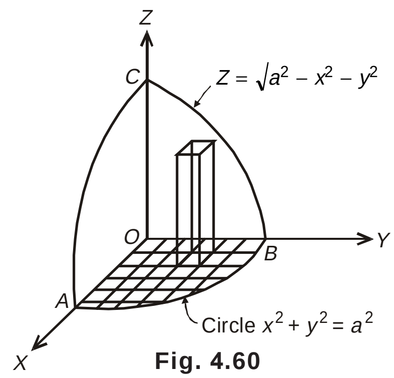

Fig. 4.60

Example 48: Evaluate $\iiint (x + y + z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution: The integration is over the region R(shaded portion) bounded by the plane $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = 1$.

The area OAB, in xy plane is bounded by the lines $x + y = 1$, $x = 0$, $y = 0$

Hence for any pt. (x, y) within this triangle, z goes from xy plane to plane ABC (*viz*. the surface of the tetrahedron) or in other words, z changes from $z = 0$ to $z = 1 - x - y$. Likewise in plane xy, y as a function x varies from $y = 0$ to $y = 1 - x$ and finally x varies from 0 to 1.

whence,
$$I = \underset{(over\,R)}{\iiint} (x + y + z)\,dx\,dy\,dz$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} (x + y + z)\,dz \right) dy \right) dx$$

$$= \int_0^a \int_0^{1-x} \left((x+y)z + \frac{z^2}{2} \right)\Bigg|_0^{1-x-y} dy\,dx$$

$$= \int_0^a \int_0^{1-x} \left[(x+y)(1-x-y) + \frac{(1-x-y)^2}{2} \right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)(1+x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}\left[1-(x+y)^2\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3} \right]_0^{1-x} dx,$$

$$= \frac{1}{2}\int_0^1 \left[(1-x) - \left(\frac{1}{3} - \frac{x^3}{3} \right) \right] dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12} \right] = \frac{1}{8}$$

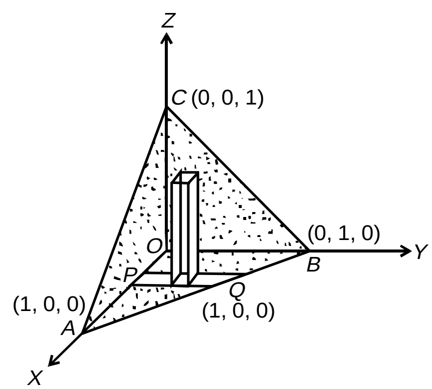

Fig. 4.61

ASSIGNMENT 8

1. Find the volume of the tetrahedron bounded by co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, by using triple integration
2. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

5.12. VOLUMES OF SOLIDS OF REVOLUTION AS A DOUBLE INTEGRAL

Let P(x, y) be any point in a region R enclosing an elementary area $dx\ dy$ around it. This elementary area on revolution about x-axis form a ring of volume,

$$\delta V = \pi[(y + \delta y)^2 - y^2]\ \delta x = 2\pi y \delta x \delta y \quad \ldots(1)$$

Hence the total volume of the solid formed by revolution of this region R about x-axis is,

$$V = \iint_R 2\pi y\,dx\,dy \quad \ldots(2)$$

Similarly, if the same region is revolved about y-axis, then the required volume becomes

$$V = \iint_R 2\pi x\,dx\,dy \quad \ldots(3)$$

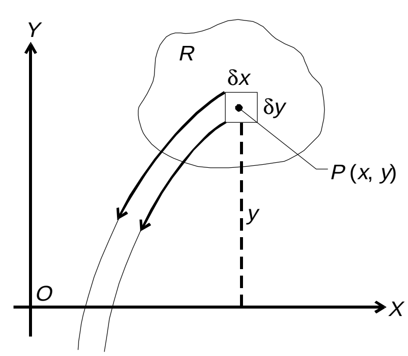

Fig. 4.62

Expressions for above volume in polar coordinates **about the initial** line and **about the pole** are $\iint_R 2\pi r^2 \sin\theta\, dr\, d\theta$ and $\iint_R 2\pi r^2 \cos\theta\, dr\, d\theta$ respectively.

Example 49: Find by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about y-axis.

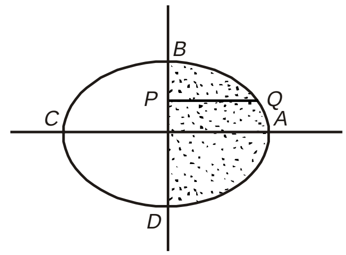

Fig. 4.63

Solution: As the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is symmetrical about the y-axis, the volume generated by the left and the right halves overlap.

Hence we shall consider the revolution of the right-half ABD for which x-varies from 0 to $a\sqrt{1-\frac{y^2}{b^2}}$ and y-varies from $-b$ to b.

$$\therefore \quad V = \int_{-b}^{b}\int_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} 2\pi x\, dx\, dy$$

$$= 2\pi\int_{-b}^{b}\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy = \frac{\pi a^2}{b^2}\int_{-b}^{b}\left(b^2-y^2\right)dy$$

$$= 2\pi\frac{a^2}{b^2}\int_0^b\left(b^2-y^2\right)dy = \frac{2\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_0^b$$

$$= \frac{4}{3}\pi a^2 b.$$

Example 50: The area bounded by the parabola $y^2 = 4x$ and the straight lines $x = 1$ and $y = 0$, in the first quadrant is revolved about the line $y = 2$. Find by double integration the volume of the solid generated.

Solution: Draw the standard parabola $y^2 = 4x$ to which the straight line $y = 2$ meets in the point $P(1, 2)$, Fig. 5.64.

Now the dotted portion *i.e.*, the area enclosed by parabola, the line $x = 1$ and $y = 0$ is revolved about the line $y = 2$.

$\therefore$ The required volume,

$$V = \int_0^1\int_0^{2\sqrt{x}} 2\pi(2-y)\,dx\,dy$$

$$= 2\pi\int_0^1\left[2y-\frac{y^2}{2}\right]_0^{2\sqrt{x}} dx = 2\pi\int_0^1\left(4\sqrt{x}-2x\right)dx$$

$$= 2\pi\left[\frac{8}{3}x^{3/2}-x^2\right]_0^1 = \frac{10\pi}{3}$$

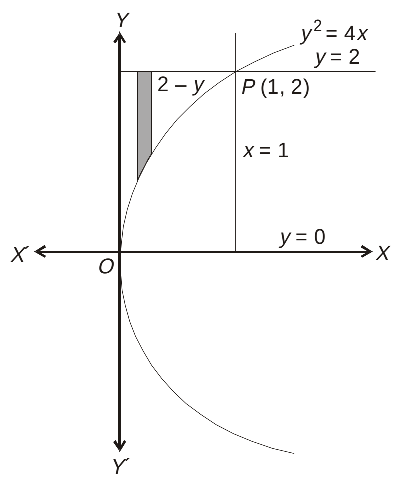

Fig. 4.64

Example 51: Calculate by double integration, the volume generated by the revolution of the cardiod $r = a(1 - \cos\theta)$ about its axis.

Soluton: On considering the upper half of the cardiod, because due to symmetry the lower half generates the same volume.

$$\therefore \quad V = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

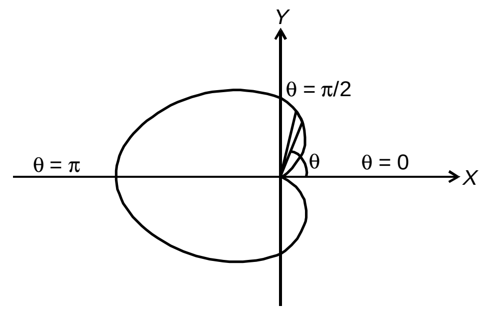

Fig. 4.65

Example 52: By using double integral, show that volume generated by revolution of cardiod $r = a(1 + \cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.

Solution: The required volume

$$= \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi \int_0^{\pi} \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} \sin\theta\, d\theta$$

$$= 2\pi \int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3} \left[-\frac{(1+\cos\theta)^4}{4} \right]_0^{\pi}$$

$$= -\frac{2\pi a^3}{3} \left[0 - \frac{2^4}{4} \right] = \frac{8\pi a^3}{3}.$$

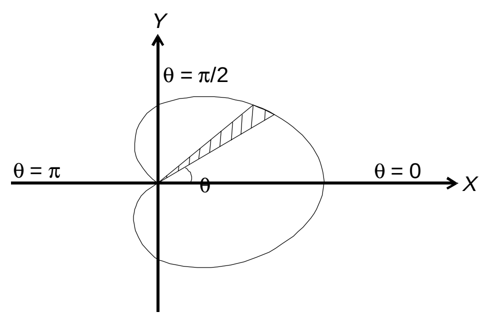

Fig. 4.66

ASSIGNMENT 9

1. Find by double integration the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis.
2. Find the volume generated by revolving a quadrant of the circle $x^2 + y^2 = a^2$, about its diameter.
3. Find the volume generated by the revolution of the curve $y^2(2a - x) = x^3$, about its asymptote through four right angles.
4. Find the volume of the solid obtained by the revolution of the leminiscate $r^2 = a^2 \cos 2\theta$ about the initial line.

4.13. CHANGE OF VARIABLE IN TRIPLE INTEGRAL

For transforming elementary area or the volume from one sets of coordinate to another, the necessary role of 'Jacobian' or 'functional determinant' comes into picture.

(*a*) Triple Integral Under General Transformation

Here $$\iiint_{R(x,y,z)} f(x,y,z)\,dx\,dy\,dz = \iiint_{R'(u,v,w)} F(u,v,w)\,|J|\,du\,dv\,dw; \text{ where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)} (\neq 0) \quad \ldots(1)$$

Since in the case of three variables $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ be continuous together with their first partial derivatives, the Jacobian of u, v, w with respect to x, y, z is defined by

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix}$$

(*b*) Triple Integral in Cylindrical Coordinates

Here $$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} F(r,\theta,z)|J|\,dr\,d\theta\,dz, \text{ where } |J| = r$$

The position of a point P in space in cylindrical coordinates is determined by the three numbers r, θ, z where r and θ are polar co-ordinates of the projection of the point P on the xy-plane and z is the z coordinate of P *i.e.*, distance of the point (P) from the xy-plane with the plus sign if the point (P) lies above the xy-plane, and minus sign if below the xy-plane (Fig. 5.67).

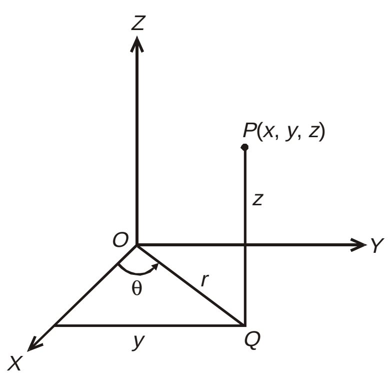

Fig. 4.67

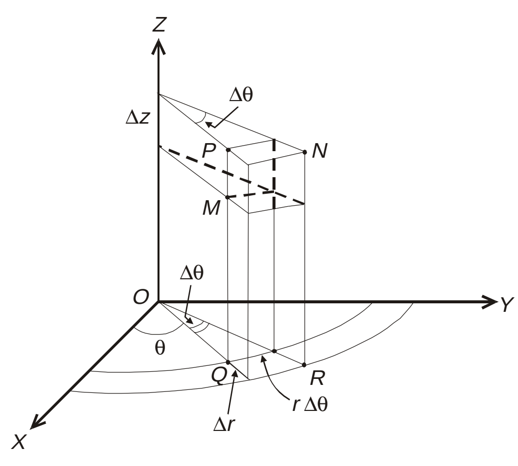

Fig. 4.68

In this case, divide the given three dimensional region R' (r, θ, z) into elementary volumes by coordinate surfaces $r = r_i$, $\theta = \theta_j$, $z = z_k$ (viz. half plane adjoining z-axis, circular cylinder axis coincides with Z-zxis, planes perpenducular to z-axis). The curvilinear 'prism' shown in Fig. 5. 68 is a volume element of which elementary base area is $r\,\Delta r\Delta\theta$ and height Δz, so that $\Delta v = r\,\Delta r\,\Delta\theta\,\Delta z$.

Here θ is the angle between OQ and the positive x-axis, r is the distance OQ and z is the distance QP. From the Fig. 5.62, it is evident that

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \text{ and so that,}$$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \qquad \ldots(2)$$

Hence, the triple integral of the function F(r, θ, z) over R′ becomes

$$V = \iiint_{R'(r,\theta,z)} F(r,\theta,z)\,r\,dr\,d\theta\,dz \qquad \ldots(3)$$

(*c*) Triple Integral in Spherical Polar Coordinates

Here $V = \iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R F(r,\theta,\phi)\,|J|\,dr\,d\theta\,d\phi$, where $|J| = r^2\sin\theta$

The position of a point P in space in spherical coordinates is determined by the three variables r, θ, ϕ where r is the distance of the point (P) from the origin and so called radius vector, θ is the angle between the radius vector on the xy-plane and the x-axis to count from this axis in a positive sense viz. counter-clockwise.

For any point in space in spherical coordinates, we have

$$0 \le r \le \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Divide the region 'R' into elementary volumes ΔV by coordinate surfaces, r = constant (sphere), θ = constant (conic surfaces with vertices at the origin), ϕ = constant (half planes passing through the Z-axis).

To within infinitesimal of higher order, the volume element Δv may be considered a parallelopiped with edges of length Δr, $r\,\Delta\theta$, $r\sin\theta\,\Delta\phi$. Then the volume element becomes $\Delta V = r^2\sin\theta\,\Delta r\,\Delta\theta\,\Delta\phi$.

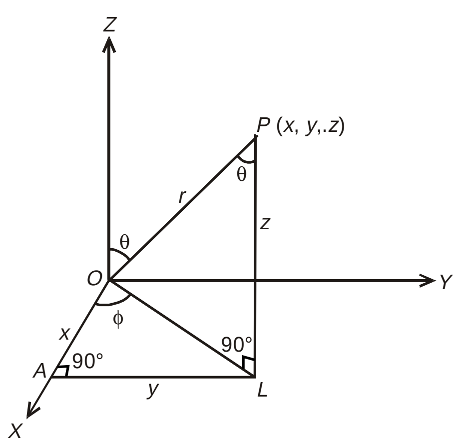

Fig. 4.69

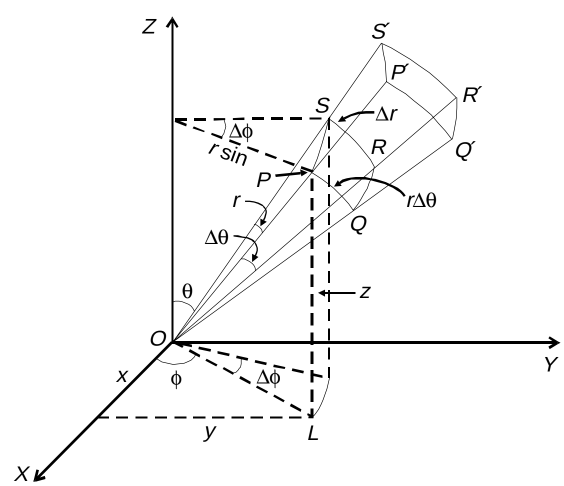

Fig. 4.70

For calculation purpose, it is evident from the Fig. 5.69 that in triangles, *OAL* and *OPL*,

$$x = \text{OL} \cos\phi = \text{OP} \cos(90 - \theta) \,.\, \cos\phi = r\sin\theta \cos\phi,$$

$$y = \text{OL} \sin\phi = \text{OP} \sin\theta \,.\, \sin\phi = r\sin\theta \sin\phi,$$

$$z = r\cos\theta.$$

Thus,
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} = r^2 \sin\theta$$

Problems Volume as a Triple Integral in Cylindrical Co-ordinates

Example 53: Find the volume intercepted between the paraboloid $x^2 + y^2 = 2az$ and the cylinder $x^2 + y^2 - 2ax = 0$.

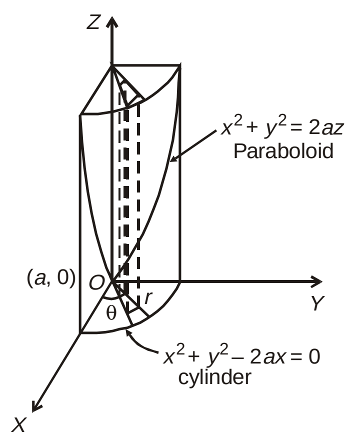

Fig. 4.71

Solution: Let V be required volume of the cylinder $x^2 + y^2 - 2ax = 0$ intercepted by the paraboloid $x^2 + y^2 = 2az$.

Transforming the given system of equations to polar-cylindrical co-ordinates.

Let $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{matrix}\right\}$ sothat $V(x, y, z) = V(r, \theta, z)$

By above substitution the equation of the paraboloid becomes $r^2 = 2az \Rightarrow z = \dfrac{r^2}{2a}$ and the cylinder $x^2 + y^2 = 2ax$ gives $r^2 - 2ar\cos\theta = 0 \Rightarrow r(r - 2a\cos\theta) = 0$ with $r = 0$ and $r = 2a\cos\theta$.

Thus, it is clear from the Fig. 5.71 that z varies from 0 to $\dfrac{r^2}{2a}$ and r as a function of θ varies from 0 to $2a\cos\theta$ with θ as limits 0 to 2π. Geometry clearly shows the volume covered under the +ve octant only, i.e. $\dfrac{1}{4}$th of the full volume.

$$\underset{(x,y,z)}{V} = \underset{(r,\theta,z)}{V'} = 4\int_{0}^{\theta=\pi/2}\int_{r=0}^{r=2a\cos\theta}\int_{z=0}^{z=r^2/2a} r\,dz\,dr\,d\theta, \text{ as } |J| = r$$

$$= 4\int_0^{\pi/2}\left(\int_0^{2a\cos\theta} r\left[z\right]_0^{r^2/2a} r\,dr\right)d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^{2a\cos\theta} \frac{r^3}{2a}\,dr\right)d\theta$$

$$= 4\frac{1}{2a}\int_0^{\pi/2} \left.\frac{r^4}{4}\right|_0^{2a\cos\theta} d\theta$$

$$= 4\frac{1}{2a}\int_0^{\pi/2}\frac{2^4a^4}{4}\cos^4\theta\,d\theta$$

$$= 2^3a^3\frac{(4-1)(4-3)}{4\times 2}\frac{\pi}{2}$$

$$= \frac{3\pi a^3}{2}.$$

Example 54: Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = b^2$. Also find the integral in case when $a = 2$ and $b = 2$.

Solution: On using the cylindrical polar co-ordinates (r, θ, z) with $x = r\cos\theta$, $y = r\sin\theta$, so that the equations of the cylinder and that of the paraboloid are $r = b$ and $z = \frac{r^2}{a}$ respectively.

See Fig. 5.72, only one-fourth of the common volume is shown.

Hence in the common region, z varies from $z = 0$ to $z = \frac{r^2}{a}$ and r and θ varies on the circle from 0 to b and 0 to $\frac{\pi}{2}$ respectively.

∴ The desired volume

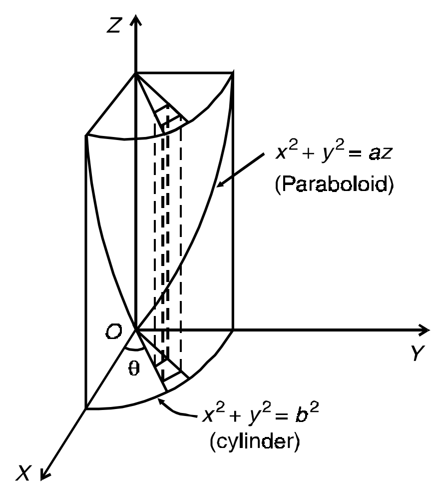

Fig. 4.72

$$V = 4\int_0^{\pi/2}\int_0^{b}\int_0^{r^2/a} r\,dr\,d\theta\,dz$$

$$= 4\int_0^{\pi/2}\left(\int_0^{b} r\,dr\left(\int_0^{r^2/a} dz\right)\right)d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^{b} r\left(\frac{r^2}{a}\right)dr\right)d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\left(\frac{r^4}{4}\Big|_0^b\right)d\theta$$

$$= \frac{4}{a}\times\frac{b^4}{4}\theta\Big|_0^{\pi/2} = \frac{\pi b^2}{2a}$$

As a particular case, when $a = 2$, $b = 2$, then

$$V = \frac{\pi(2)^4}{2\times 2} = 4\pi$$

Problmes on Volume in Polar Spherical Co-ordinates

Example 55: Find the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2$

OR

Find the volume cut by the cone $x^2 + y^2 = z^2$ from the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: For the given sphere, $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2$, the centre of the sphere is (0, 0, 0) and the vertex of the cone is origin. Therefore, the volume common to the two bodies is symmetrical about the plane $z = 0$, i.e. the required volume, $V = 2\iiint dxdydz$

In spherical co-ordinates, we have $\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}; J = r^2\sin\theta$

Thus, $x^2 + y^2 + z^2 = a^2$ becomes $r^2 = a^2$ *i.e.*, $r = a$

and $x^2 + y^2 = z^2$ becomes $r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta$

i.e., $\sin^2\theta = \cos^2\theta$ *i.e.* $\theta = \pi/4$.

Clearly, the volume shown in the figure (Fig. 5.73) is one-fourth, i.e. in first quadrant only and, in the common region,

$$\left.\begin{aligned} &r \text{ varies from 0 to } a, \\ &\theta \text{ varies from 0 to } \frac{\pi}{4}, \\ &\phi \text{ varies from 0 to } \frac{\pi}{2} \end{aligned}\right\}$$

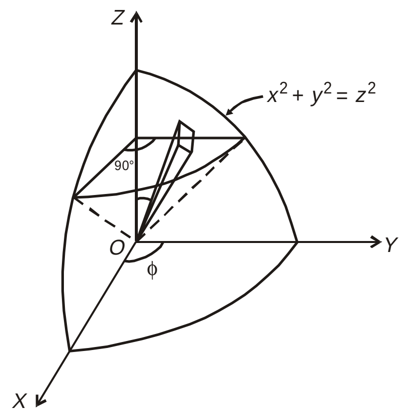

Fig. 4.73

Hence the required volume,

$$V = 2\left[4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^a r^2\sin\theta\, dr\, d\theta\, d\phi\right]$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/4}\left(\int_0^a r^2 dr\right)\sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/4}\left(\frac{r^3}{3}\right)_0^a \sin\theta\, d\theta\, d\phi$$

$$= \frac{8}{3}a^3\int_0^{\pi/2}[-\cos\theta]_0^{\pi/4}\, d\phi$$

$$= \frac{8}{3}a^3\left(1 - \frac{1}{\sqrt{2}}\right)\int_0^{\pi/2} d\phi$$

$$= \frac{4\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

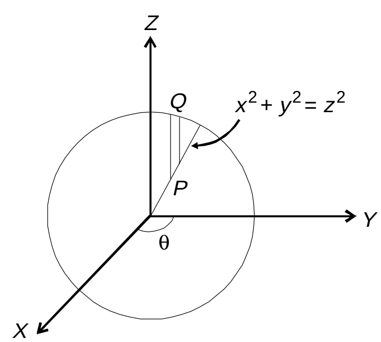

Fig. 4.74

Alternately: In polar-cylindrical co-ordinates, intersection of the two curves $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$ results in $z^2 + z^2 = a^2$ or $z^2 = \frac{a^2}{2}$.

Further, $x^2 + y^2 = a^2 - z^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2} \Rightarrow r = \frac{a}{\sqrt{2}}$, i.e. r varies from 0 to $\frac{a}{\sqrt{2}}$

Hence, $$V = 2\int_0^{2\pi}\int_0^{a/\sqrt{2}}\left(\sqrt{a^2 - r^2} - r\right) r\, dr\, d\theta$$

| ∵ P lies on the cone whereas Q lies on the sphere as a function of (r, θ)

$$= 2\int_0^{a/\sqrt{2}} \left(r\sqrt{a^2 - r^2} - r^2\right)\left(\int_0^{2\pi} d\theta\right) dr$$

$$= 4\pi\left[-\frac{1}{3}\left(a^2 - r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^{\frac{a}{\sqrt{2}}} \quad \left[\text{since } r\left(a^2 - r^2\right)^{\frac{1}{2}} = \frac{-1}{3}\left(-3r\left(a^2 - r^2\right)^{\frac{1}{2}}\right) = \frac{-1}{3} d\left(a^2 - r^2\right)^{\frac{3}{2}}\right]$$

$$= 4\pi\left[-\frac{1}{3}\frac{a^3}{2\sqrt{2}} - \frac{1}{3}\frac{a^3}{2\sqrt{2}} + \frac{a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3}\left[1 - \frac{1}{\sqrt{2}}\right]$$

Example 56: By changing to shperical polar co-ordinate system, prove that $\displaystyle\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz = \frac{\pi}{4} abc$ **where** $V = \left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$

Solution: Taking $\left.\begin{aligned}\frac{x}{a} &= u,\\ \frac{y}{b} &= v,\\ \frac{z}{c} &= w\end{aligned}\right\}$, so that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \quad \Rightarrow \quad u^2 + v^2 + w^2 \le 1$

Now transformation co-efficient,

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\therefore \qquad V = \iiint_{V(x,y,z)} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$$

$$= \iiint_{V'(u,v,w)} \sqrt{1 - u^2 - v^2 - w^2}\,(abc)\, du\, dv\, dw$$

To transform to polar spherical co-rodinate system, let $\left.\begin{aligned} u &= r\sin\theta\cos\phi,\\ v &= r\sin\theta\sin\phi,\\ w &= r\cos\theta \end{aligned}\right\}$

Then $V'_{(u,\, v,\, w)} = \{(u, v, w): u^2 + v^2 + w^2 \le 1,\ u \ge 0,\ v \ge 0,\ w \ge 0\}$ reduces to

$V''_{(r,\, \theta,\, \phi)} = \{r^2 \le 1 \quad i.e., \quad 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi\}$

$$\therefore \qquad = \iiint_{V'(u,v,w)} \sqrt{1 - u^2 - v^2 - w^2}\, abc\, du\, dv\, dw$$

$$= \iiint_{V''(r,\theta,\phi)} abc\sqrt{1-r^2}\,|J|\,dr\,d\theta\,d\phi \qquad \text{where } |J| = r^2\sin\theta$$

$$\Rightarrow \qquad V''_{(r,\theta,\phi)} = abc\int_{\phi=0}^{\phi=2\pi}\left(\int_0^{\pi}\left(\int_0^1 \sqrt{1-r^2}\, r^2 dr\right)\sin\theta\, d\theta\right)d\phi$$

Now put $r = \sin t$ so that $dr = \cos t\, dt$ and for $\left.\begin{matrix} r = 0,\ t = 0, \\ r = 1,\ t = \dfrac{\pi}{2} \end{matrix}\right\}$

$$\therefore \qquad V''_{(r,\theta,\phi)} = abc \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{\pi/2} \cos t \sin^2 t \cos t\, dt\right)\sin\theta\, d\theta\right)d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\left[\frac{(2-1)\cdot(2-1)}{(2+2)(4-2)}\frac{\pi}{2}\right]\sin\theta\, d\theta\right)d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\left(\frac{1}{4}\frac{1}{2}\frac{\pi}{2}\right)\sin\theta\, d\theta\right)d\phi$$

$$= \frac{\pi abc}{16}\int_0^{2\pi}[-\cos\theta]_0^{\pi}\, d\phi$$

$$= \frac{\pi abc}{16}\int_0^{2\pi} 2\, d\phi = \frac{\pi abc}{8}\int_0^{2\pi} d\phi = \frac{\pi^2 abc}{4}.$$

Example 57: By change of variable in polar co-ordinate, prove that

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}.$$

OR

Evaluate the integral being extended to octant of the sphere $x^2 + y^2 + z^2 = 1$.

OR

Evaluate above integral by changing to polar spherical co-ordinate system.

Solution: Simple Evaluation:

$$I = \int_0^1 dx\int_0^{\sqrt{1-x^2}} dy\int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}}$$

Treating $\dfrac{1}{\sqrt{(1-x^2-y^2)-z^2}}$ as $\dfrac{1}{\sqrt{a^2-z^2}}$

$$I = \int_0^1 dx\int_0^{\sqrt{1-x^2}}\left(\left|\sin^{-1}\frac{z}{a}\right|_0^{\sqrt{1-x^2-y^2}}\right)dy$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \left(\left| \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right|_0^{\sqrt{1-x^2-y^2}} \right) dy, \text{ as } a = \sqrt{1-x^2-y^2}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0 \right) dy$$

$$= \frac{\pi}{2} \int_0^1 \left((y)_0^{\sqrt{1-x^2}} \right) dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1, \text{ using } \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= \frac{\pi}{2} \left[0 + \frac{1}{2} \frac{\pi}{2} \right] = \frac{\pi^2}{8}$$

By change of variable to polar spherical co-ordinates, the region of integration

$$V = \{(x, y, z);\ x^2 + y^2 + z^2 \le 1;\ x \ge 0,\ z \ge 0,\ y \ge 0.\}$$

becomes $$I = \left\{ (r, \theta, \phi);\ r^2 \le 1, \text{ i.e. } 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2} \right\}$$

where $$\left. \begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta \end{aligned} \right\}$$

Now $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \text{coefficient of transformation} = r^2 \sin\theta.$$

whence $$\iiint_V \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \left(\sin\theta \left(\int_0^1 \frac{r^2}{\sqrt{1-r^2}} \right) dr \right) d\theta$$

Let $r = \sin t$ so that $dr = \cos t\, dt$. Further, when $\left. \begin{aligned} r = 0,\ t = 0, \\ r = 1,\ t = \frac{\pi}{2} \end{aligned} \right\}$

$\therefore$ $$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cdot \cos t\, dt$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta \left[\frac{1}{2} \cdot \frac{\pi}{2} \right];$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi (-\cos\theta)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}\phi\Big|_0^{\pi/2} = \frac{\pi^2}{8}.$$

Example 58: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by changing to polar co-ordinates.

Solution: We discuss this problem under change of variables.

Take $\frac{x}{a} = X,\ \frac{y}{b} = Y,\ \frac{z}{c} = Z$ so that $J = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = abc$

$\therefore$ The required volume,

$$V = \iiint dx\, dy\, dz = \iiint |J|\, dX\, dY\, dZ$$

$= abc\iiint dX\, dY dZ$, taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$.

Change this new system (X, Y, Z) to spherical polar co-ordinates (r, θ, ϕ) by taking

$$\left.\begin{aligned} X &= r\sin\theta\cos\phi,\\ Y &= r\sin\theta\sin\phi,\\ Z &= r\cos\theta \end{aligned}\right\} \text{ so that } J' = \frac{\partial(X, Y, Z)}{\partial(r, \theta, \phi)} = r^2\sin\theta,$$

$$V = abc\iiint |J|\, dr\, d\theta\, d\phi = abc\iiint r^2 \sin\theta\, dr\, d\theta\, d\phi$$

taken throughout the sphere $r^2 \le 1$, i.e. $0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

On considering the symmetry,

$$V = abc \cdot 8\int_0^{\pi/2}\left(\int_0^{\pi/2}\left(\int_0^1 r^2 dr\right)\sin\theta\, d\theta\right) d\phi$$

$$= 8\,abc\int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{r^3}{3}\Big|_0^1 \sin\theta\, d\theta\right) d\phi$$

$$= \frac{8}{3}abc\int_0^{\pi/2} [-\cos\theta]_0^{\pi/2} d\phi$$

$$= \frac{8}{3}abc\int_0^{\pi/2} 1 \cdot d\phi$$

$$= \frac{8}{3}abc\phi\Big|_0^{\pi/2} = \frac{8}{3}abc\frac{\pi}{2} = \frac{4}{3}\pi abc$$

Miscellaneous Problem

Example 59: Evaluate the surface integral $I = \iint_S \left(x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy\right)$.

where S is the surface bounded by $z = 0$, $z = b$, $x^2 + y^2 = a^2$.

OR

By transformation to a triple Integral, evaluate $I = \iint_S \left(x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy\right)$, where S is the surface bounded by $z = 0$, $z = b$, $x^2 + y^2 = a^2$.

Solution: On making use of Green's Theorem,

$$I = \int_{-a}^{a}\int_{0}^{b}\left(\sqrt{a^2 - y^2}\right)^3 dz\,dy - \int_{-a}^{a}\int_{0}^{b}\left(-\sqrt{a^2 - y^2}\right)^3 dz\,dy$$

$$+\int_{-a}^{a}\int_{0}^{b} x^2\sqrt{a^2 - x^2}\,dz dx - \int_{-a}^{a}\int_{-a}^{a} x^2\left(-\sqrt{a^2 - x^2}\right)dz\,dx$$

$$+\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\left(a^2 - y^2\right)b\,dx\,dy - \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 0\,dx\,dy$$

Using Divergence Theorem,

$$I = \iiint_V \left(3x^2 + x^2 + x^2\right)dx\,dy\,dz$$

$$= 4\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left(\int_0^b dz\right)dy\right]5x^2\,dx$$

$$= 4\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} b\,dy\right]5x^2\,dx$$

$$= 20\,b\int_0^a x^2\sqrt{a^2 - x^2}\,dx$$

$$= \frac{5}{4}\pi a^4 b\,.$$

***Note*:** As direct calculation of the integral may prove to be instructive. The evaluation of the integral can be carried out by calculating the sum of the integrals evaluated over the projections of the surface S on the co-ordinate planes. Thus, which upon evaluation is seen to check with the result already obtained. It should be noted that the angles α, β, γ are mode by the exterior normals in the +ve direction of the co-ordinate axes.

ANSWERS

Assignmet 1

1. $\left(\frac{\pi^2}{4}\right)$

2. $\frac{a^4}{3}$

3. $\frac{1}{ab}$

6. $\frac{\pi}{4}$

Assignment 2

1. $\int_0^a\left(\int_0^x \frac{x}{x^2+y^2}\,dy\right)dx$

2. $\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

3. $\int_a^{a\sin\alpha}\int_0^{y\cdot\cos\alpha} f(x,y)\,dx\,dy + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

4. $\int_0^{ma}\int_{\frac{y}{l}}^{\frac{y}{m}} f(x,y)\,dx\,dy + \int_{ma}^{la} f(x,y)\,dx\,dy$

Assignment 3

1. $\frac{4a^2}{3}$

2. $\frac{3}{2}\pi\left(b^4-a^4\right)$

3. $a^2\left(\frac{3}{4}\pi+\frac{4}{3}\right)$

Assignment 4

2. $\frac{1}{10}$ sq. units

Assignment 5

1. $\frac{\pi a^4}{8}$ units

2. $\frac{a^3}{12}(\pi+2)$ units

3. $\frac{2\pi}{9}$ units

4. $\frac{\pi}{4}$ units

Assignment 6

1. 1

2. $\frac{8}{9}a^3bc\left(3+2ab^2+2ac^2\right)$

3. 8π

4. $\frac{8}{9}\log 2-\frac{19}{9}$

Assignment 7

1. $\frac{1}{6lmn}$

2. $abc\left(\frac{\pi}{4}-\frac{13}{24}\right)$

Assignment 8

1. $abc/6$

2. $\dfrac{3\pi a^3}{2}$

Assignment 9

1. $\dfrac{4\pi ab^2}{3}$

2. $\dfrac{2}{3}\pi a^2$

3. $2\pi^2 a^3$

4. $\dfrac{\pi a^3}{4}\left\{\dfrac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)-\dfrac{1}{3}\right\}$