Partial Derivatives

1 Functions of two or more variables

In many situations a quantity (variable) of interest depends on two or more other quantities (variables), e.g.

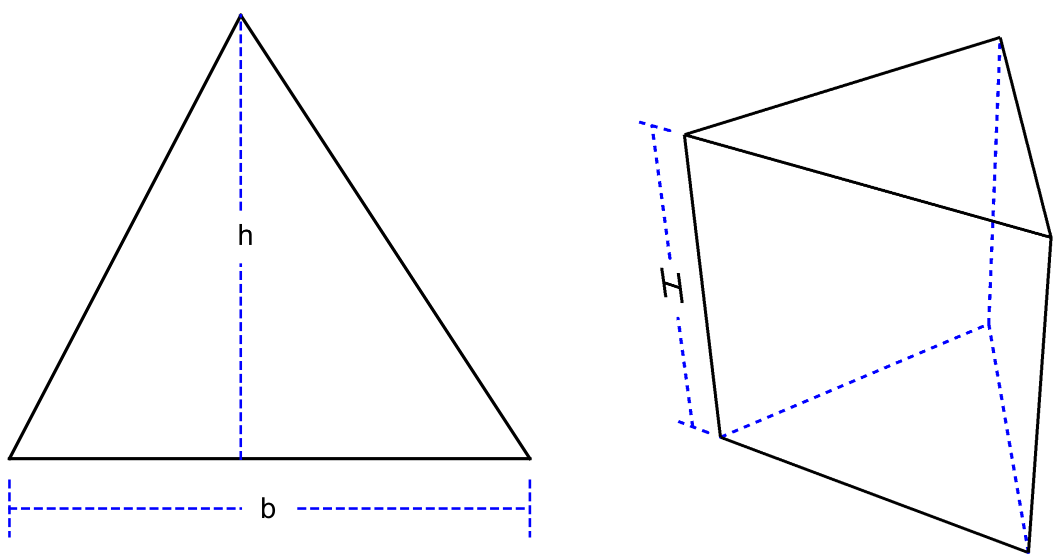


Figure 1: b is the base length of the triangle, h is the height of the triangle, H is the height of the cylinder.

The area of the triangle and the base of the cylinder: $A = \frac{1}{2}bh$

The volume of the cylinder: $V = AH = \frac{1}{2}bhH$

The arithmetic average $\bar{x}$ of n real numbers $x_1, \ldots, x_n$

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

We say

A is a function of the two variables b and h.
V is a function of the three variables b, h and H.
$\bar{x}$ is a function of the n variables $x_1, ..., x_n$.

The expression $z = f(x, y)$ means that z is a function of x and y;

$w = f(x, y, z)$; $u = f(x_1, x_2, \ldots, x_n)$.

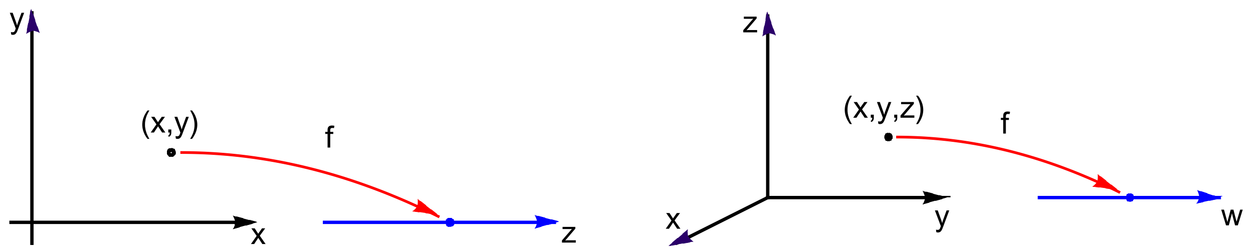


Figure 2: A function f assigns a unique number $z = f(x, y)$, or $w = f(x, y, z)$ to a point in (x, y)-plane or (x, y, z)-space.

The independent variables of a function may be restricted to lie in some set $\mathcal{D}$ which we call the **domain** of f, and denote $\mathcal{D}(f)$. The **natural domain** consists of all points for which a function defined by a formula gives a real number.

Definition. A function f of two variables, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set $\mathcal{D}$ in the xy-plane.

A function f of n variables, $x_1, ..., x_n$, is a rule that assigns a unique real number $f(x_1, ..., x_n)$ to each point $(x_1, ..., x_n)$ in some set $\mathcal{D}$ in the n-dimensional $x_1...x_n$-space, denoted $\mathbb{R}^n$.

Definition. The **graph** of a function $z = f(x, y)$ in xyz-space is a set of points $P = \big(x, y, f(x, y)\big)$ where (x, y) belong to $\mathcal{D}(f)$.

In general such a graph is a surface in 3-space.

Examples. Find the natural domain of f, identify the graph of f as a surface in 3-space and sketch it.

1. $f(x, y) = 0$;

2. $f(x, y) = 1$;

3. $f(x, y) = x$;

4. $f(x, y) = ax + by + c$;

5. $f(x, y) = x^2 + y^2$;

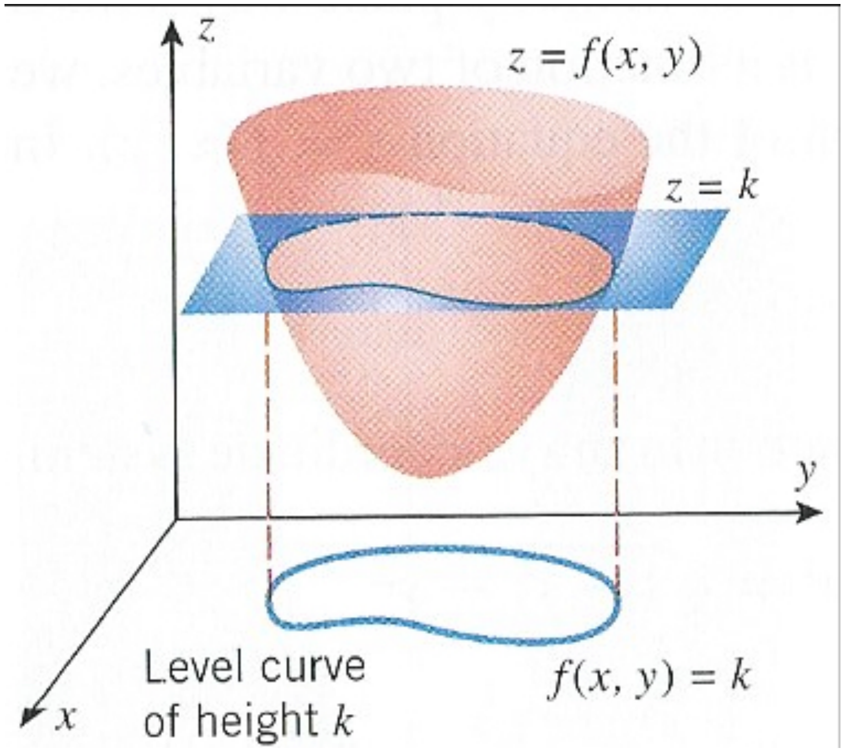
6. $f(x, y) = \sqrt{1 - x^2 - y^2}$;

7. $f(x, y) = \sqrt{1 + x^2 + y^2}$;

8. $f(x, y) = \sqrt{x^2 + y^2 - 1}$;

9. $f(x, y) = -\sqrt{x^2 + y^2}$;

2 Level curves



If $z = f(x, y)$ is cut by $z = k$, then at all points on the intersection we have $f(x, y) = k$.

This defines a curve in the xy-plane which is the projection of the intersection onto the xy-plane, and is called the **level curve of height** k or the **level curve with constant** k.

A set of level curves for $z = f(x, y)$ is called a **contour plot** or **contour map** of f.

Examples.

1. $f(x, y) = ax + by + c$;
2. $f(x, y) = x^2 + y^2$;
3. $f(x, y) = \sqrt{1 - x^2 - y^2}$;
4. $f(x, y) = \sqrt{1 + x^2 + y^2}$;
5. $f(x, y) = \sqrt{x^2 + y^2 - 1}$;
6. $f(x, y) = -\sqrt{x^2 + y^2}$;

7. $f(x, y) = y^2 - x^2$. It is the hyperbolic paraboloid (saddle surface).

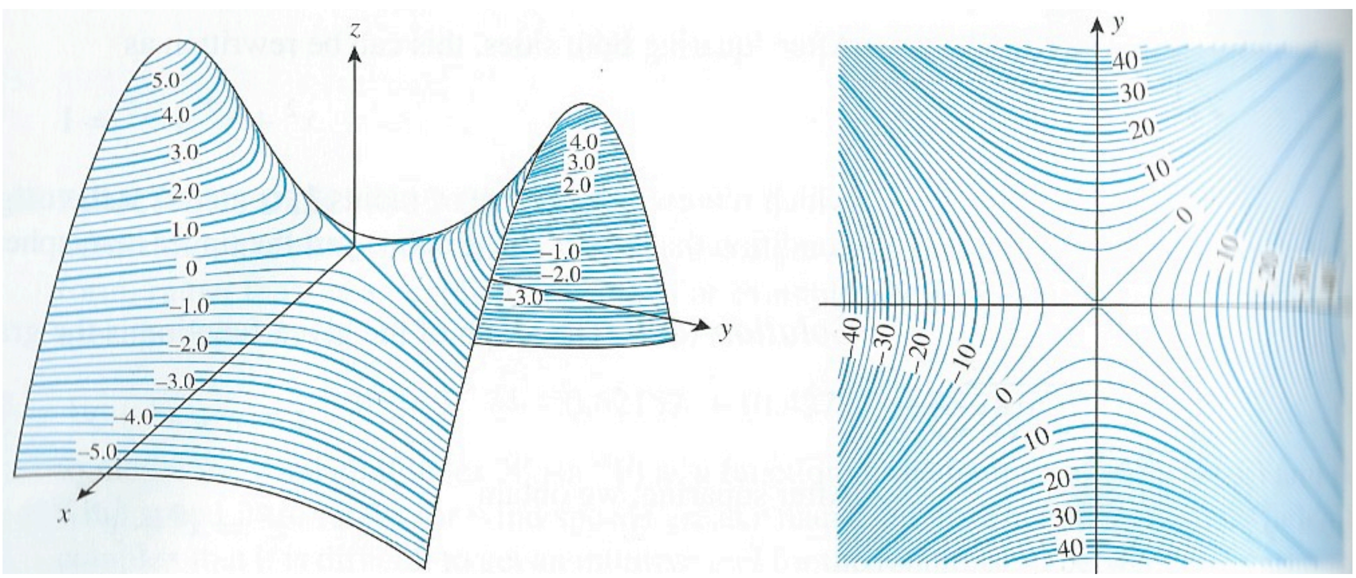


Figure 3: The hyperbolic paraboloid and its contour map.

There is no "direct" way to graph a function of three variables. The graph would be a curved 3-dimensional space (a 3-dim *manifold* if it is smooth), in 4-space. But $f(x, y, z) = k$ defines a surface in 3-space which we call the **level surface with constant** k.

Examples.

1. $f(x, y, z) = x^2 + y^2 + z^2$;

2. $f(x, y, z) = z^2 - x^2 - y^2$;

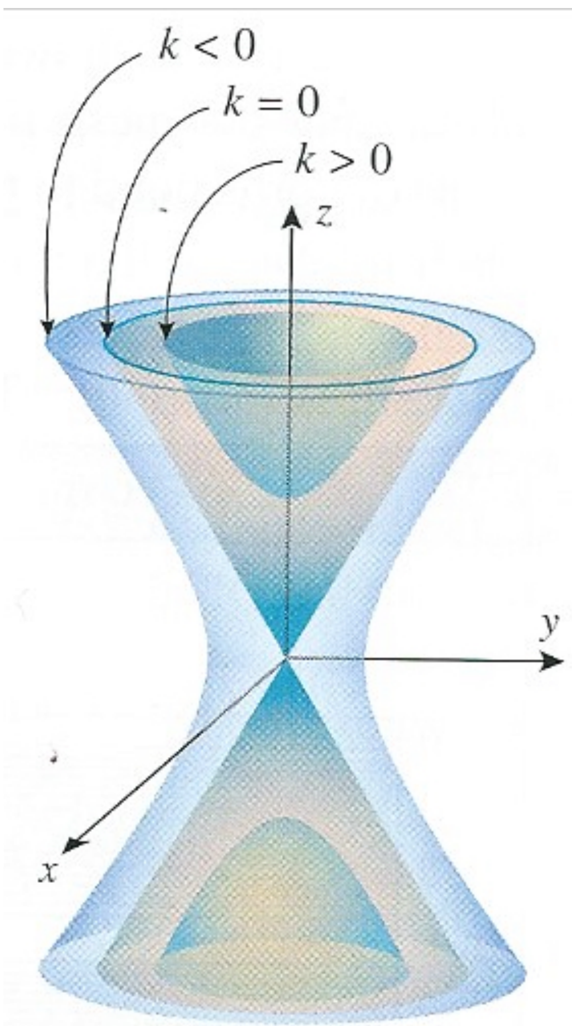
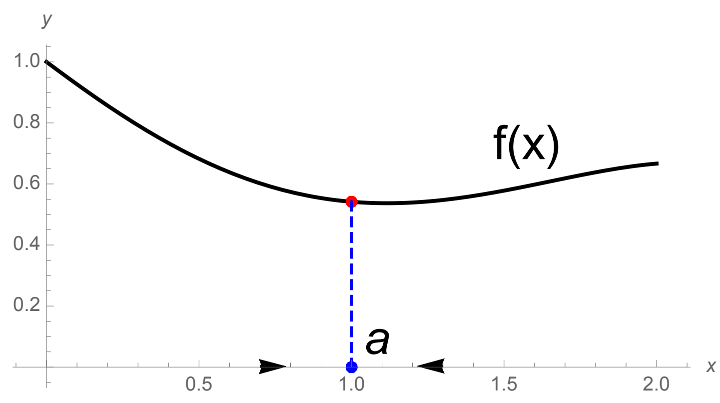
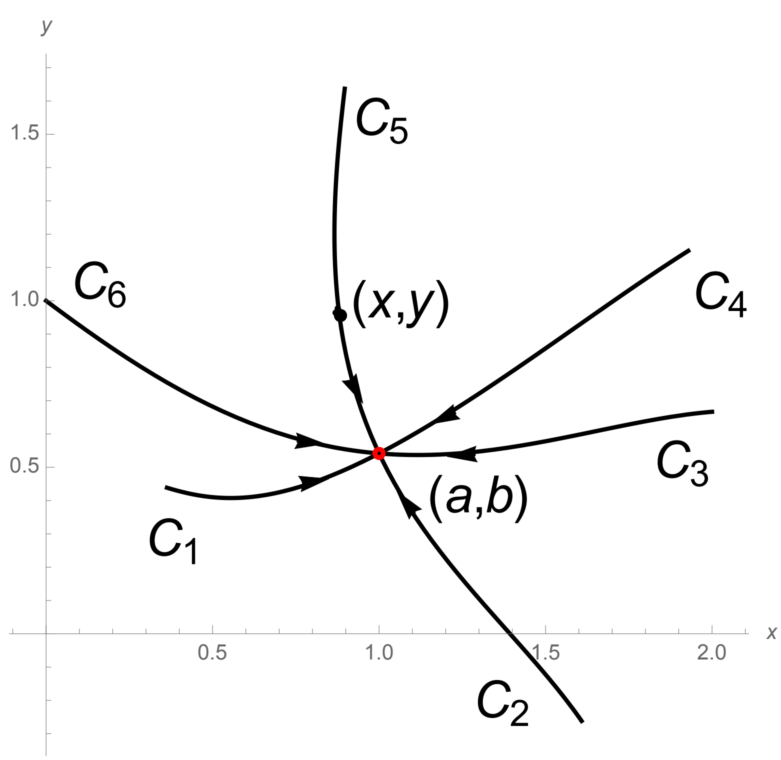


Figure 4: Level surfaces of $f(x, y, z) = z^2 - x^2 - y^2$

3 Limits and Continuity



There are two one-sided limits for $y = f(x)$.



For $z = f(x, y)$ there are infinitely many curves along which one can approach (a, b).
This leads to the notion of the limit of $f(x, y)$ along a curve C.
If all these limits coincide then $f(x, y)$ has a limit at (a, b), and the limit is equal to $f(a, b)$ then f is continuous at (a, b).

4 Partial Derivatives

Recall that for a function $f(x)$ of a single variable the derivative of f at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the instantaneous rate of change of f at a, and is equal to the slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$.

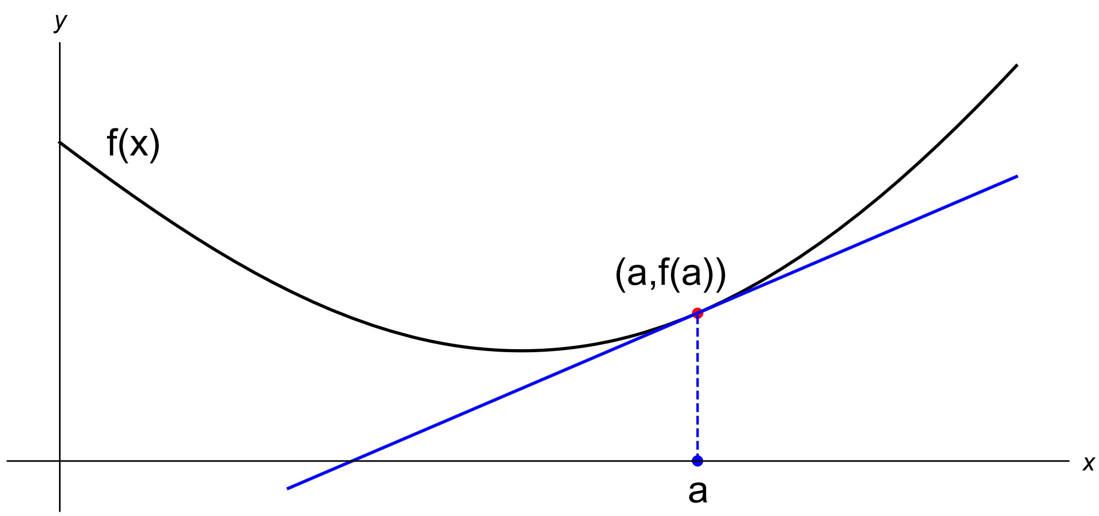


Figure 5: Equation of the tangent line: $y = f(a) + f'(a)(x - a)$.

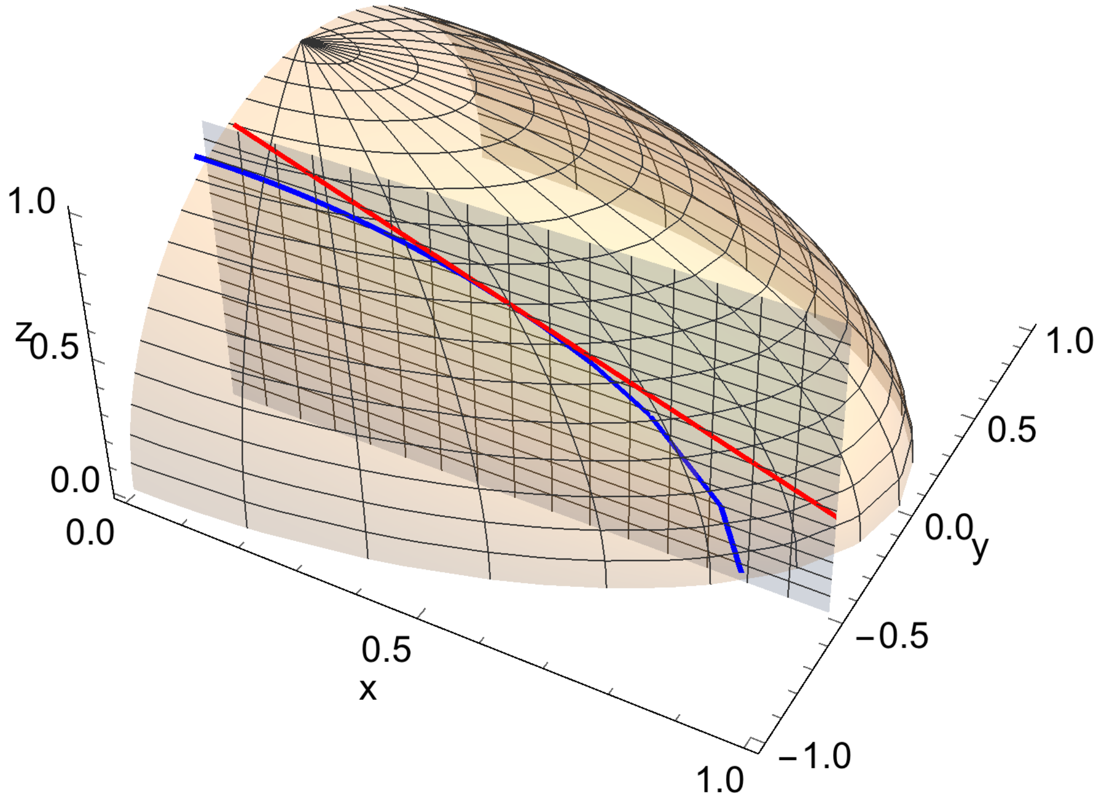
Consider $f(x, y)$. If we fix $y = b$ where b is a number from the domain of f then $f(x, b)$ is a function of a single variable x and we can calculate its derivative at some $x = a$. This derivative is called the **partial derivative** of $f(x, y)$ with respect to x at (a, b) and is denoted by

$$f_x(a, b) \text{ or by } \frac{\partial f(a, b)}{\partial x}$$

$$f_x(a, b) = \frac{\partial f(a, b)}{\partial x} = \frac{d}{dx}\big[f(x, b)\big]\Big|_{x=a} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\text{If } f(x, y) = x \text{ then } \frac{\partial x}{\partial x} = 1, \text{ and if } f(x, y) = y \text{ then } \frac{\partial y}{\partial x} = 0$$

Geometrically, given the surface $z = f(x, y)$, we consider its intersection with the plane $y = b$ which is a curve. This curve is the graph of the function $f(x, b)$, and therefore the partial derivative $f_x(a, b)$ is the slope of the tangent line to the curve at $(a, b, f(a, b))$



Equation of the tangent line: $x = t$, $y = b$, $z = f(a, b) + f_x(a, b)(t - a)$

We call $f_x(a, b)$ the **slope of the surface in the x-direction** at (a, b)

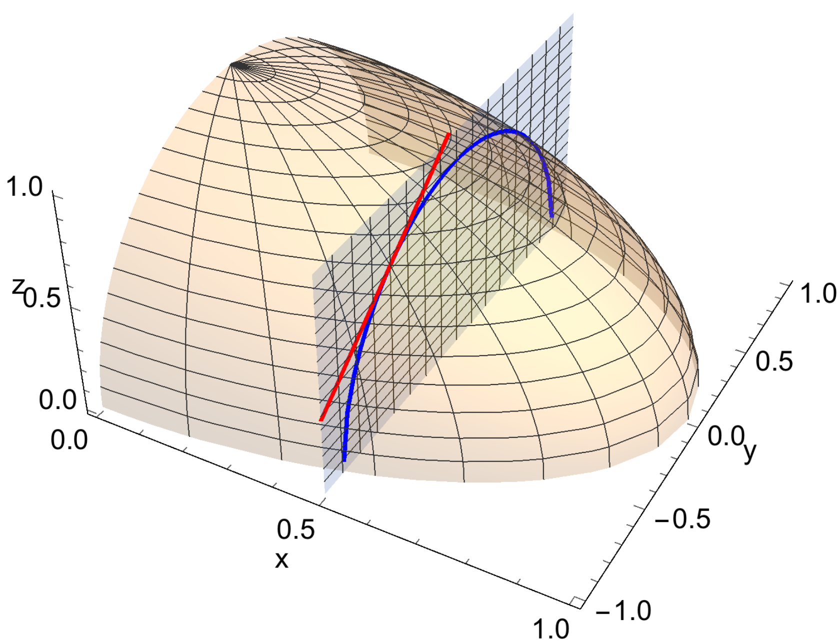
Similarly, if we fix $x = a$ where a is a number from the domain of f then $f(a, y)$ is a function of a single variable y and we can calculate its derivative at some $y = b$. This derivative is called the **partial derivative** of $f(x, y)$ with respect to y at (a, b) and is denoted by

$$f_y(a, b) \text{ or by } \frac{\partial f(a, b)}{\partial y}$$

$$f_y(a, b) = \frac{\partial f(a, b)}{\partial y} = \frac{d}{dy}\big[f(a, y)\big]\Big|_{y=b} = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

$$\text{If } \ f(x, y) = x \ \text{ then } \ \frac{\partial x}{\partial y} = 0, \text{ and if } \ f(x, y) = y \ \text{ then } \ \frac{\partial y}{\partial y} = 1$$

The intersection of the surface $z = f(x, y)$ with the plane $x = a$ is a curve which is the graph of the function $f(a, y)$, and therefore the partial derivative $f_y(a, b)$ is the slope of the tangent line to the curve at $(a, b, f(a, b))$



Equation of the tangent line: $x = a$, $y = t$, $z = f(a, b) + f_y(a, b)(t - a)$

We call $f_y(a, b)$ the **slope of the surface in the y-direction** at (a, b)

If we allow (a, b) to vary, the partial derivatives become functions of two variables:

$$a \to x\,,\ b \to y \ \text{ and } \ f_x(a,b) \to f_x(x,y),\ \ f_y(a,b) \to f_y(x,y)$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}\,, \quad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Partial derivative notation: if $z = f(x, y)$ then

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \partial_x f = \partial_x z\,, \quad f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \partial_y f = \partial_y z$$

Example.

$$z = f(x,y) = \ln \frac{\sqrt[3]{2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18}}{2}$$

Find $f_x(x, y)$, $f_y(x, y)$, $f(3, -2)$, $f_x(3, -2)$, $f_y(3, -2)$

For $w = f(x, y, z)$ there are three partial derivatives $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$

Example.

$$f(x,y,z) = \sqrt{z^2 + y - x + 2\cos(3x - 2y)}$$

Find

$f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$,
$f(2, 3, -1)$, $f_x(2, 3, -1)$, $f_y(2, 3, -1)$, $f_z(2, 3, -1)$

In general, for $w = f(x_1, x_2, \ldots, x_n)$ there are n partial derivatives:

$$\frac{\partial w}{\partial x_1}, \quad \frac{\partial w}{\partial x_2}, \quad \cdots \quad , \quad \frac{\partial w}{\partial x_n}$$

Example.

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Find

$$\frac{\partial r}{\partial x_1}, \quad \frac{\partial r}{\partial x_2}, \quad \frac{\partial r}{\partial x_9}, \quad \frac{\partial r}{\partial x_i}, \quad \frac{\partial r}{\partial x_{n-1}}, \quad n \geq 9\,,\ i \leq n$$

Second-order derivatives: f_{xx}, f_{xy}, f_{yx}, f_{yy}

$$\begin{array}{ccccc} & & & & f_{xx} \\ & & & \nearrow & \\ & & f_x & \rightarrow & f_{xy} \\ & \nearrow & & & \\ f & & & & \\ & \searrow & & & \\ & & f_y & \rightarrow & f_{yx} \\ & & & \searrow & \\ & & & & f_{yy} \end{array}$$

Notation

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

f_{xy} and f_{yx} are called the **mixed second-order partial derivatives**. f_x and f_y can be called first-order partial derivative.

Example.

$$z = 2e^{y-\frac{\pi}{2}} \sin x - 3e^{x-\frac{\pi}{4}} \cos y$$

Find

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial y \partial x},$$

$$\frac{\partial z}{\partial x}(\frac{\pi}{4}, \frac{\pi}{2}), \quad \frac{\partial z}{\partial y}(\frac{\pi}{4}, \frac{\pi}{2}), \quad \frac{\partial^2 z}{\partial x \partial y}(\frac{\pi}{4}, \frac{\pi}{2}), \quad \frac{\partial^2 z}{\partial y \partial x}(\frac{\pi}{4}, \frac{\pi}{2})$$

Equality of mixed partial derivatives

Theorem. Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disc, then $f_{xy} = f_{yx}$ on that disc.

Higher-order derivatives

Third-order, fourth-order, and higher-order derivatives are obtained by successive differentiation.

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right)$$

$$f_{xyxz} = \frac{\partial^4 f}{\partial z \partial x \partial y \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial^3 f}{\partial x \partial y \partial x}\right)$$

For higher-order derivatives the equality of mixed partial derivatives also holds if the derivatives are continuous.

In what follows we always assume that the order of partial derivatives is irrelevant for functions of any number of independent variables.

5 Differentiability, differentials and local linearity

For $f(x, y)$, the symbol Δf, called the **increment** of f, denotes the change

$$\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$$

For small Δx, Δy

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

Definition. A function $f(x, y)$ is said to be **differentiable** at (a, b) provided $f_x(a, b)$ and $f_y(a, b)$ both exist and

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(a, b)\Delta x - f_y(a, b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

For $f(x, y, z)$

$$\Delta f = f(a + \Delta x, b + \Delta y, c + \Delta z) - f(a, b, c)$$

For small Δx, Δy, Δz

$$\Delta f \approx f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z$$

and $f(x, y, z)$ is **differentiable** at (a, b, c) if

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \frac{\Delta f - f_x(a, b, c)\Delta x - f_y(a, b, c)\Delta y - f_z(a, b, c)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + +(\Delta z)^2}} = 0$$

Theorem. If a function is differentiable at a point, then it is continuous at that point.

Theorem. If all first-order derivatives of f exist and are continuous at a point, then f is differentiable at a point.

Differentials

If $z = f(x, y)$ is differentiable at (a, b) we let

$$dz = f_x(a, b)dx + f_y(a, b)dy$$

denote a new function with dependent variable dz and independent variables dx, dy. It is called the **total differential of z (or f) at** (a, b). It is a linear function of dx and dy.

Note that $\Delta z \approx dz$ if $\Delta x = dx$ and $\Delta y = dy$

If we allow (a, b) to vary, the differential becomes a function of four variables, dx, dy, x, y:

$$a \to x\,,\ b \to y\ \Rightarrow\ dz = f_x(x, y)dx + f_y(x, y)dy$$

Definition. If $f(x, y)$ is differentiable at (a, b) then

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **local linear approximation of f at** (a, b).
Its graph is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$

Example. $f(x, y) = \sqrt{x^2 + y^2}$. Compute $f(3.04, 3.98)$, and estimate the error if a calculator gives $f(3.04, 3.98) \approx 5.00819$

If $w = f(x, y, z)$, the total differential of w (or f) at (a, b, c) is

$$dw = f_x(a, b, c)dx + f_y(a, b, c)dy + f_z(a, b, c)dz$$

or if $a \to x\,,\ b \to y\,,\ c \to z$

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

The local linear approximation of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

6 The Chain Rule

Recall

$$y = f(x(t)) \;\Rightarrow\; \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

because

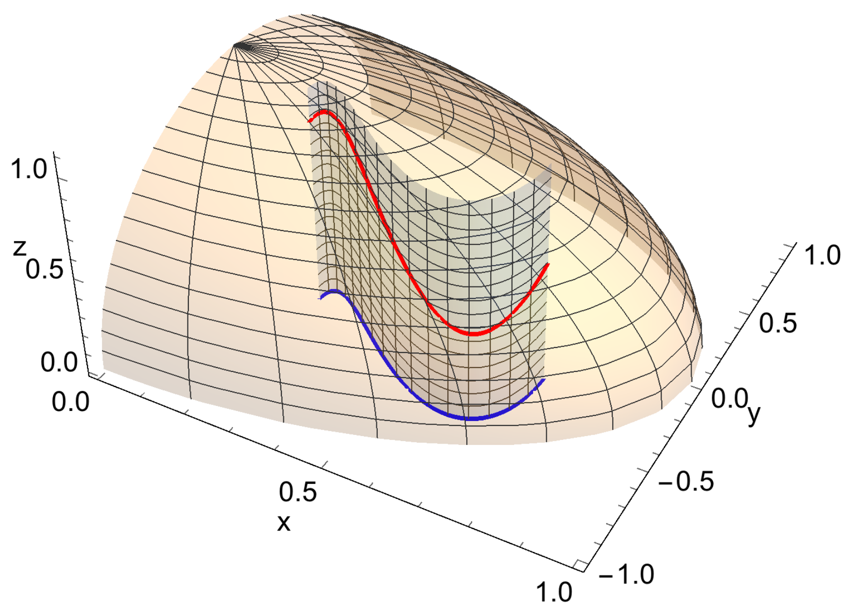
$$\Delta y \approx \frac{dy}{dx}\Delta x\,, \quad \Delta x \approx \frac{dx}{dt}\Delta t$$

Let $z = f(x, y)$ and $x = x(t)$, $y = y(t)$. Then $z = f(x(t), y(t))$ is a function of the single variable t.

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y\,, \quad \Delta x \approx \frac{dx}{dt}\Delta t\,,\; \Delta y \approx \frac{dy}{dt}\Delta t$$

and therefore

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\,\frac{dx}{dt} + \frac{\partial z}{\partial y}\,\frac{dy}{dt}$$



Example. $z = \sqrt{4 - x^2 - y^2}$, $x = 1 + \cos t$, $y = \sin t$

Similarly, if $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$. Then $w = f(x(t), y(t), z(t))$ is a function of the single variable t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$ and $x_1 = x_1(t)$, $x_2 = x_2(t)$, ... , $x_n = x_n(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots \frac{\partial w}{\partial x_n}\frac{dx_n}{dt} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\frac{dx_i}{dt}$$

Implicit differentiation

Let $z = f(x, y)$ and $y = y(x)$. Then

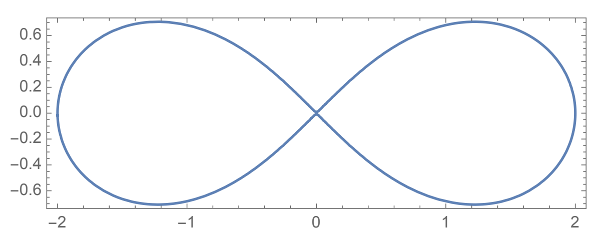
$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Suppose $y(x)$ is such that $f(x, y(x)) = \text{const}$. Then, $\frac{dz}{dx} = 0$ and

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{f_x}{f_y} \quad \text{if} \quad f_y \neq 0$$

Example. The lemniscate is defined by the equation

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$



Find dy/dx.

The chain rule for partial derivatives

1. Let $y = f(x)$ and $x = x(u, v)$
Then $y = f(x(u, v))$ is a function of u and v, and

$$\Delta y \approx \frac{dy}{dx}\Delta x\,, \quad \Delta x \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v$$

Thus,

$$\frac{\partial y}{\partial u} = \frac{dy}{dx}\frac{\partial x}{\partial u}\,, \qquad \frac{\partial y}{\partial v} = \frac{dy}{dx}\frac{\partial x}{\partial v}$$

2. Let $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$
Then $x = f(x(u, v), y(u, v)$ is a function of u and v, and

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y\,, \quad \Delta x \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v\,, \quad \Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$

Thus,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\,, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

3. Let $w = f(x, y, z)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}\,, \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

4. Let $w = f(x_1, ..., x_n)$ and $x_1 = x_1(u_1, ..., u_m)$, ... , $x_n = x_n(u_1, ..., u_m)$

$$\frac{\partial w}{\partial u_\alpha} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\frac{\partial x_i}{\partial u_\alpha}\,, \quad \alpha = 1, \ldots, m$$

Example. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ where

$$z = \cos\frac{x}{2}\,\sin 2y\,; \quad x = 3u - 2v\,, \quad y = u^2 - 2v^3$$

Example. The wave equation: Consider a string of length L that is stretched taut between $x = 0$ and $x = L$ on an x-axis, and suppose that the string is set into vibratory motion by "plucking" it at time $t = 0$. The displacement of a point on the string depends both on x and t: $u(x,t)$. One-dimensional wave equation for small displacements

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Show that

$$u(x,t) = f(x - ct) + g(x + ct)$$

is a solution to the equation. In fact it is the general solution.