CHAPTER 6

TORSION

In this chapter, we treat the problem of torsion of prismatic bars with noncircular cross sections. We treat both linearly elastic and fully plastic torsion. For prismatic bars with circular cross sections, the torsion formulas are readily derived by the method of mechanics of materials. However, for noncircular cross sections, more general methods are required. In the following sections, we treat noncircular cross sections by several methods, one of which is the semiinverse method of Saint-Venant (Boresi and Chong, 2000). General relations are derived that are applicable for both the linear elastic torsion problem and the fully plastic torsion problem. To aid in the solution of the resulting differential equation for some linear elastic torsion problems, the Saint-Venant solution is used in conjunction with the Prandtl elastic-membrane (soap-film) analogy.

The semiinverse method of Saint-Venant is comparable to the mechanics of materials method in that certain assumptions, based on an understanding of the mechanics of the problem, are introduced initially. Sufficient freedom is allowed so that the equations describing the torsion boundary value problem of solids may be employed to determine the solution more completely. For the case of circular cross sections, the method of Saint-Venant leads to an exact solution (subject to appropriate boundary conditions) for the torsion problem. Because of its importance in engineering, the torsion problem of circular cross sections is discussed first.

6.1 TORSION OF A PRISMATIC BAR OF CIRCULAR CROSS SECTION

Consider a solid cylinder with cross-sectional area A and length L. Let the cylinder be subjected to a twisting couple **T** applied at the right end (Figure 6.1). An equilibrating torque acts on the left end. The vectors that represent the torque are directed along the z axis, the centroidal axis of the shaft. Under the action of the torque, an originally straight generator of the cylinder AB will deform into a helical curve AB^*. However, because of the radial symmetry of the circular cross section and because a deformed cross section must appear to be the same from both ends of the torsion member, plane cross sections of the torsion member normal to the z axis remain plane after deformation and all radii remain straight. Furthermore, for small displacements, each radius remains inextensible. In other words, the torque **T** causes each cross section to rotate as a rigid body about the z axis (axis of the couple); this axis is called the *axis of twist*. The rotation β of a given section, relative to the plane $z = 0$, will depend on its distance from the plane $z = 0$. For small deformations,

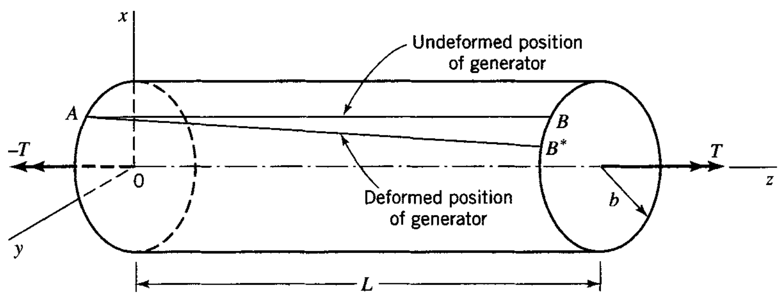

FIGURE 6.1 Circular cross section torsion member.

following Saint-Venant, we assume that the amount of rotation of a given section depends linearly on its distance z from the plane $z = 0$. Thus,

$$\beta = \theta z \tag{6.1}$$

where θ is the angle of twist per unit length of the shaft. Under the conditions that plane sections remain plane and Eq. 6.1 holds, we now seek to satisfy the equations of elasticity; that is, we employ the semiinverse method of seeking the elasticity solution.

Since cross sections remain plane and rotate as rigid bodies about the z axis, the displacement component w, parallel to the z axis, is zero. To calculate the (x, y) components of displacements u and v, consider a cross section at distance z from the plane $z = 0$. Consider a point in the circular cross section (Figure 6.2) with radial distance $0P$. Under the deformation, radius $0P$ rotates into the radius $0P^*$ ($0P^* = 0P$). In terms of the angular displacement β of the radius, the displacement components (u, v) are

$$\begin{aligned} u &= x^* - x = 0P[\cos(\beta + \phi) - \cos\phi] \\ v &= y^* - y = 0P[\sin(\beta + \phi) - \sin\phi] \end{aligned} \tag{6.2}$$

Expanding $\cos(\beta + \phi)$ and $\sin(\beta + \phi)$ and noting that $x = 0P \cos \phi$ and $y = 0P \sin \phi$, we may write Eqs. 6.2 in the form

$$\begin{aligned} u &= x(\cos\beta - 1) - y\sin\beta \\ v &= x\sin\beta + y(\cos\beta - 1) \end{aligned} \tag{6.3}$$

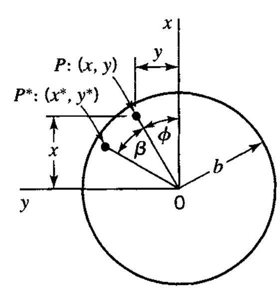

FIGURE 6.2 Angular displacement β.

Restricting the displacement to be small (since then $\sin \beta \approx \beta$ and $\cos \beta \approx 1$), we obtain with the assumption that $w = 0$,

$$u = -y\beta, \quad v = x\beta, \quad w = 0 \tag{6.4}$$

to first-degree terms in β. Substitution of Eq. 6.1 into Eqs. 6.4 yields

$$u = -\theta yz, \quad v = \theta xz, \quad w = 0 \tag{6.5}$$

On the basis of the foregoing assumptions, Eqs. 6.5 represent the displacement components of a point in a circular shaft subjected to a torque **T**.

Substitution of Eqs. 6.5 into Eqs. 2.81 yields the strain components (if we ignore temperature effects)

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = 0, \quad 2\epsilon_{zx} = \gamma_{zx} = -\theta y, \quad 2\epsilon_{zy} = \gamma_{zx} = \theta x \tag{6.6}$$

Since the strain components are derived from admissible displacement components, compatibility is automatically satisfied. (See Section 2.8; see also Boresi and Chong, 2000, Section 2.16.) With Eqs. 6.6, Eqs. 3.32 yield the stress components for linear elasticity

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0, \quad \sigma_{zx} = -\theta G y, \quad \sigma_{zy} = \theta G x \tag{6.7}$$

Equations 6.7 satisfy the equations of equilibrium, provided the body forces are zero (Eqs. 2.45).

To satisfy the boundary conditions, Eqs. 6.7 must yield no forces on the lateral surface of the bar; on the ends, they must yield stresses such that the net moment is equal to **T** and the resultant force vanishes. Since the direction cosines of the unit normal to the lateral surface are $(l, m, 0)$ (see Figure 6.3), the first two of Eqs. 2.10 are satisfied identically. The last of Eqs. 2.10 yields

$$l\sigma_{zx} + m\sigma_{zy} = 0 \tag{6.8}$$

By Figure 6.3,

$$l = \cos\phi = \frac{x}{b}, \quad m = \sin\phi = \frac{y}{b} \tag{6.9}$$

Substitution of Eqs. 6.7 and 6.9 into Eqs. 6.8 yields

$$-\frac{xy}{b} + \frac{xy}{b} = 0$$

Therefore, the boundary conditions on the lateral surface are satisfied.

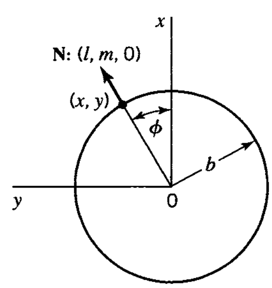

FIGURE 6.3 Unit normal vector.

On the ends, the stresses must be distributed so that the net moment is **T**. Therefore, summation of moments on each end with respect to the z axis yields (Figure 6.4)

$$\sum M_z = T = \int_A (x\sigma_{zy} - y\sigma_{zx})dA \tag{6.10}$$

Substitution of Eqs. 6.7 into Eq. 6.10 yields

$$T = G\theta\int_A \left(x^2 + y^2\right)dA = G\theta\int_A r^2 dA \tag{6.11}$$

Since the last integral is the polar moment of inertia ($J = \pi b^4/2$) of the circular cross section, Eq. 6.11 yields

$$\theta = \frac{T}{GJ} \tag{6.12}$$

which relates the angular twist θ per unit length of the shaft to the magnitude T of the applied torque. The factor GJ is the torsional rigidity (or torsional stiffness) of the member.

Because compatibility and equilibrium are satisfied, Eqs. 6.7 represent the solution of the elasticity problem. However, in applying torsional loads to most torsion members of circular cross section, the distributions of σ_{zx} and σ_{zy} on the member ends probably do not satisfy Eqs. 6.7. In these cases, it is assumed that σ_{zx} and σ_{zy} undergo a redistribution with distance from the ends of the bar until, at a distance of a few bar diameters from the ends, the distributions are essentially given by Eqs. 6.7. This concept of redistribution of the applied end stresses with distance from the ends is known as the Saint-Venant principle (Boresi and Chong, 2000).

Since the solution of Eqs. 6.7 indicates that σ_{zx} and σ_{zy} are independent of z, the stress distribution is the same for all cross sections. Thus, the stress vector $\boldsymbol{\tau}$ for any point P in a cross section is given by the relation

$$\boldsymbol{\tau} = -\theta G y\mathbf{i} + \theta G x\mathbf{j} \tag{6.13}$$

The stress vector $\boldsymbol{\tau}$ lies in the plane of the cross section, and it is perpendicular to the radius vector $\mathbf{r}$ joining point P to the origin 0. By Eq. 6.13, the magnitude of $\boldsymbol{\tau}$ is

$$\tau = \theta G\sqrt{x^2 + y^2} = \theta G r \tag{6.14}$$

Hence, τ is a maximum for $r = b$; that is, τ attains a maximum value of $\theta G b$.

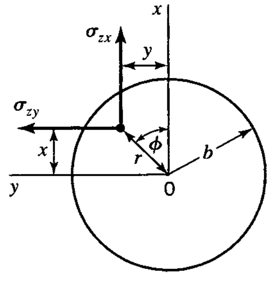

FIGURE 6.4 Shear stresses (σ_{zx}, σ_{zy}).

Substitution of Eq. 6.12 into Eq. 6.14 yields the result

$$\tau = \frac{Tr}{J} \tag{6.15}$$

which relates the magnitude τ of the shear stress to the magnitude T of the torque. This result holds also for cylindrical bars with hollow circular cross sections (Figure 6.5), with inner radius a and outer radius b; for this cross section $J = \pi(b^4 - a^4)/2$ and $a \leq r \leq b$.

6.1.1 Design of Transmission Shafts

Torsional shafts are used frequently to transmit power from a power plant to a machine; an application is noted in Figure 6.6, where an electric motor is used to drive a centrifugal pump. By dynamics, the power P, measured in watts [N • m/s], transmitted by a shaft is defined by the relation

$$P = T\omega \tag{a}$$

where T is the torque applied to the shaft and ω is the angular velocity [rad/s] of the rotating shaft. The frequency [Hz] of rotation of the shaft is denoted by f. Thus,

$$\omega = 2\pi f \tag{b}$$

Equations (a) and (b) yield

$$T = \frac{P}{2\pi f} \tag{c}$$

If the power P and frequency f are specified, Eq. (c) determines the design torque for the shaft. The dimensions of the shaft are dictated by the mode of failure, the strength of the material associated with the mode of failure, the required factor of safety, and the shaft cross section shape.

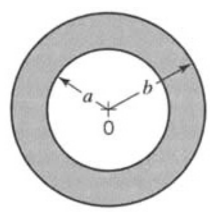

FIGURE 6.5 Hollow circular cross section.

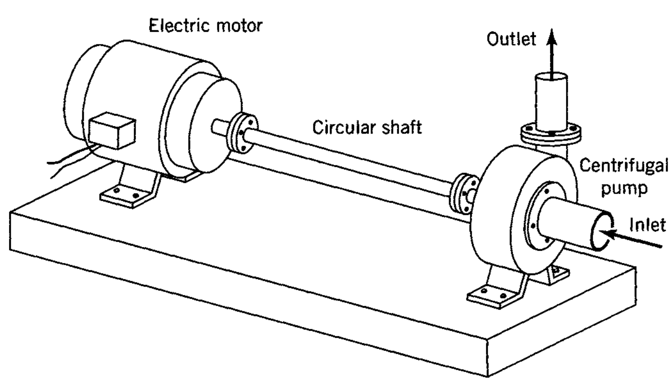

FIGURE 6.6 Transmission of power through a circular shaft.

EXAMPLE 6.1
Shaft with Hollow Circular Cross Section

A steel shaft has a hollow circular cross section (see Figure 6.5), with radii a = 22 mm and b = 25 mm. It is subjected to a twisting moment T = 500 N • m.

(a) Determine the maximum shear stress in the shaft.

(b) Determine the angle of twist per unit length.

Solution

(a) The polar moment of inertia of the cross section is

$$J = \pi(b^4 - a^4)/2 = \pi(25^4 - 22^4)/2 = 245{,}600 \text{ mm}^4 = 24.56 \times 10^{-8} \text{ m}^4$$

Hence, by Eq. 6.15,

$$\tau_{max} = Tb/J = 500 \times 0.025/24.56 \times 10^{-8} = 50.9 \text{ MPa}$$

(b) By Eq. 6.12, with G = 77 GPa,

$$\theta = T/GJ = 500/(77 \times 10^9 \times 24.56 \times 10^{-8}) = 0.0264 \text{ rad/m}$$

EXAMPLE 6.2
Circular Cross Section Drive Shaft

Two pulleys, one at B and one at C, are driven by a motor through a stepped drive shaft ABC, as shown in Figure E6.2. Each pulley absorbs a torque of 113 N • m. The stepped shaft has two lengths $AB = L_1$ = 1 m and $BC = L_2$ = 1.27 m. The shafts are made of steel (Y = 414 MPa, G = 77 GPa). Let the safety factor be SF = 2.0 for yield by the maximum shear-stress criterion.

(a) Determine suitable diameter dimensions d_1 and d_2 for the two shaft lengths.

(b) With the diameters selected in part (a), calculate the angle of twist β_c of the shaft at C.

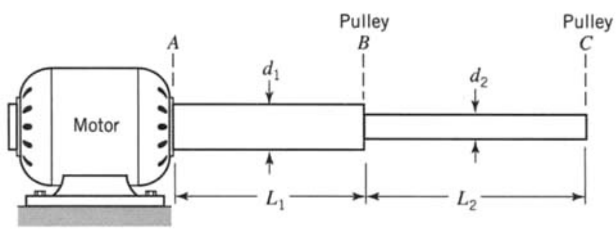

FIGURE E6.2 Circular cross section shaft.

Solution

Since each pulley removes 113 N • m, shaft AB must transmit a torque T_1 = 226 N • m, and shaft BC must transmit a torque T_2 = 113 N • m. Also, the maximum permissible shear stress in either shaft length is (by Eq. 4.12) $\tau_{max} = \tau_Y/SF = 0.25Y$ = 103.5 MPa.

(a) By Eq. 6.15, we have $\tau_{max} = 2T/(\pi r_1^3)$. Consequently, we have

$$r_1 = [2T/(\pi\tau_{max})]^{1/3} = [2 \times 226/(\pi \times 103.5 \times 10^6)]^{1/3} = 0.0112 \text{ m}$$

Hence, the diameter $d_1 = 2r_1$ = 0.0224 m = 22.4 mm. Similarly, we find $d_2 = 2r_2 = 2 \times 0.00886$ m = 0.0177 m = 17.7 mm. Since these dimensions are not standard sizes, we choose d_1 = 25.4 mm and d_2 = 19.05 mm, since these sizes (1.0 and 0.75 in., respectively) are available in U.S. customary units.

(b) By Eq. 6.12, the unit angle of twist in the shaft length AB is

$$\theta_1 = T_1/(GJ_1) = 2T_1/(G\pi r_1^4) = (2 \times 226)/(77 \times 10^9 \times \pi \times 0.0127^4) = 0.07183 \text{ rad/m}$$

Similarly, we obtain θ_2 = 0.1135 rad/m. Therefore, the angle of twist at C is

$$\beta_c = 1.0 \times 0.07183 + 1.27 \times 0.1135 = 0.216 \text{ rad} = 12.4°.$$

EXAMPLE 6.3
Design Torque for a Hollow Torsion Shaft

The torsion member shown in Figure E6.3 is made of an aluminum alloy that has a shear yield strength $\tau_Y = 190$ MPa and a shear modulus $G = 27.0$ GPa. The length of the member is $L = 2.0$ m. The outer diameter of the shaft is $D_o = 60.0$ mm and the inner diameter is $D_i = 40.0$ mm. Two design criteria are specified for the shaft. First, the factor of safety against general yielding must be at least $SF = 2.0$. Second, the angle of twist must not exceed 0.20 rad. Determine the maximum allowable design torque T for the shaft.

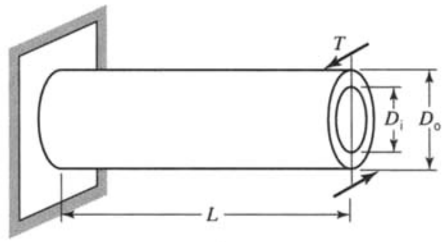

FIGURE E6.3

Solution

(a) Consider first the case of general yielding. At general yielding, the maximum shear stress in the shaft must be equal to the shear yield strength $\tau_Y = 190$ MPa. Hence, by Eq. 6.15, the design torque T is

$$(SF)T = \frac{\tau_Y J}{D_o/2}$$

or

$$T = \frac{(190 \times 10^6)J}{(0.060)}$$

By Figure E6.3,

$$J = \frac{\pi}{2}\left[\left(\frac{D_o}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4\right] = \frac{\pi}{2}\left[(0.030)^4 - (0.020)^4\right]$$

or

$$J = 1.021 \times 10^{-6} \mathrm{m}^4$$

Hence,

$$T = 3.233 \text{ kN} \bullet \text{m}$$

(b) For a limiting angle of twist of $\psi = \theta L = 0.20$ rad, the design torque is obtained by Eq. 6.12 as

$$T = GJ\theta = \frac{(27 \times 10^9)(1.021 \times 10^{-6})(0.20)}{2.0}$$

or

$$T = 2.757 \text{ kN} \bullet \text{m}$$

Thus, the required design torque is limited by the angle of twist and is $T = 2.757$ kN • m.

EXAMPLE 6.4
Solid Shaft with Abrupt Change in Cross Section

The torsion member shown in Figure E6.4*a* is made of steel (G = 77.5 GPa) and is subjected to torsional loads as shown. Neglect the effect of stress concentrations at the abrupt change in cross section at section B and assume that the material remains elastic.

(a) Determine the maximum shear stress in the member.

(b) Determine the angle of twist ψ of sections A, B, and C, relative to the left end O of the member.

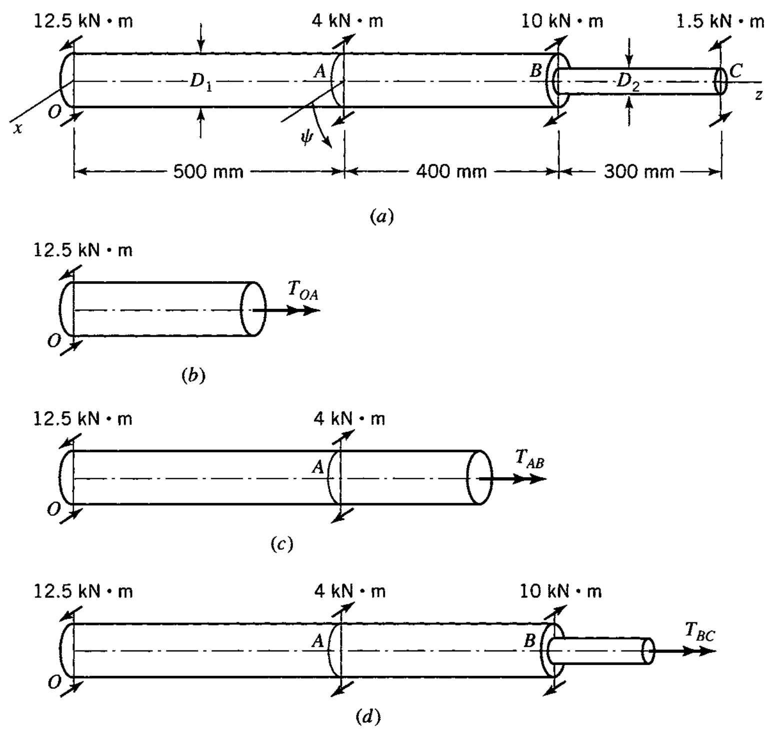

FIGURE E6.4

Solution

(a) Note that the member is in torsional equilibrium and that the twisting moment is constant in the segments OA, AB, and BC of the member. The moments in segments OA, AB, and BC are obtained by moment equilibrium with the free-body diagrams shown in Figures E6.4*b*, *c*, and *d*. Thus,

$$T_{OA} = -12.5 \text{ kN} \cdot \text{m}$$
$$T_{AB} = -8.5 \text{ kN} \cdot \text{m}$$
$$T_{BC} = 1.5 \text{ kN} \cdot \text{m}$$

Since the magnitude of T_{OA} is larger than that of T_{AB}, the maximum shear stress in the segment OAB occurs in segment OA. Hence, the maximum shear stress in the member occurs either in segment OA or segment BC. In segment OA, by Eq. 6.15,

$$\tau_{OA} = \frac{T_{OA}(D_1/2)}{J_{OA}} = 63.66 \text{ MPa}$$

In length BC, by Eq. 6.15,

$$\tau_{BC} = \frac{T_{BC}(D_2/2)}{J_{BC}} = 61.12 \text{ MPa}$$

Hence, the maximum shear stress in the member is $\tau = 63.66$ MPa in segment OA.

(b) The angle of twist is given by Eq. 6.12 as

$$\psi = \theta L = \frac{TL}{GJ} \tag{a}$$

where the positive direction of rotation is shown in Figure E6.4*a*. For section A, Eq. (a) yields

$$\psi_A = \frac{T_{OA}L_{OA}}{GJ_{OA}} = -0.00821 \text{ rad} \tag{b}$$

The negative sign for ψ_A indicates that section A rotates clockwise relative to section O.

For section B, the angle of twist is

$$\psi_B = \psi_A + \psi_{BA} \tag{c}$$

where ψ_{BA} is the angle of twist of section B relative to section A. Thus by Eqs. (a)–(c)

$$\psi_B = -0.01268 \text{ rad} \tag{d}$$

Similarly, the angle of twist of section C is

$$\psi_C = \psi_B + \psi_{CB} \tag{e}$$

where ψ_{CB} is the angle of twist of section C relative to section B. Thus by Eqs. (a), (d), and (e)

$$\psi_C = -0.00322 \text{ rad}$$

In summary, the angle of twist at section A is 0.00821 rad clockwise relative to section O, and the angles of twist at sections B and C are 0.01268 rad and 0.00322 rad, both clockwise relative to section O.

EXAMPLE 6.5 *Shaft-Speed Reducer System*

A solid shaft is frequently used to transmit power to a speed reducer and then from the speed reducer to other machines. For example, assume that an input power of 100 kW at a frequency of 100 Hz is transmitted by a solid shaft of diameter D_{in} to a speed reducer. The frequency is reduced to 10 Hz and the output power is transferred to a solid shaft of diameter D_{out}. Both input and output shafts are made of a ductile steel ($\tau_Y = 220$ MPa). A safety factor of $SF = 2.5$ is specified for the design of each shaft. The output power is also 100 kW, since the speed reducer is highly efficient. Determine the diameters of the input and output shafts. Assume that fatigue is negligible.

Solution

Since fatigue is not significant, general yielding is the design failure mode. By the relation among power, frequency, and torque, the input torque T_{in} and the output torque T_{out} are, respectively,

$$T_{in} = \frac{P}{2\pi f_{in}} = 159.2 \text{ N} \cdot \text{m} \tag{a}$$

$$T_{out} = \frac{P}{2\pi f_{out}} = 1592 \text{ N} \cdot \text{m} \tag{b}$$

For a safety factor of 2.5, the diameters D_{in} and D_{out} are given by Eqs. 6.15, (a), and (b) as follows:

$$(SF)T_{in} = \frac{\tau_Y J_{in}}{R_{in}} = \frac{\tau_Y \pi \left(D_{in}^4\right)/32}{(D_{in}/2)}$$

$$(SF)T_{out} = \frac{\tau_Y J_{out}}{R_{out}} = \frac{\tau_Y \pi \left(D_{out}^4\right)/32}{(D_{out}/2)}$$

Therefore,

$$D_{in} = 20.96 \text{ mm}$$
$$D_{out} = 45.17 \text{ mm}$$

Note that although the two shafts transmit the same power, the high-speed shaft has a much smaller diameter. So, if weight is to be kept to a minimum, power should be transmitted at the highest possible frequency. Weight can also be reduced by using a hollow shaft.

6.2 SAINT-VENANT'S SEMIINVERSE METHOD

The analysis for the torsion of noncircular cross sections proceeds in much the same fashion as for circular cross sections. However, in the case of noncircular cross sections, Saint-Venant assumed more generally that w is a function of (x, y), the cross-section coordinates. Then, the cross section does not remain plane but *warps*; that is, different points in the cross section, in general, undergo different displacements in the z direction.

Consider a torsion member with a uniform cross section of general shape as shown in Figure 6.7. Axes (x, y, z) are taken as for the circular cross section (Figure 6.1). The applied shear stress distribution on the ends $(\sigma_{zx}, \sigma_{zy})$ produces a torque **T**. In general, any number of stress distributions on the end sections may produce a torque **T**. According to Saint-Venant's principle, the stress distribution on sections sufficiently far removed from the ends depends principally on the magnitude of **T** and not on the stress distribution on the ends. Thus, for sufficiently long torsion members, the end stress distribution does not affect the stress distributions in a large part of the member.

In Saint-Venant's semiinverse method we start by approximating the displacement components resulting from torque **T**. This approximation is based on observed geometric changes in the deformed torsion member.

6.2.1 Geometry of Deformation

As with circular cross sections, Saint-Venant assumed that every straight torsion member with constant cross section (relative to axis z) has an axis of twist, about which each cross section rotates approximately as a rigid body. Let the z axis in Figure 6.7 be the axis of twist.

For the torsion member in Figure 6.7, let $0A$ and $0B$ be line segments in the cross section for $z = 0$, which coincide with the x and y axes, respectively. After deformation, by rigid-body displacements, we may translate the new position of 0, that is, 0*, back to

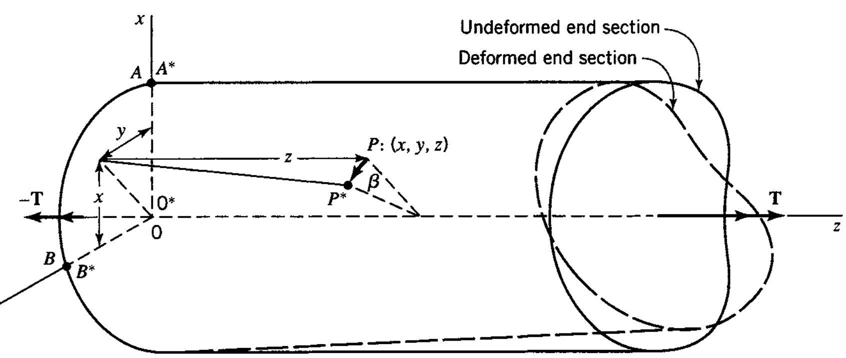

FIGURE 6.7 Torsion member.

coincide with 0, align the axis of twist along the z axis, and rotate the deformed torsion member until the projection of 0^*A^* on the (x, y) plane coincides with the x axis. Because of the displacement (w displacement) of points in each cross section, 0^*A^* does not, in general, lie in the (x, y) plane. However, the amount of warping is small for small displacements; therefore, line $0A$ and curved line 0^*A^* are shown as coinciding in Figure 6.7. Experimental evidence indicates also that the distortion of each cross section in the z direction is essentially the same. This distortion is known as *warping*. Furthermore, experimental evidence indicates that the cross-sectional dimensions of the torsion member are not changed significantly by the deformations, particularly for small displacements. In other words, deformation in the plane of the cross section is negligible. Hence, the projection of 0^*B^* on the (x, y) plane coincides approximately with the y axis, indicating that ϵ_{xy} ($\gamma_{xy} = 2\epsilon_{xy}$) is approximately zero (see Section 2.7, particularly, Eq. 2.74).

Consider a point P with coordinates (x, y, z) in the undeformed torsion member (Figure 6.7). Under deformation, P goes into P^*. The point P, in general, is displaced by an amount w parallel to the z axis because of the warping of the cross section and by amounts u and v parallel to the x and y axes, respectively. The cross section in which P lies rotates through an angle β with respect to the cross section at the origin. This rotation is the principal cause of the (u, v) displacements of point P. These observations led Saint-Venant to assume that $\beta = \theta z$, where θ is the angle of twist per unit length, and therefore that the displacement components take the form

$$u = -\theta yz, \quad v = \theta xz, \quad w = \theta\psi(x, y) \tag{6.16}$$

where ψ is the warping function (compare Eqs. 6.16 for the general cross section with Eqs. 6.5 for the circular cross section). The function $\psi(x, y)$ may be determined such that the equations of elasticity are satisfied. Since we have assumed continuous displacement components (u, v, w), the small-displacement compatibility conditions (Eqs. 2.83) are automatically satisfied.

The state of strain at a point in the torsion member is given by substitution of Eqs. 6.16 into Eqs. 2.81 to obtain

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = 0 \\
2\epsilon_{zx} &= \gamma_{zx} = \theta\left(\frac{\partial\psi}{\partial x} - y\right) \\
2\epsilon_{zy} &= \gamma_{zy} = \theta\left(\frac{\partial\psi}{\partial y} + x\right)
\end{aligned} \tag{6.17}$$

If the equation for γ_{zx} is differentiated with respect to y, the equation for γ_{zy} is differentiated with respect to x, and the second of these resulting equations is subtracted from the first, the warping function ψ may be eliminated to give the relation

$$\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{zy}}{\partial x} = -2\theta \tag{6.18}$$

If the torsion problem is formulated in terms of $(\gamma_{zx}, \gamma_{zy})$, Eq. 6.18 is a geometrical condition (compatibility condition) to be satisfied for the torsion problem.

6.2.2 Stresses at a Point and Equations of Equilibrium

For torsion members made of isotropic materials, stress–strain relations for either elastic (the first of Eqs. 6.17 and Eqs. 3.32) or inelastic conditions indicate that

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0 \tag{6.19}$$

The stress components (σ_{zx}, σ_{zy}) are nonzero. If body forces and acceleration terms are neglected, these stress components may be substituted into Eqs. 2.45 to obtain equations of equilibrium for the torsion member:[1]

$$\frac{\partial \sigma_{zx}}{\partial z} = 0 \tag{6.20}$$

$$\frac{\partial \sigma_{zy}}{\partial z} = 0 \tag{6.21}$$

$$\frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \tag{6.22}$$

Equations 6.20 and 6.21 indicate that $\sigma_{zx} = \sigma_{xz}$ and $\sigma_{zy} = \sigma_{yz}$ are independent of z. These stress components must satisfy Eq. 6.22, which expresses a necessary and sufficient condition for the existence of a stress function $\phi(x, y)$ (the so-called Prandtl stress function) such that

$$\begin{aligned} \sigma_{zx} &= \frac{\partial \phi}{\partial y} \\ \sigma_{zy} &= -\frac{\partial \phi}{\partial x} \end{aligned} \tag{6.23}$$

Thus, the torsion problem is transformed into the determination of the stress function ϕ. Boundary conditions put restrictions on ϕ.

6.2.3 Boundary Conditions

Because the lateral surface of a torsion member is free of applied stress, the resultant shear stress τ on the surface S of the cross section must be directed tangent to the surface (Figures 6.8*a* and *b*). The two shear stress components σ_{zx} and σ_{zy} that act on the cross-sectional element with sides dx, dy, and ds may be written in terms of $\boldsymbol{\tau}$ (Figure 6.8*b*) in the form

$$\begin{aligned} \sigma_{zx} &= \tau \sin \alpha \\ \sigma_{zy} &= \tau \cos \alpha \end{aligned} \tag{6.24}$$

where, according to Figure 6.8*a*,

$$\sin \alpha = \frac{dx}{ds}, \quad \cos \alpha = \frac{dy}{ds} \tag{6.25}$$

Since the component of $\boldsymbol{\tau}$ in the direction of the normal $\mathbf{n}$ to the surface S is zero, projections of $\boldsymbol{\sigma}_{zx}$ and $\boldsymbol{\sigma}_{zy}$ in the normal direction (Figure 6.8*b*) yield, with Eq. 6.25,

$$\begin{aligned} \sigma_{zx} \cos \alpha - \sigma_{zy} \sin \alpha &= 0 \\ \sigma_{zx} \frac{dy}{ds} - \sigma_{zy} \frac{dx}{ds} &= 0 \end{aligned} \tag{6.26}$$

[1]This approach was taken by Prandtl. See Section 7.3 of Boresi and Chong (2000).

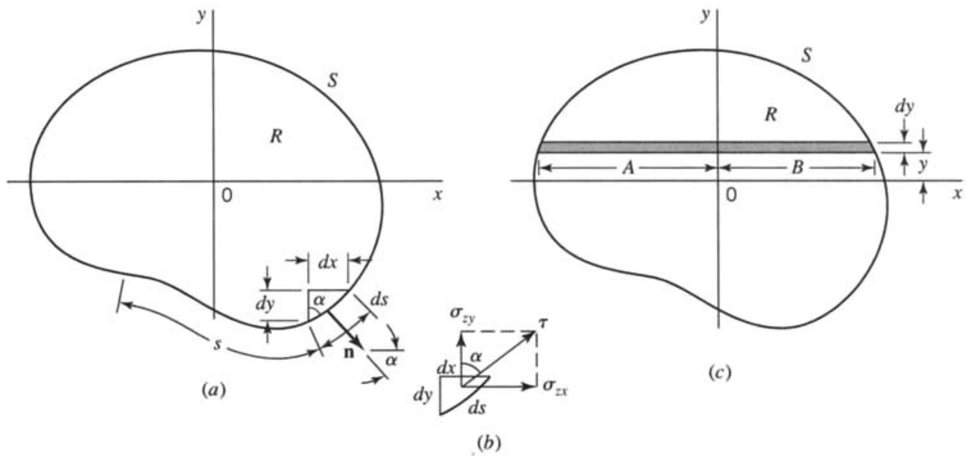

FIGURE 6.8 Cross section of a torsion member.

Substituting Eqs. 6.23 into Eq. 6.26, we find

$$\frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} = \frac{d\phi}{ds} = 0$$

or

$$\phi = \text{constant on the boundary } S \tag{6.27}$$

Since the stresses are given by partial derivatives of ϕ (see Eqs. 6.23), it is permissible to take this constant to be zero; thus, we select

$$\phi = 0 \text{ on the boundary } S \tag{6.28}$$

The preceding argument can be used to show that the shear stress

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} \tag{6.29}$$

at any point in the cross section is directed tangent to the contour ϕ = constant through the point.

The distributions of σ_{zx} and σ_{zy} on a given cross section must satisfy the following equations:

$$\sum F_x = 0 = \int \sigma_{zx}\, dx\, dy = \int \frac{\partial\phi}{\partial y}\, dx\, dy \tag{6.30}$$

$$\sum F_y = 0 = \int \sigma_{zy}\, dx\, dy = -\int \frac{\partial\phi}{\partial x}\, dx\, dy \tag{6.31}$$

$$\begin{aligned}\sum M_z = T &= \int (x\sigma_{zy} - y\sigma_{zx})\, dx\, dy \\ &= -\int \left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} \right) dx\, dy\end{aligned} \tag{6.32}$$

In satisfying the second equilibrium equation, consider the strip across the cross section of thickness dy as indicated in Figure 6.8*c*. Because the stress function does not vary in the y direction for this strip, the partial derivative can be replaced by the total derivative. For the strip, Eq. 6.31 becomes

$$
\begin{aligned}
dy\int\frac{\partial\phi}{\partial x}\,dx &= dy\int\frac{d\phi}{dx}\,dx = dy\int_{\phi(A)}^{\phi(B)} d\phi \\
&= dy[\phi(B) - \phi(A)] = 0
\end{aligned}
\tag{6.33}
$$

since ϕ is equal to zero on the boundary. The same is true for every strip so that $\sum F_y = 0$ is satisfied. In a similar manner, Eq. 6.30 is verified. In Eq. 6.32, consider the term

$$
-\int x\frac{\partial\phi}{\partial x}\,dx\,dy
$$

which for the strip in Figure 6.8*c* becomes

$$
-dy\int x\frac{d\phi}{dx}\,dx = -dy\int_{\phi(A)}^{\phi(B)} x\,d\phi \tag{6.34}
$$

Evaluating the latter integral by parts and noting that $\phi(B) = \phi(A) = 0$, we obtain

$$
-dy\int_{\phi(A)}^{\phi(B)} x\,d\phi = -dy\left(x\phi\Big|_A^B - \int_{x_A}^{x_B}\phi\,dx\right) = dy\int_{x_A}^{x_B}\phi\,dx \tag{6.35}
$$

Summing for the other strips and repeating the process using strips of thickness dx for the other term in Eq. 6.32, we obtain the relation

$$
T = 2\iint\phi\,dx\,dy \tag{6.36}
$$

The stress function ϕ can be considered to represent a surface over the cross section of the torsion member. This surface is in contact with the boundary of the cross section (see Eq. 6.28). Hence, Eq. 6.36 indicates that the torque is equal to twice the volume between the stress function and the plane of the cross section.

Note: Equations 6.18, 6.23, 6.28, and 6.36, as well as other equations in this section, have been derived for torsion members that have uniform cross sections that do not vary with z, that have simply connected cross sections, that are made of isotropic materials, and that are loaded so that deformations are small. These equations are used to obtain solutions for torsion members; they do not depend on any assumption regarding material behavior except that the material is isotropic; therefore, they are valid for any specified material response (elastic or inelastic).

Two types of typical material response are considered in this chapter: linearly elastic response and elastic–perfectly plastic response (Figure 4.4*a*). The linearly elastic response leads to the linearly elastic solution of torsion, whereas the elastic–perfectly plastic response leads to the fully plastic solution of torsion of a bar for which the entire cross section yields. The material properties associated with various material responses are determined by appropriate tests. Usually, as noted in Chapter 4, we assume that the material properties are determined by either a tension test or torsion test of a cylinder with thin-wall annular cross section.

6.3 LINEAR ELASTIC SOLUTION

Stress–strain relations for linear elastic behavior of an isotropic material are given by Hooke's law (see Eqs. 3.32). By Eqs. 3.32 and 6.23, we obtain

$$\sigma_{zx} = \frac{\partial\phi}{\partial y} = G\gamma_{zx}$$
$$\sigma_{zy} = -\frac{\partial\phi}{\partial x} = G\gamma_{zy} \tag{6.37}$$

Substitution of Eqs. 6.37 into Eq. 6.18 yields

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta \tag{6.38}$$

If the unit angle of twist θ is specified for a given torsion member and ϕ satisfies the boundary condition indicated by Eq. 6.28, then Eq. 6.38 uniquely determines the stress function $\phi(x, y)$. Once ϕ has been determined, the stresses are given by Eqs. 6.23 and the torque is given by Eq. 6.36. The elasticity solution of the torsion problem for many practical cross sections requires special methods (Boresi and Chong, 2000) for determining the function ϕ and is beyond the scope of this book. As indicated in the following paragraphs, an indirect method may be used to obtain solutions for certain types of cross sections, although it is not a general method.

Let the boundary of the cross section for a given torsion member be specified by the relation

$$F(x, y) = 0 \tag{6.39}$$

Furthermore, let the torsion member be subjected to a specified unit angle of twist and define the stress function by the relation

$$\phi = BF(x, y) \tag{6.40}$$

where B is a constant. This stress function is a solution of the torsion problem, provided $F(x, y) = 0$ on the lateral surface of the bar and

$$\partial^2F/\partial x^2 + \partial^2F/\partial y^2 = \text{constant}$$

Then, the constant B may be determined by substituting Eq. 6.40 into Eq. 6.38. With B determined, the stress function ϕ for the torsion member is uniquely defined by Eq. 6.40. This indirect approach may, for example, be used to obtain the solutions for torsion members whose cross sections are in the form of a circle, an ellipse, or an equilateral triangle.

6.3.1 Elliptical Cross Section

Let the cross section of a torsion member be bounded by an ellipse (Figure 6.9). The stress function ϕ for the elliptical cross section may be written in the form

$$\phi = B\left(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1\right) \tag{6.41}$$

since $F(x, y) = x^2/h^2 + y^2/b^2 - 1 = 0$ on the boundary (Eq. 6.39). Substituting Eq. 6.41 into Eq. 6.38, we obtain

$$B = -\frac{h^2b^2G\theta}{h^2 + b^2} \tag{6.42}$$

in terms of the geometrical parameters (h, b), shear modulus G, and unit angle of twist θ. With ϕ determined, the shear stress components for the elliptical cross section are, by Eqs. 6.23,

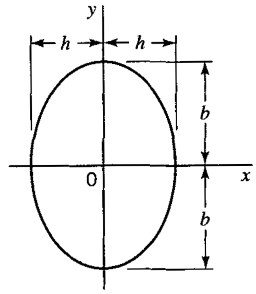

FIGURE 6.9 Ellipse.

$$\sigma_{zx} = \frac{\delta\phi}{\partial y} = \frac{2By}{b^2} = -\frac{2h^2G\theta y}{h^2+b^2} \tag{6.43}$$

$$\sigma_{zy} = -\frac{\partial\phi}{\partial x} = -\frac{2Bx}{h^2} = \frac{2b^2G\theta x}{h^2+b^2} \tag{6.44}$$

The maximum shear stress τ_{max} occurs at the boundary nearest the centroid of the cross section. Its value is

$$\tau_{max} = \sigma_{zy(x=h)} = \frac{2b^2hG\theta}{h^2+b^2} \tag{6.45}$$

The torque T for the elliptical cross section torsion member is obtained by substituting Eq. 6.41 into Eq. 6.36. Thus, we obtain

$$T = \frac{2B}{h^2}\int x^2dA + \frac{2B}{b^2}\int y^2dA - 2B\int dA = \frac{2B}{h^2}I_y + \frac{2B}{b^2}I_x - 2BA$$

Determination of I_x, I_y and A in terms of (b, h) allows us to write

$$T = -\pi Bhb \tag{6.46}$$

The torque may be expressed either in terms of τ_{max} or θ by means of Eqs. 6.42, 6.45, and 6.46. Thus,

$$\tau_{max} = \frac{2T}{\pi bh^2}, \quad \theta = \frac{T(b^2+h^2)}{G\pi b^3h^3} \tag{6.47}$$

where $G\pi b^3h^3/(b^2+h^2) = GJ$ is called the torsional rigidity (stiffness) of the section and the torsional constant for the cross section is

$$J = \pi b^3h^3/(b^2+h^2)$$

6.3.2 Equilateral Triangle Cross Section

Let the boundary of a torsion member be an equilateral triangle (Figure 6.10). The stress function is given by the relation

$$\phi = \frac{G\theta}{2h}\left(x - \sqrt{3}y - \frac{2h}{3}\right)\left(x + \sqrt{3}y - \frac{2h}{3}\right)\left(x + \frac{h}{3}\right) \tag{6.48}$$

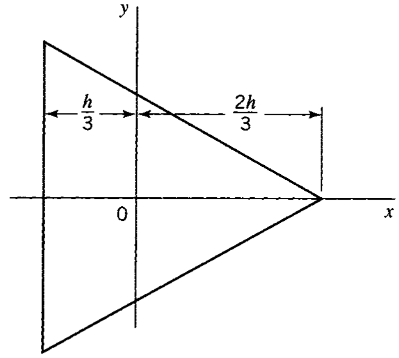

FIGURE 6.10 Equilateral triangle.

Proceeding as for the elliptical cross section, we find

$$\tau_{max} = \frac{15\sqrt{3}T}{2h^3}, \quad \theta = \frac{15\sqrt{3}T}{Gh^4} \tag{6.49}$$

where $Gh^4/15\sqrt{3} = GJ$ is called the torsional rigidity of the section. Hence, the torsional constant for the cross section is

$$J = h^4/(15\sqrt{3})$$

6.3.3 Other Cross Sections

There are many torsion members whose cross sections are so complex that exact analytical solutions are difficult to obtain. However, approximate solutions may be obtained by Prandtl's membrane analogy (see Section 6.4). An important class of torsion members includes those with thin walls. Included in the class of thin-walled torsion members are open and box-sections. Approximate solutions for these types of section are obtained in Sections 6.5 and 6.7 by means of the Prandtl membrane analogy.

6.4 THE PRANDTL ELASTIC-MEMBRANE (SOAP-FILM) ANALOGY

In this section, we consider a solution of the torsion problem by means of an analogy proposed by Prandtl (1903). The method is based on the similarity of the equilibrium equation for a membrane subjected to lateral pressure and the torsion (stress function) equation (Eq. 6.38). Although this method is of historical interest, it is rarely used today to obtain quantitative results. It is discussed here primarily from a heuristic viewpoint, in that it is useful in the visualization of the distribution of shear-stress components in the cross section of a torsion member.

To set the stage for our discussion, consider an opening in the (x, y) plane that has the same shape as the cross section of the torsion bar to be investigated. Cover the opening with a homogeneous elastic membrane, such as a soap film, and apply pressure to one side of the membrane. The pressure causes the membrane to bulge out of the (x, y) plane, forming a curved surface. If the pressure is small, the slope of the membrane will also be small. Then, the lateral displacement $z(x, y)$ of the membrane and the Prandtl torsion stress function $\phi(x, y)$ satisfy the same equation in (x, y). Hence, the displacement $z(x, y)$ of the membrane is mathematically equivalent to the stress function $\phi(x, y)$, provided that $z(x, y)$ and

$\phi(x, y)$ satisfy the same boundary conditions. This condition requires the boundary shape of the membrane to be identical to the boundary shape of the cross section of the torsion member. In the following discussion, we outline the physical and mathematical procedures that lead to a complete analogy between the membrane problem and the torsion problem.

As already noted, the Prandtl membrane analogy is based on the equivalence of the torsion equation (Eq. 6.38 is repeated here for convenience)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \tag{6.50}$$

and the elastic membrane equation (to be derived in the next paragraph)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{S}$$

where z denotes the lateral displacement of an elastic membrane subjected to a lateral pressure p in terms of force per unit area and an initial (large) tension S (Figure 6.11) in terms of force per unit length.

For the derivation of the elastic membrane equation, consider an element $ABCD$ of dimensions dx, dy of the elastic membrane shown in Figure 6.11. The net vertical force resulting from the tension S acting along edge AD of the membrane is (if we assume small displacements so that $\sin \alpha \approx \tan \alpha$)

$$-S\,dy \sin\alpha \approx -S\,dy \tan\alpha = -S\,dy\frac{\partial z}{\partial x}$$

and, similarly, the net vertical force resulting from the tension S (assumed to remain constant for sufficiently small values of p) acting along edge BC is

$$S\,dy \tan\left(\alpha + \frac{\partial \alpha}{\partial x}dx\right) = S\,dy\frac{\partial}{\partial x}\left(z + \frac{\partial z}{\partial x}dx\right)$$

Similarly, for edges AB and DC, we obtain

$$-S\,dx\frac{\partial z}{\partial y}, \quad S\,dx\frac{\partial}{\partial y}\left(z + \frac{\partial z}{\partial y}dy\right)$$

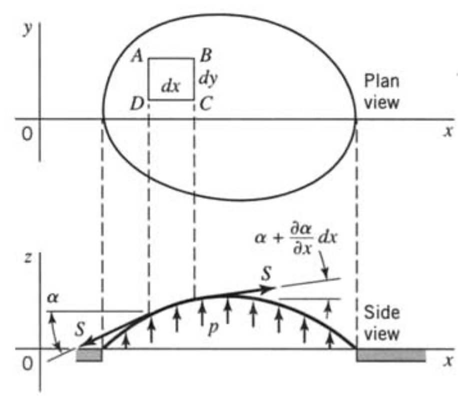

FIGURE 6.11 Deformation of a pressurized elastic membrane.

Consequently, the summation of force in the vertical direction yields for the equilibrium of the membrane element $dx\,dy$

$$S\frac{\partial^2 z}{\partial x^2}dx\,dy + S\frac{\partial^2 z}{\partial y^2}dx\,dy + p\,dx\,dy$$

or

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{S} \tag{6.51}$$

By comparison of Eqs. 6.50 and 6.51, we arrive at the following analogous quantities:

$$z = c\phi, \quad \frac{p}{S} = c2G\theta \tag{6.52}$$

where c is a constant of proportionality. Hence,

$$\frac{z}{p/S} = \frac{\phi}{2G\theta}, \quad \phi = \frac{2G\theta S}{p}z \tag{6.53}$$

Accordingly, the membrane displacement z is proportional to the Prandtl stress function ϕ, and since the shear-stress components σ_{zx}, σ_{zy} are equal to the appropriate derivatives of ϕ with respect to x and y (see Eqs. 6.23), it follows that the stress components are proportional to the derivatives of the membrane displacement z with respect to the (x, y) coordinates (Figure 6.11). In other words, the stress components at a point (x, y) of the bar are proportional to the slopes of the membrane at the corresponding point (x, y) of the membrane. Consequently, the distribution of shear-stress components in the cross section of the bar is easily visualized by forming a mental image of the slope of the corresponding membrane. Furthermore, for simply connected cross sections,[2] since z is proportional to ϕ, by Eqs. 6.36 and 6.53, we note that the twisting moment T is proportional to the volume enclosed by the membrane and the (x, y) plane (Figure 6.11). For multiply connected cross section, additional conditions arise (Section 6.6; see also Boresi and Chong, 2000).

An important aspect of the elastic membrane analogy is that valuable deductions can be made by merely visualizing the shape that the membrane must take. For example, if a membrane covers holes machined in a flat plate, the corresponding torsion members have equal values of $G\theta$; therefore, the stiffnesses (see Eqs. 6.47 and 6.49) of torsion members made of materials having the same G are proportional to the volumes between the membranes and flat plate. For cross sections with equal area, one can deduce that a long narrow rectangular section has the least stiffness and a circular section has the greatest stiffness.

Important conclusions may also be drawn with regard to the magnitude of the shear stress and hence to the cross section for minimum shear stress. Consider the angle section shown in Figure 6.12*a*. At the external corners A, B, C, E, and F, the membrane has zero slope and the shear stress is zero; therefore, external corners do not constitute a design problem. However, at the reentrant corner at D (shown as a right angle in Figure 6.12*a*), the corresponding membrane would have an infinite slope, which indicates an infinite

[2]A region R is *simply connected* if every closed curve within it or on its boundary encloses only points in R. For example, the solid cross section in Figure 6.8*a* (region R) is simply connected (as are all the cross sections in Section 6.3), since any closed curve in R or on its boundary contains only points in R. However, a region R between two concentric circles is not simply connected (see Figure 6.5), since its inner boundary $r = a$ encloses points not in R. A region or cross section that is not simply connected is called *multiply connected*.

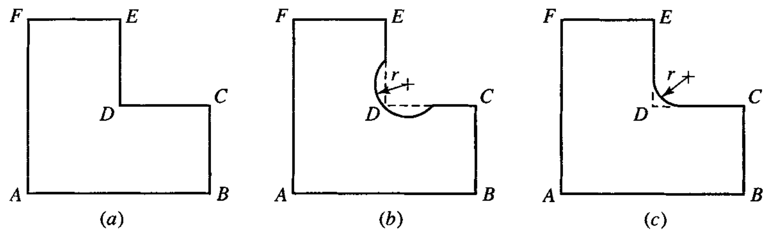

FIGURE 6.12 Angle sections of a torsion member. (*a*) Poor. (*b*) Better. (*c*) Best.

shear stress in the torsion member. In practical problems, the magnitude of the shear stress at D would be finite but very large compared to that at other points in the cross section.

6.4.1 Remark on Reentrant Corners

If a torsion member with cross section shown in Figure 6.12*a* is made of a ductile material and it is subjected to static loads, the material in the neighborhood of D yields and the load is redistributed to adjacent material, so that the stress concentration at point D is not particularly important. If, however, the material is brittle or the torsion member is subjected to fatigue loading, the shear stress at D limits the load-carrying capacity of the member. In such a case, the maximum shear stress in the torsion member may be reduced by removing some material as shown in Figure 6.12*b*. Preferably, the member should be redesigned to alter the cross section (Figure 6.12*c*). The maximum shear stress would then be about the same for the two cross sections shown in Figures 6.12*b* and 6.12*c* for a given unit angle of twist; however, a torsion member with the cross section shown in Figure 6.12*c* would be stiffer for a given unit angle of twist.

6.5 NARROW RECTANGULAR CROSS SECTION

The cross sections of many members of machines and structures are made up of narrow rectangular parts. These members are used mainly to carry tension, compression, and bending loads. However, they may be required also to carry secondary torsional loads. For simplicity, we use the elastic membrane analogy to obtain the solution of a torsion member whose cross section is in the shape of a narrow rectangle.

Consider a bar subjected to torsion. Let the cross section of the bar be a solid rectangle with width $2h$ and depth $2b$, where $b >> h$ (Figure 6.13). The associated membrane is shown in Figure 6.14.

Except for the region near $x = \pm b$, the membrane deflection is approximately independent of x. Hence, if we assume that the membrane deflection is independent of x and parabolic with respect to y, the displacement equation of the membrane is

$$z = z_0\left[1 - \left(\frac{y}{h}\right)^2\right] \tag{6.54}$$

where z_0 is the maximum deflection of the membrane. Note that Eq. 6.54 satisfies the condition $z = 0$ on the boundaries $y = \pm h$. Also, if p/S is a constant in Eq. 6.51, the parameter z_0 may be selected so that Eq. 6.54 represents a solution of Eq. 6.51. Consequently, Eq. 6.54 is an approximate solution of the membrane displacement. By Eq. 6.54 we find

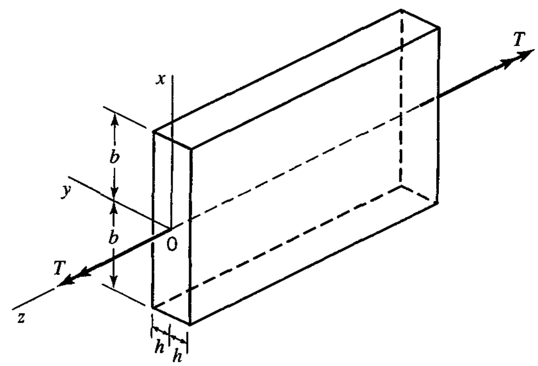

FIGURE 6.13 Narrow rectangular torsion member.

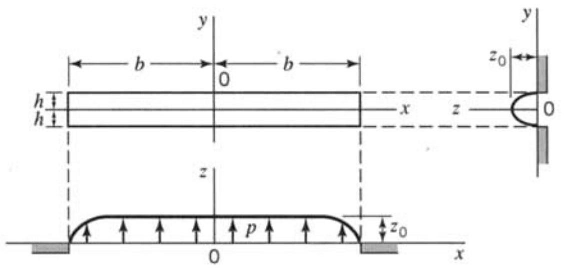

FIGURE 6.14 Membrane for narrow rectangular cross section.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{2z_0}{h^2} \tag{6.55}$$

By Eqs. 6.55, 6.51, and 6.52, we may write $-2z_0/h^2 = -2cG\theta$ and Eq. 6.54 becomes

$$\phi = G\theta h^2\left[1 - \left(\frac{y}{h}\right)^2\right] \tag{6.56}$$

Consequently, Eqs. 6.23 yield

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = -2G\theta y, \quad \sigma_{zy} = -\frac{\partial \phi}{\partial x} = 0 \tag{6.57}$$

and we note that the maximum value of σ_{zx} is

$$\tau_{max} = 2G\theta h, \quad \text{for } y = \pm h \tag{6.58}$$

Equations 6.36 and 6.56 yield

$$T = 2\int_{-b}^{b}\int_{-h}^{h} \phi \, dx \, dy = \frac{1}{3}G\theta(2b)(2h)^3 = GJ\theta \tag{6.59}$$

where

$$J = \frac{1}{3}(2b)(2h)^3 \tag{6.60}$$

is the torsional constant and GJ is the torsional rigidity. Note that the torsional constant J is small compared to the polar moment of inertia $J_O = [(2b)(2h)^3 + (2h)(2b)^3]/12$; see Table B.1.

In summary, we note that the solution is approximate and, in particular, the boundary condition for $x = \pm b$ is not satisfied. From Eqs. 6.58 and 6.59 we obtain

$$\tau_{max} = \frac{3T}{(2b)(2h)^2} = \frac{2Th}{J}, \quad \theta = \frac{3T}{G(2b)(2h)^3} = \frac{T}{GJ} \tag{6.61}$$

6.5.1 Cross Sections Made Up of Long Narrow Rectangles

Many rolled composite sections are made up of joined long narrow rectangles. For these cross sections, it is convenient to define the torsional constant J by the relation

$$J = C\frac{1}{3}\sum_{i=1}^{n}(2b_i)(2h_i)^3 \tag{6.62}$$

where C is a correction coefficient. If $b_i > 10h_i$ for each rectangular part of the composite cross section (see Table 6.1 in Section 6.6), then $C \approx 1$. For many rolled sections, b_i may be less than $10h_i$ for one or more of the rectangles making up the cross section. In this case, it is recommended that $C = 0.91$. When $n = 1$ and $b > 10h$, $C = 1$ and Eq. 6.62 is identical to Eq. 6.60. For $n > 1$, Eqs. 6.61 take the form

$$\tau_{max} = \frac{2Th_{max}}{J}, \quad \theta = \frac{T}{GJ} \tag{6.63}$$

where h_{max} is the maximum value of the h_i.

Cross-sectional properties for typical torsion members are given in the manual published by the American Institute of Steel Construction, Inc. (AISC, 1997). The formulas for narrow rectangular cross sections may also be used to approximate narrow curved members. See Example 6.6.

EXAMPLE 6.6 ***Torsion of a Member with Narrow Semicircular Cross Section***

Consider a torsion member of narrow semicircular cross section (Figure E6.6), with constant thickness $2h$ and mean radius a. The mean circumference is $2b = \pi a$. We consider the member to be equivalent to a slender rectangular member of dimension $2h \times \pi a$. Then, for a twisting moment T applied to the member, by Eqs. 6.61, we approximate the maximum shear stress and angle of twist per unit length as follows:

$$\tau_{max} = \frac{2Th}{J}, \quad J = \frac{\pi a}{3}(2h)^3$$

Hence,

$$\tau_{max} = \frac{3T}{4\pi a h^2} \quad \text{and} \quad \theta = \frac{T}{GJ} = \frac{3T}{8\pi G a h^3}$$

Alternatively, we may express θ in terms of τ_{max} as $\theta = \tau_{max}/2Gh$.

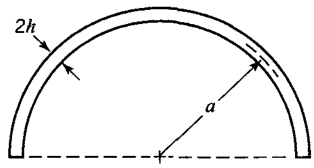

FIGURE E6.6

6.6 TORSION OF RECTANGULAR CROSS SECTION MEMBERS

In Section 6.5 the problem of a torsion bar with narrow rectangular cross section was approximated by noting the deflection of the corresponding membrane. In this section we again consider the rectangular section of width $2h$ and depth $2b$, but we discard the restriction that $h << b$ (Figure 6.15).

By visualizing the membrane corresponding to the cross section in Figure 6.15, we note that the torsion stress function ϕ must be even in both x and y. Also, from Eqs. 6.38 and 6.28, the torsion problem is defined by

$$\begin{aligned} \nabla^2\phi &= -2G\theta \quad \text{over the cross section} \\ \phi &= 0 \quad \text{around the perimeter} \end{aligned} \tag{a}$$

where
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

By Eq. 6.56, we see that $G\theta(h^2 - x^2)$ is a particular solution of the first of Eqs. (a). Accordingly, we take the stress function ϕ in the form

$$\phi = G\theta(h^2 - x^2) + V(x,y) \tag{b}$$

where $V(x, y)$ is an even function of (x, y). Substitution of Eq. (b) into Eqs. (a) yields

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{over the cross section} \\ V &= \begin{cases} 0 & \text{for } x = \pm h \\ G\theta(x^2 - h^2) & \text{for } y = \pm b \end{cases} \end{aligned} \tag{c}$$

We seek solutions of Eqs. (c) by the method of separation of variables. Thus we take

$$V = f(x)\,g(y) \tag{d}$$

where $f(x)$ and $g(y)$ are functions of x and y, respectively. The first of Eqs. (c) and (d) yield

$$\nabla^2 V = gf'' + g''f = 0$$

where primes denote derivatives with respect to x or y. For this equation to be satisfied, we must have

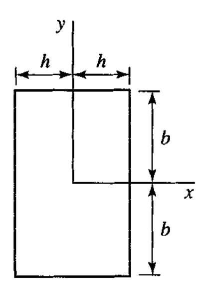

FIGURE 6.15 Rectangular cross section.

$$\frac{f''}{f} = -\frac{g''}{g} = -\lambda^2$$

where λ is a positive constant. Hence,

$$f'' + \lambda^2 f = 0$$
$$g'' - \lambda^2 g = 0$$

The solutions of these equations are

$$f = A\cos\lambda x + B\sin\lambda x$$
$$g = C\cosh\lambda y + D\sinh\lambda y$$

Because V must be even in x and y, it follows that $B = D = 0$. Consequently, from Eq. (d) the function V takes the form

$$V = A\cos\lambda x\cosh\lambda y \tag{e}$$

where A denotes an arbitrary constant.

To satisfy the second of Eqs. (c), Eq. (e) yields the result

$$\lambda = \frac{n\pi}{2h}, \quad n = 1, 3, 5, \ldots$$

To satisfy the last of Eqs. (c) we employ the method of superposition and we write

$$V = \sum_{n=1,3,5,\ldots}^{\infty} A_n \cos\frac{n\pi x}{2h}\cosh\frac{n\pi y}{2h} \tag{f}$$

Equation (f) satisfies $\nabla^2 V = 0$ over the cross-sectional area. Equation (f) also automatically satisfies the boundary condition for $x = \pm h$. The boundary condition for $y = \pm b$ yields the condition [see Eq. (c)]

$$\sum_{n=1,3,5,\ldots}^{\infty} C_n \cos\frac{n\pi x}{2h} = G\theta\left(x^2 - h^2\right) = F(x) \tag{g}$$

where

$$C_n = A_n \cosh\frac{n\pi b}{2h} \tag{h}$$

By the theory of Fourier series, we multiply both sides of Eq. (g) by $\cos(n\pi x/2h)$ and integrate between the limits $-h$ and $+h$ to obtain the coefficients C_n as follows:

$$C_n = \frac{1}{h}\int_{-h}^{h} F(x)\cos\frac{n\pi x}{2h}\,dx$$

Because $F(x)\cos(n\pi x/2h) = G\theta(x^2 - h^2)\cos(n\pi x/2h)$ is symmetrical about $x = 0$, we may write

$$C_n = \frac{2G\theta}{h}\int_0^h \left(x^2 - h^2\right)\cos\frac{n\pi x}{2h}\,dx$$

or

$$C_n = \frac{2G\theta}{h}\int_0^h x^2 \cos\frac{n\pi x}{2h}\,dx - 2G\theta h\int_0^h \cos\frac{n\pi x}{2h}\,dx$$

Integration yields

$$C_n = \frac{-32G\theta h^2(-1)^{(n-1)/2}}{n^3\pi^3} \tag{i}$$

Hence, Eqs. (f), (h), and (i) yield

$$A_n = -\frac{32G\theta h^2(-1)^{(n-1)/2}}{n^3\pi^3\cosh\frac{n\pi b}{2h}}$$

and

$$\phi = G\theta\left(h^2 - x^2\right) - \frac{32G\theta h^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\cos\frac{n\pi x}{2h}\cosh\frac{n\pi y}{2h}}{n^3\cosh\frac{n\pi b}{2h}} \tag{j}$$

Note that since $\cosh x = 1 + x^2/2! + x^4/4! + \cdots$, the series in Eq. (j) goes to zero if $b/h \to \infty$ (that is, if the section is very narrow). Then Eq. (j) reduces to

$$\phi \approx G\theta\left(h^2 - x^2\right)$$

This result verifies the assumption employed in Section 6.5 for the slender rectangular cross section.

By Eqs. 6.23 and (j), we obtain

$$\begin{aligned}
\sigma_{zx} &= \frac{\partial\phi}{\partial y} = -\frac{16G\theta h}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\cos\frac{n\pi x}{2h}\sinh\frac{n\pi y}{2h}}{n^2\cosh\frac{n\pi b}{2h}} \\
\sigma_{zy} &= -\frac{\partial\phi}{\partial x} = 2G\theta x - \frac{16G\theta h}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\sin\frac{n\pi x}{2h}\cosh\frac{n\pi y}{2h}}{n^2\cosh\frac{n\pi b}{2h}}
\end{aligned} \tag{k}$$

By Eqs. 6.36 and (j), the twisting moment is

$$T = 2\int_{-b}^{b}\int_{-h}^{h}\phi\,dx\,dy = C\theta = GJ\theta \tag{l}$$

where GJ is the torsional rigidity and J is the torsional constant given by

$$J = 2\int_{-b}^{b}\int_{-h}^{h}\left(h^2 - x^2\right)dx\,dy - \frac{64h^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^3\cosh\frac{n\pi b}{2h}}\int_{-b}^{b}\int_{-h}^{h}\left(\cos\frac{n\pi x}{2h}\cosh\frac{n\pi y}{2h}\right)dx\,dy$$

Integration yields

$$J = \frac{(2h)^3(2b)}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{h}{b}\right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh\frac{n\pi b}{2h}\right] \tag{m}$$

The factor outside the brackets on the right side of Eq. (m) is an approximation for a thin rectangular cross section, because the series goes to zero as b/h becomes large.

Commonly, Eq. (m) is written in the form

$$J = k_1(2h)^3(2b)$$

where

$$k_1 = \frac{1}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{h}{b}\right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh\frac{n\pi b}{2h}\right]$$

The torque-rotation equation [Eq. (l)] can then be written in the more compact form

$$T = G\theta k_1(2h)^3(2b) \tag{n}$$

Values of k_1 for several ratios of b/h are given in Table 6.1.

To determine the maximum shear stress in the rectangular torsion member, we consider the case $b > h$; see Figure 6.15. The maximum slope of the stress function, and by analogy the membrane, for the rectangular section occurs at $x = \pm h$, $y = 0$. At the two points for which $x = \pm h$, $y = 0$, the first of Eqs. (k) gives $\sigma_{zx} = 0$. Therefore, σ_{zy} is the maximum shear stress at $x = \pm h$, $y = 0$. By the second of Eqs. (k),

$$\sigma_{zy}\Big|_{\substack{x=h\\y=0}} = \tau_{max} = 2G\theta h - \frac{16G\theta h}{\pi^2}\left[\frac{1}{\cosh\frac{\pi b}{2h}} + \frac{1}{9\cosh\frac{3\pi b}{2h}} + \cdots\right]$$

or

$$\tau_{max} = 2G\theta hk \tag{o}$$

where

$$k = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \left(\frac{1}{n^2\cosh\frac{n\pi b}{2h}}\right)$$

TABLE 6.1 Torsional Parameters for Rectangular Cross Sections

b/h	1.0	1.5	2.0	2.5	3.0	4.0	6.0	10	∞
k_1	0.141	0.196	0.229	0.249	0.263	0.281	0.299	0.312	0.333
k_2	0.208	0.231	0.246	0.256	0.267	0.282	0.299	0.312	0.333

Then by Eqs. (n) and (o), we may express τ_{max} as

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2}$$

where

$$k_2 = \frac{k_1}{k}$$

Values of k_2 for several ratios of b/h are listed in Table 6.1.

A summary of the results for rectangular cross sectional torsion members is given by the following equations:

$$\begin{aligned} T &= GJ\theta \\ J &= k_1(2b)(2h)^3 \\ \tau_{max} &= \frac{T}{k_2(2b)(2h)^2} = 2G\theta h\frac{k_1}{k_2} \end{aligned} \tag{6.64}$$

where values of k_1 and k_2 are given in Table 6.1 for various values of b/h.

EXAMPLE 6.7 ***Torsional Constant for a Wide-Flange Section***

The nominal dimensions of a steel wide-flange section (W760 × 220) are shown in Figure E6.7. The beam is subjected to a twisting moment $T = 5000$ N • m.

(a) Determine the maximum shear stress τ_{max} and its location. Ignore the fillets and stress concentrations.

(b) Determine the angle of twist per unit length for the applied twisting moment.

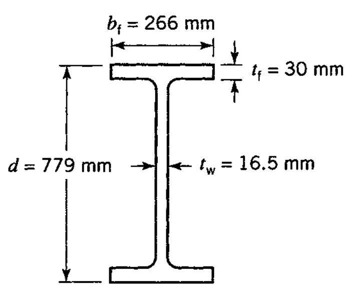

FIGURE E6.7

Solution

For the flanges $b/h = 8.867 < 10$. So, for a flange, $k_1 = 0.308$ by interpolation from Table 6.1. Therefore, for two flanges

$$J_f = 2\left[k_1(b_f)(t_f)^3\right] = 4{,}424{,}100 \text{ mm}^4$$

For the web, $b/h = 43.58 > 10$. Therefore, for the web $k_1 = 0.333$ and

$$J_w = k_1\left(d - 2t_f\right)\left(t_w^3\right) = 1{,}076{,}600 \text{ mm}^4$$

Hence, the torsional constant for the section is

$$J = J_f + J_w = 5{,}500{,}700 \text{ mm}^4 = 5.501 \times 10^{-6} \text{ m}^4$$

(**a**) By Eq. 6.63, the maximum shear stress is

$$\tau_{max} = \frac{2Th_{max}}{J} = \frac{2(5000)(0.015)}{5.501 \times 10^{-6}} = 27.27 \text{ MPa}$$

and it is located along the vertical line of symmetry on the outer edge of the top and bottom flanges.

(**b**) By the second of Eqs. 6.63 or the first of Eqs. 6.64, the angle of twist per unit length is

$$\theta = \frac{T}{GJ} = \frac{5000}{\left(200 \times 10^9\right)\left(5.501 \times 10^{-6}\right)} = 0.00454 \text{ rad/m}$$

EXAMPLE 6.8 ***Rectangular Section Torsion Member***

A rod with rectangular cross section is used to transmit torque to a machine frame (Figure E6.8). It has a width of 40 mm. The first 3.0-m length of the rod has a depth of 60 mm, and the remaining 1.5-m length has a depth of 30 mm. The rod is made of steel for which $G = 77.5$ GPa. For $T_1 = 750$ N • m and $T_2 =$ 400 N • m, determine the maximum shear stress in the rod. Determine the angle of twist of the free end.

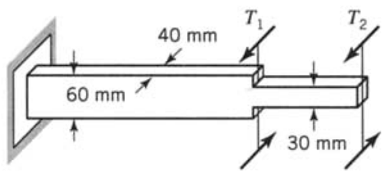

FIGURE E6.8

Solution

For the left portion of the rod,

$$\frac{b}{h} = \frac{30}{20} = 1.5$$

From Table 6.1, we find $k_1 = 0.196$ and $k_2 = 0.231$. For the right portion of the rod,

$$\frac{b}{h} = \frac{20}{15} = 1.33$$

Linear interpolation between the values 1.0 and 1.5 in Table 6.1 gives $k_1 = 0.178$ and $k_2 = 0.223$.

The torque in the left portion of the rod is $T = T_1 + T_2 = 1.15$ kN • m; the maximum shear stress in this portion of the rod is

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} = 51.9 \text{ MPa}$$

The torque in the right portion of the rod is $T_2 = 400$ N • m; the maximum shear stress in this portion of the rod is

$$\tau_{max} = 49.8 \text{ MPa}$$

Hence, the maximum shear stress occurs in the left portion of the rod and is equal to 51.9 MPa.

The angle of twist β is equal to the sum of the angles of twist for the left and right portions of the rod. Thus,

$$\beta = \sum \frac{TL}{GJ} = 0.0994 \text{ rad}$$

6.7 HOLLOW THIN-WALL TORSION MEMBERS AND MULTIPLY CONNECTED CROSS SECTIONS

In general, the solution for a torsion member with a multiply connected cross section is more complex than that for the solid (simply connected cross section) torsion member. For simplicity, we refer to the torsion member with a multiply connected cross section as a hollow torsion member. The complexity of the solution can be illustrated for the hollow torsion member in Figure 6.16. No shear stresses act on the lateral surface of the hollow region of the torsion member; therefore, the stress function and the membrane must have zero slope over the hollow region (see Eqs. 6.23 and Section 6.4). Consequently, the associated elastic membrane may be given a zero slope over the hollow region by machining a flat plate to the dimensions of the hollow region and displacing the plate a distance z_1, as shown in Figure 6.16. However, the distance z_1 is not known. Furthermore, only one value of z_1 is valid for specified values of p and S.

The solution for torsion members having thin-wall noncircular sections is based on the following simplifying assumption. Consider the thin-wall torsion member in Figure 6.17*a*. The plateau (region of zero slope) over the hollow area and the resulting membrane are shown in Figure 6.17*b*. If the wall thickness is small compared to the other dimensions of the cross section, sections through the membrane, made by planes parallel to the z axis and perpendicular to the outer boundary of the cross section, are approximately straight lines. It is *assumed that these intersections are straight lines.* Because the shear stress is given by the slope of the membrane, this simplifying assumption leads to the condition that the shear stress is constant through the thickness. However, the shear stress around the boundary is not constant, unless the thickness t is constant. This is apparent by Figure 6.17*b* since $\tau = \partial\phi/\partial n$, where n is normal to a membrane contour curve z = constant. Hence, by Eqs. 6.53 and Figure 6.17*b*, $\tau = (2G\theta S/p)\partial z/\partial n = (2G\theta S/p)\tan\alpha$. Finally, by Eq. 6.52,

$$\tau = \frac{1}{c}\tan\alpha = \frac{1}{c}\sin\alpha \qquad \text{(since } \alpha \text{ is assumed to be small)} \tag{6.65}$$

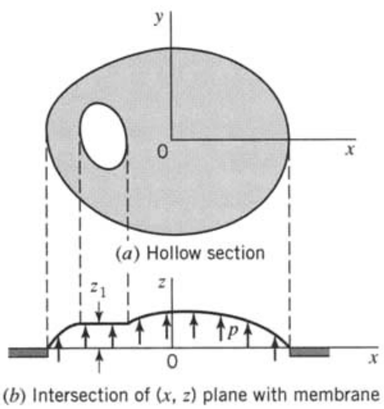

FIGURE 6.16 Membrane for hollow torsion member. (*a*) Hollow section. (*b*) Intersection of (*x*, *z*) plane with membrane.

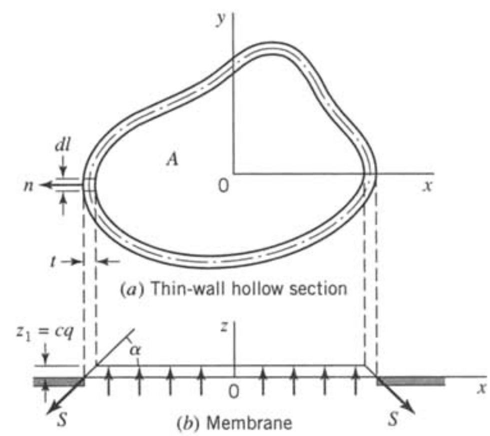

FIGURE 6.17 Membrane for thin-wall hollow torsion member. (*a*) Thin-wall hollow section. (*b*) Membrane.

The quantity $q = \tau t$, with dimensions [F/L], is commonly referred to as *shear flow.* As indicated in Figure 6.17*b*, the shear flow is constant around the cross section of a thin-wall hollow torsion member and is equal to ϕ. Since $q = \tau t$ is constant, the shear stress τ varies with the thickness t, with the maximum shear stress occurring at minimum t. For a thin-wall hollow torsion member with perimeter segments $l_1, l_2, \ldots$, of constant thickness $t_1, t_2, \ldots$, the corresponding shear stresses are $\tau_1 = q/t_1$, $\tau_2 = q/t_2, \ldots$ (assuming that stress concentrations between segments are negligible).

Since ϕ is proportional to z (Eq. 6.52), by Eq. 6.36, the torque is proportional to the volume under the membrane. Thus, we have approximately ($z_1 = c\phi_1$)

$$T = 2A\phi_1 = \frac{2Az_1}{c} = 2Aq = 2A\tau t \tag{6.66}$$

in which A is the area enclosed by the mean perimeter of the cross section (see the area enclosed by the dot-dashed line in Figure 6.17*a*). A relation between τ, G, θ, and the dimensions of the cross section may be derived from the equilibrium conditions in the z direction. Thus,

$$\sum F_z = pA - \oint S \sin\alpha \, dl = 0$$

and, by Eqs. 6.65 and 6.52,

$$\frac{1}{A}\oint \tau \, dl = \frac{p}{cS} = 2G\theta \tag{6.67}$$

where l is the length of the mean perimeter of the cross section and S is the tensile force per unit length of the membrane.

Equations 6.66 and 6.67 are based on the simplifying assumption that the wall thickness is sufficiently small so that the shear stress may be assumed to be constant through the wall thickness. For the cross section considered in Example 6.9, the resulting error is negligibly small when the wall thickness is less than one-tenth of the minimum cross-sectional dimension.

With $q = \tau t$ being constant, it is instructive to write Eq. 6.67 in the form

$$\theta = \frac{1}{2GA}\oint \tau \, dl = \frac{q}{2GA}\oint \frac{dl}{t} \tag{a}$$

where, in general, thickness t is a pointwise function of l. For a cross section of constant thickness, $\oint dl/t = l/t$, where l is the circumferential length of the constant-thickness cross section. For a circumference with segments $l_1, l_2, \ldots$, of constant thickness $t_1, t_2, \ldots$,

$$\oint \frac{dl}{t} = \frac{l_1}{t_1} + \frac{l_2}{t_2} + \cdots$$

Then, Eq. (a) may be written as

$$\theta = \frac{q}{2GA}\left(\frac{l_1}{t_1} + \frac{l_2}{t_2} + \cdots\right)$$

By Eqs. 6.66 and (a), we may eliminate q to obtain

$$T = GJ\theta$$

where

$$J = \frac{4A^2}{\oint dl/t}$$

and GJ is the torsional stiffness of a general hollow cross section.

Also, since q is constant, for a hollow cross section with segments $l_1, l_2, \ldots,$ of constant thickness $t_1, t_2, \ldots,$ Eq. 6.66 may be written as

$$T = 2Aq = 2A\tau_1 t_1 = 2A\tau_2 t_2 = \cdots$$

where $\tau_1, \tau_2, \ldots$ are the shear stresses in the cross section segments $l_1, l_2, \ldots$.

For a thin hollow tube with constant thickness, the shear stress τ is constant both through the thickness and around the perimeter. From Eq. 6.67, we have

$$\theta = \frac{\tau l}{2AG}$$

Noting that, from Eq. 6.66, $\tau = T/2At$, we can write the load–rotation relation for the tube as

$$\theta = \frac{Tl}{4GtA^2} \tag{b}$$

If Eq. (b) is written in the conventional form $\theta = T/GJ$, then we see that the torsion constant for the thin-wall tube with constant thickness is

$$J = \frac{4tA^2}{l}$$

If the thin-wall tube has a circular cross section, then $A = \pi R^2$ and $l = 2\pi R$, where R is the mean radius of the tube. Hence, we see that an approximate expression for the torsion constant is given by

$$J = 2\pi R^3 t$$

As the ratio t/R becomes smaller, the quality of the approximation improves.

6.7.1 Hollow Thin-Wall Torsion Member Having Several Compartments

Thin-wall hollow torsion members may have two or more compartments. Consider the torsion member whose cross section is shown in Figure 6.18*a*. Section *a–a* through the membrane is shown in Figure 6.18*b*. The plateau over each compartment is assumed to have a different elevation z_i. If there are N compartments, there are $N + 1$ unknowns to be determined. For a specified torque T, the unknowns are the N values for the shear flow q_i and the unit angle of twist θ, which is assumed to be the same for each compartment. By Eq. 6.66 the $N + 1$ equations are given by

$$\begin{aligned} T &= 2\sum_{i=1}^{N} A_i \frac{z_i}{c} \\ &= 2\sum_{i=1}^{N} A_i q_i \end{aligned} \tag{6.68}$$

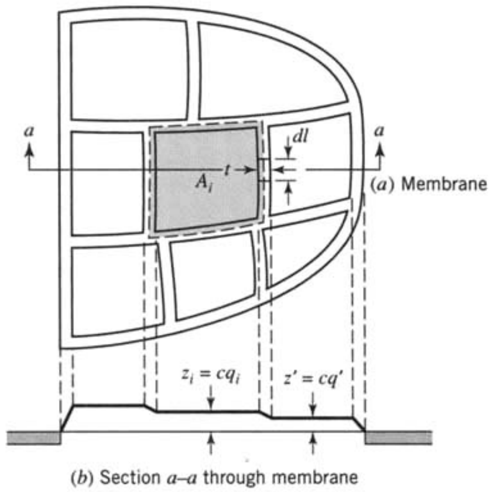

FIGURE 6.18 Multicompartment hollow thin-wall torsion member. (*a*) Membrane. (*b*) Section *a–a* through membrane.

and by N additional equations similar to Eq. 6.67

$$\theta = \frac{1}{2GA_i} \oint_{l_i} \frac{q_i - q'}{t} dl, \quad i = 1, 2, \ldots, N \tag{6.69}$$

where A_i is the area bounded by the mean perimeter for the ith compartment, q' is the shear flow for the compartment adjacent to the ith compartment where dl is located, t is the thickness where dl is located, and l_i is the length of the mean perimeter for the ith compartment. We note that q' is zero at the outer boundary. The maximum shear stress occurs where the membrane has the greatest slope, that is, where $(q_i - q')/t$ takes on its maximum value for the N compartments.

EXAMPLE 6.9
Hollow Thin-Wall Circular Torsion Member

A hollow circular torsion member has an outside diameter of 22.0 mm and an inside diameter of 18.0 mm, with mean diameter $D = 20.0$ mm and $t/D = 0.10$.

(a) Let the shear stress at the mean diameter be $\tau = 70.0$ MPa. Determine T and θ using Eqs. 6.66 and 6.67 and compare these values with values obtained using the elasticity theory. $G = 77.5$ GPa.

(b) Let a cut be made through the wall thickness along the entire length of the torsion member and let the maximum shear stress in the resulting torsion member be 70.0 MPa. Determine T and θ.

Solution

(a) The area A enclosed by the mean perimeter is

$$A = \frac{\pi D^2}{4} = 100\pi \text{ mm}^2$$

The torque, given by Eq. 6.66, is

$$T = 2A\tau t = 2(100\pi)(70)(2) = 87{,}960 \text{ N} \cdot \text{mm} = 87.96k \text{ N} \cdot \text{m}$$

Because the wall thickness is constant, Eq. 6.67 gives

$$\theta = \frac{\tau\pi D}{2GA} = \frac{70(\pi)(20)}{2(77{,}500)(100\pi)} = 0.0000903 \text{ rad/mm}$$

Elasticity values of T and θ are given by Eqs. 6.15 and 6.12. Thus, with

$$J = \frac{\pi}{32}\left(22^4 - 18^4\right) = 4040\pi \text{ mm}^4$$

we find that

$$T = \frac{\tau J}{r} = \frac{70(4040\pi)}{10} = 88{,}840 \text{ N} \bullet \text{mm} = 88.84 \text{ N} \bullet \text{m}$$

and

$$\theta = \frac{\tau}{Gr} = \frac{70}{77{,}500(10)} = 0.0000903 \text{ rad/mm}$$

The approximate solution agrees with the elasticity theory in the prediction of the unit angle of twist and yields torque that differs by only 1%. Note that the approximate solution assumes that the shear stress was uniformly distributed, whereas the elasticity solution indicates that the maximum shear stress is 10% greater than the value at the mean diameter, since the elasticity solution indicates that τ is proportional to r. Note that for a thin tube $J\ ^a\ 2\pi R^3 t = 4000\pi$ mm^4, where R is the mean radius and t is the wall thickness.

(b) When a cut is made through the wall thickness along the entire length of the torsion member, the torsion member becomes equivalent to a long narrow rectangle, for which the theory of Section 6.5 applies. Thus, with $h = 1$ and $b = 10\pi$

$$\theta = \frac{\tau_{max}}{2Gh} = \frac{70}{2(77{,}500)(1)} = 0.0004516 \text{ rad/mm}$$

$$T = \frac{8bh^2\tau_{max}}{3} = \frac{8(10\pi)(1)^2(70)}{3} = 5864 \text{ N} \bullet \text{mm} = 5.864 \text{ N} \bullet \text{m}$$

Hence, after the cut, for the same shear stress the torque is 6.7% of the torque for part (a), whereas the unit angle of twist is 5 times greater than that for part (a).

EXAMPLE 6.10 ***Two-Compartment Hollow Thin-Wall Torsion Member***

A hollow thin-wall torsion member has two compartments with cross-sectional dimensions as indicated in Figure E6.10. The material is an aluminum alloy for which $G = 26.0$ GPa. Determine the torque and unit angle of twist if the maximum shear stress, at locations away from stress concentrations, is 40.0 MPa.