$$\theta_i = \sum_{j=1}^{m}\left(\int \frac{N_j n_{ji}^M}{E_j A_j}\,dz + \int \frac{k_j V_j v_{ji}^M}{G_j A_j}\,dz + \int \frac{M_j m_{ji}^M}{E_j I_j}\,dz + \int \frac{T_j t_{ji}^M}{G_j J_j}\,dz\right) \tag{5.20b}$$

where N_j, V_j, M_j, and T_j are the internal member forces resulting from the real loads. The use of this method is illustrated in the following examples.

EXAMPLE 5.11 Cantilever Beam Deflections and Rotations

The cantilever beam in Figure E5.11 has a rectangular cross section and is subjected to a midspan load P as shown. Neglect strain energy resulting from shear.

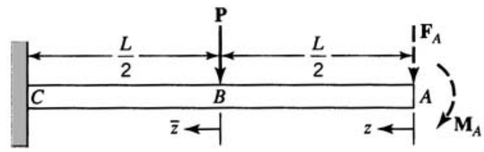

FIGURE E5.11

(a) Determine the vertical deflection and rotation of the free end of the beam by the dummy load method.

(b) Show that the same results are obtained by the dummy unit load method.

Solution

(a) The first step in the dummy load method is to apply a fictitious load F_A and a fictitious moment M_A at point A as shown in Figure E5.11. Next, we write the moment expressions for the two intervals of the beam. For interval A–B

$$M_{AB} = M_A + F_A z \tag{a}$$

For interval B–C

$$M_{BC} = M_A + F_A\left(\bar{z} + \frac{L}{2}\right) + P\bar{z} \tag{b}$$

Differentiation of Eqs. (a) and (b) with respect to the fictitious force and moment yields, for interval A–B,

$$\frac{\partial M_{AB}}{\partial F_A} = z \tag{c}$$

$$\frac{\partial M_{AB}}{\partial M_A} = 1 \tag{d}$$

and for interval B–C,

$$\frac{\partial M_{BC}}{\partial F_A} = \bar{z} + \frac{L}{2}$$

$$\frac{\partial M_{BC}}{\partial M_A} = 1 \tag{e}$$

$$\frac{\partial M_{BC}}{\partial M_A} = 1 \tag{f}$$

To find the vertical deflection at point A, we substitute Eqs. (a), (b), (c), and (e) into Eq. 5.17 and perform the integration:

$$q_A = \int_0^{L/2} \frac{M_A + F_A(z)}{EI}(z)\,dz + \int_0^{L/2} \frac{M_A + F_A(\bar{z} + L/2) + P\bar{z}}{EI}\left(\bar{z} + \frac{L}{2}\right)d\bar{z} \tag{g}$$

$$= M_A\left(\frac{L^2}{2EI}\right) + F_A\left(\frac{L^3}{3EI}\right) + P\left(\frac{5L^3}{48EI}\right) \tag{h}$$

Since, in fact, the fictitious loads F_A and M_A do not exist, they are set to zero. Then Eq. (h) yields the deflection of point A as

$$q_A = \frac{5PL^3}{48EI} \tag{i}$$

To find the rotation of the section at A, we substitute Eqs. (a), (b), (d), and (f) into Eq. 5.18 and perform the integration:

$$\theta_A = \int_0^{L/2} \frac{M_A + F_A(z)}{EI}(1)\,dz + \int_0^{L/2} \frac{M_A + F_A(\bar{z} + L/2) + P\bar{z}}{EI}(1)\,d\bar{z} \tag{j}$$

$$= M_A\left(\frac{L}{EI}\right) + F_A\left(\frac{L^2}{2EI}\right) + P\left(\frac{L^2}{8EI}\right) \tag{k}$$

Again, the fictitious loads F_A and M_A are set to zero. Then Eq. (k) yields the rotation of the section at A as

$$\theta_A = \frac{PL^2}{8EI} \tag{l}$$

(b) In the dummy unit load method, F_A and M_A are set to unity. Then the internal moment M resulting from the real force at B and the internal moments m^F and m^M (see Eqs. 5.19a and 5.19b) resulting from the unit force and unit moment at A are, for interval A–B,

$$M_{AB} = 0 \tag{m}$$

$$m^F_{AB} = 1.0(z) = z \tag{n}$$

$$m^M_{AB} = 1.0 \tag{o}$$

and for interval B–C,

$$M_{BC} = P\bar{z} \tag{p}$$

$$m^F_{BC} = 1.0\left(\bar{z} + \frac{L}{2}\right) = \bar{z} + \frac{L}{2} \tag{q}$$

$$m^M_{BC} = 1.0 \tag{r}$$

The deflection at point A is obtained by the substitution of Eqs. (m), (n), (p), and (q) into Eq. 5.20a. The result is [see Eq. (i)]

$$q_A = \int_0^{L/2} \frac{P\bar{z}}{EI}\left(\bar{z} + \frac{L}{2}\right)d\bar{z} \tag{s}$$

$$= \frac{5PL^3}{48EI} \tag{t}$$

The rotation of the section at A is obtained by the substitution of Eqs. (m), (o), (p), and (r) into Eq. 5.20b. The result is (see Eq. l)

$$\theta_A = \int_0^{L/2} \frac{P\bar{z}}{EI}(1)d\bar{z} \tag{u}$$

$$= \frac{PL^2}{8EI} \tag{v}$$

Thus, the equivalence of the dummy load approach with the dummy unit load approach is demonstrated for this example.

EXAMPLE 5.12 *Pin-Connected Truss*

The pin-connected truss in Figure E5.12 is made of an aluminum alloy for which $E = 72.0$ GPa. The magnitudes of the loads are $P = 10$ kN and $Q = 5$ kN. Members BC, CD, and DE each have cross-sectional area of 900 mm^2. The remaining members each have cross-sectional area of 150 mm^2. Determine the rotation of member BE caused by the loads P and Q.

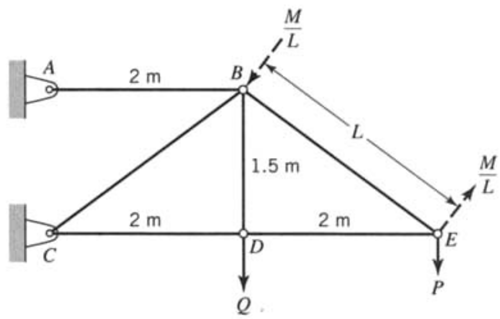

FIGURE E5.12

Solution

To determine the rotation of member BE by energy methods, a moment M must be acting on member BE. Let M be an imaginary counterclockwise moment represented by a couple with equal and opposite forces M/L ($L = BE = 2500$ mm) applied perpendicular to BE at points B and E as indicated in Figure E5.12. Equations of equilibrium give the following values for the axial forces in the members of the structure:

$$N_{AB} = \frac{4}{3}(Q + 2P) - \frac{5M}{3L}, \qquad \frac{\partial N_{AB}}{\partial M} = -\frac{5}{3L}$$

$$N_{BC} = -\frac{5}{3}(Q + P), \qquad \frac{\partial N_{BC}}{\partial M} = 0$$

$$N_{BD} = Q, \qquad \frac{\partial N_{BD}}{\partial M} = 0$$

$$N_{BE} = \frac{5P}{3} - \frac{4M}{3L}, \qquad \frac{\partial N_{BE}}{\partial M} = -\frac{4}{3L}$$

$$N_{CD} = N_{DE} = -\frac{4P}{3} + \frac{5M}{3L}, \qquad \frac{\partial N_{CD}}{\partial M} = \frac{5}{3L}$$

After the partial derivatives $\partial N_j/\partial M$ have been taken, the magnitude of M in the N_j is set to zero. The values of N_j and $\partial N_j/\partial M$ are then substituted into Eq. 5.18 to give

$$\begin{aligned}\theta_{BE} &= \sum_{j=1}^{6} \frac{N_j L_j}{E_j A_j} \frac{\partial N_j}{\partial M} = \frac{N_{AB} L_{AB}}{EA_{AB}} \frac{\partial N_{AB}}{\partial M} + \frac{N_{BC} L_{BC}}{EA_{BC}} \frac{\partial N_{BC}}{\partial M} \\ &\quad + \frac{N_{BD} L_{BD}}{EA_{BD}} \frac{\partial N_{BD}}{\partial M} + \frac{N_{BE} L_{BE}}{EA_{BE}} \frac{\partial N_{BE}}{\partial M} \\ &\quad + 2\frac{N_{CD} L_{CD}}{EA_{CD}} \frac{\partial N_{CD}}{\partial M}\end{aligned}$$

$$\begin{aligned}\theta_{BE} &= \frac{4(25{,}000)(2000)}{3(72{,}000)(150)}\left[-\frac{5}{3(2500)}\right] + \frac{5(10{,}000)(2500)}{3(72{,}000)(150)}\left[-\frac{4}{3(2500)}\right] \\ &\quad - \frac{2(4)(10{,}000)(2000)}{3(72{,}000)(900)}\left[\frac{5}{3(2500)}\right] = -0.004115 - 0.002058 - 0.000549 \\ &= -0.00672 \text{ rad}\end{aligned}$$

The negative sign for θ_{BE} indicates that the angle change is clockwise; that is, the angle change has a sign opposite to that assumed for M.

EXAMPLE 5.13
Curved Beam Loaded Perpendicular to Its Plane

The semicircular curved beam of radius R in Figure E5.13 has a circular cross section of radius r. The curved beam is indicated by its centroidal axis to simplify the figure. It is fixed at 0 and lies in the (x, y) plane with center of curvature at C on the x axis. Load P parallel to the z axis acts at a section $\pi/2$ from the fixed end. Determine the z component of the deflection of the free end. Assume that R/r is sufficiently large for U_S to be negligible.

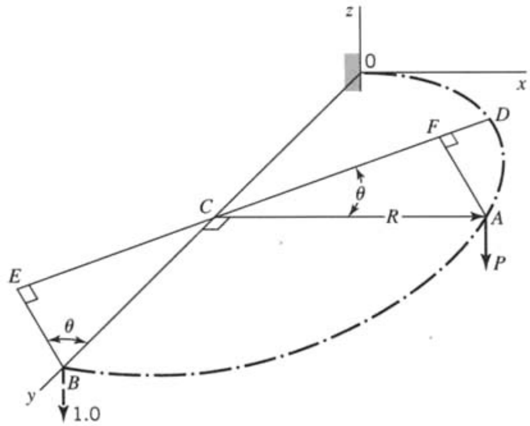

FIGURE E5.13

Solution

To find the z component of the deflection of the free end of the curved beam, a dummy unit load parallel to the z axis is applied at B as indicated in Figure E5.13. Consider a section D of the curved beam at an angle θ measured from section A at the load P. The internal moment and torque at section D resulting from forces at A and B are

$$\begin{aligned} M_D &= P(AF) = PR\sin\theta \\ T_D &= P(DF) = PR(1-\cos\theta) \\ m_D^F &= 1.0(BE) = R\cos\theta \\ t_D^F &= 1.0(DC+CE) = R(1+\sin\theta) \end{aligned}$$

These values are substituted into Eqs. 5.20a and 5.20b to give

$$\begin{aligned} q_B &= \int_0^{\pi/2} \left[\frac{PR\sin\theta(R\cos\theta)}{EI} + \frac{PR(1-\cos\theta)[R(1+\sin\theta)]}{GJ} \right] R\,d\theta \\ &= \frac{1}{2}\frac{PR^3}{EI} + \frac{PR^3}{GJ}\left(\frac{\pi}{2}-\frac{1}{2}\right) \\ &= \frac{2PR^3}{\pi E r^4}[1+(1+\nu)(\pi-1)] \end{aligned}$$

EXAMPLE 5.14
Stiffness of a Coil Spring

A coil spring is formed by winding a wire (or circular rod) of diameter d into a helix with diameter D, number of coils n, and pitch angle β (see Figure E5.14a in which $n = 3$). Assume that the material has modulus of elasticity E and shear modulus G. Determine the stiffness of the spring under a concentric axial load P. Ignore end effects; that is, ignore the method by which the axial load is applied to the ends of the spring. Assume that the ratio d/D is small enough that the equations for bending and torsion of straight members (Chapter 1) apply.

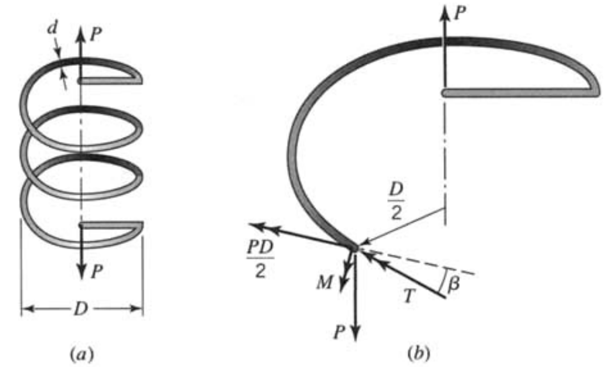

FIGURE E5.14

Solution Even though the force P acts at the location and in the direction of interest, we use the dummy unit load method to find deflection of the spring. The concentric axial load P causes a shear force V and a normal force N that act on the cross section of the wire. The effect of these internal forces is expected to be small and we dismiss them from consideration here.

The load P has a lever arm of $D/2$ relative to the centroidal axis of the wire. Hence it produces a moment of magnitude $PD/2$. The components of this moment relative to the cross-sectional plane of the wire are a bending moment M and a torque T (see the enlarged view of a portion of the top coil in Figure 5.14*b*) given by

$$M = \frac{PD\,\sin\beta}{2}$$
$$T = \frac{PD\,\cos\beta}{2} \tag{a}$$

If end effects are ignored, M and T are each constant over the full length of the spring.

The corresponding moment and torque in the spring caused by a concentric unit axial load on the spring are

$$m = \frac{D\,\sin\beta}{2}$$
$$t = \frac{D\,\cos\beta}{2} \tag{b}$$

Substitution of Eqs. (a) and (b) into Eq. 5.20a gives

$$q = \int_0^L \left(\frac{PD^2\sin^2\beta}{4EI} + \frac{PD^2\cos^2\beta}{4GJ}\right) ds$$

The differential ds is an element of arc length of the wire. The limit of integration L is the total length of wire in the spring, which is approximated as $L = n\pi D$. Since all quantities are constant with respect to s, the integration is trivial. Hence, the axial deflection of the spring is

$$q = \frac{n\pi PD^3}{4}\left(\frac{\sin^2\beta}{EI} + \frac{\cos^2\beta}{GJ}\right) \tag{c}$$

The spring stiffness k is found by dividing the axial load P by the deflection q from Eq. (c). After substitution for A, I, and J in terms of d, this results in

$$k = \frac{EGd^4}{8nD^3(2G\,\sin^2\beta + E\,\cos^2\beta)}$$

The relative effects of the bending and torsional moments on the stiffness of the spring can now be examined. For instance, in a closely wound spring the pitch angle β is small and the effect of bending moment M is small. For this case, $\sin^2\beta \approx 0$, $\cos^2\beta \approx 1$, and only the torque T contributes to strain energy in the spring. Hence, the stiffness is given by

$$k_T = \frac{Gd^4}{8nD^3}$$

5.5 STATICALLY INDETERMINATE STRUCTURES

As we observed in Section 5.4, a statically determinate structure (Figure 5.8) may be made statically indeterminate by the addition of a member (member BD in Figure 5.9). Alternatively, a statically indeterminate structure may be rendered statically determinate if certain members, internal actions, or supports are removed. For example, the truss in Figure 5.9 is rendered statically determinate if member BD (or equally well member AC) is removed. Such a member in a statically indeterminate structure is said to be *redundant*, since after its removal the structure will remain in static equilibrium under arbitrary loads. In general, statically indeterminate structures contain one or more redundant members or supports. For simplicity a redundant member or redundant support is often referred to only as a redundant, without additional qualification.

Generally in the analysis of structures, internal actions in each member of the structure must be determined. For statically indeterminate structures, the equations of static equilibrium are not sufficient to determine these internal actions. For example, in Figure 5.11*a*, the propped cantilever beam has four unknown support reactions, whereas there are only three equations of equilibrium for a planar structure. If the support at B were removed, the beam would function as a simple cantilever beam. Hence, we may consider the support at B to be redundant and, if it is removed, the beam is rendered statically determinate. The choice of the redundant is arbitrary.

If we consider the support at B to be redundant, additional information is required to determine the magnitude of the reaction R (see Figure 5.11*c*). As we shall see, the fact that

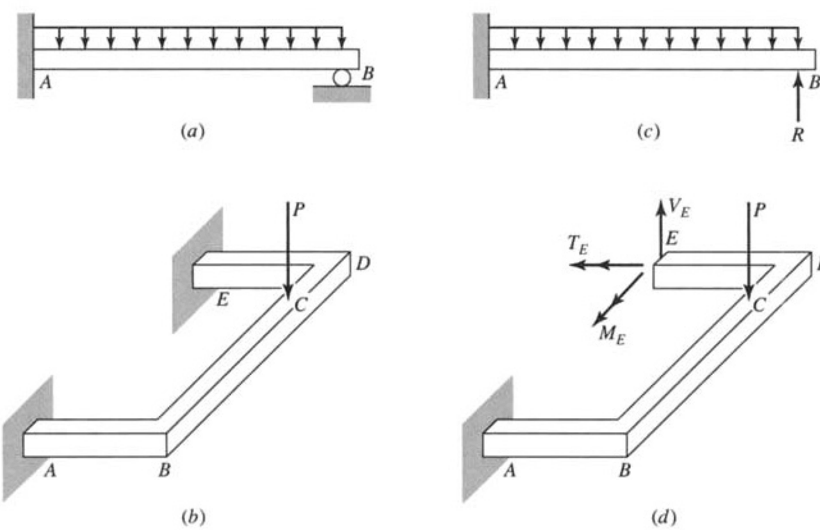

FIGURE 5.11 Structures with redundant supports.

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the support at B prevents the tip of the beam from displacing vertically may be used, in conjunction with Castigliano's theorem on deflections, to obtain the additional equation needed to determine the redundant reaction R.

Likewise, the three support reactions at A (or E) for member $ABCDE$ in Figure 5.11*b* can be chosen as the redundants. Hence, either the support at A or E (but not both) may be removed to render the structure statically determinate. Let us assume that the support reactions at E are chosen as the redundants (Figure 5.11*d*). The three redundant reactions are a vertical force V_E, which prevents vertical deflection at E; a bending moment M_E, which prevents bending rotation of the section at E; and a torque T_E, which prevents torsional rotation of the section at E. The fact that vertical deflection, bending rotation, and torsional rotation are prevented at section E may be used, in conjunction with Castigliano's theorem on deflections, to obtain the additional equations needed to determine the support reactions at E.

The structures in Figure 5.12 do not contain redundant reactions but do contain redundant members. In Figure 5.12*a*, the member BE (or CD) of the truss is redundant. Hence, the truss is statically indeterminate. If either member BE or member CD is removed, the truss is rendered statically determinate. Likewise, the member ABC of the statically indeterminate structure in Figure 5.12*b* is redundant. It may be removed to render the structure statically determinate.

Since the truss of Figure 5.12*a* is pin-joined, the redundant member BE is subject to an internal axial force. Hence, the only redundant internal force for the truss is the tension in member BE (Figure 5.12*c*). However, the redundant member ABC of the structure in Figure 5.12*d* may support three internal reactions: the axial force N, shear V, and moment M. The additional equations (in addition to the equations of static equilibrium) required to determine the additional unknowns (the redundant internal actions caused by redundant members) in statically indeterminant structures may be obtained by the application of Castigliano's theorem on deflections.

In particular, we can show that

$$\frac{\partial U}{\partial F_i} = 0 \tag{5.21}$$

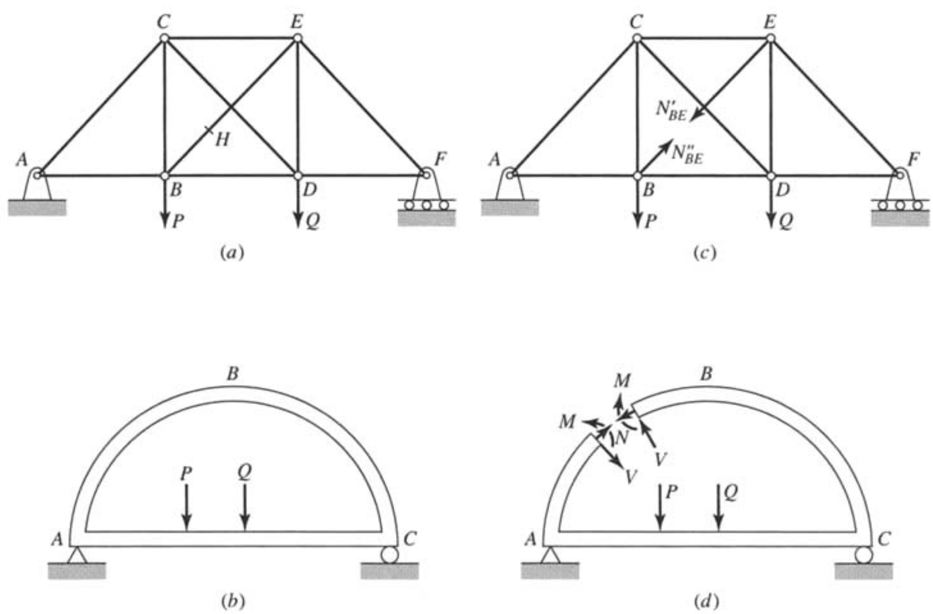

FIGURE 5.12 Structures with redundant members.

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for every internal redundant force or external redundant reaction (F_1, F_2, ...) in the structure, and

$$\frac{\partial U}{\partial M_i} = 0 \tag{5.22}$$

for every internal redundant moment or external redundant moment (M_1, M_2, ...) in the structure. Equations 5.21 and 5.22 are readily verified for the structures in Figure 5.11. The beam in Figure 5.11*a* has a redundant external reaction R at B. Since the deflection at point B is zero, Eq. 5.17 gives $q_R = \partial U/\partial R = 0$, which agrees with Eq. 5.21. The structure in Figure 5.11*b* has three internal redundant reactions (V_E, M_E, T_E) at section E, as indicated in Figure 5.11*d*. Since the deflection and rotations at E remain zero as the structure is loaded, Eqs. 5.17 and 5.18 yield the results $\partial U/\partial V_E = \partial U/\partial M_E = \partial U/\partial T_E = 0$, which agree with Eqs. 5.21 and 5.22.

It is not directly apparent that Eqs. 5.21 and 5.22 are valid for the internal redundant member forces in the structures in Figure 5.12. To show that they are valid, let N_{BE} be the redundant internal action for the pin-joined truss (Figure 5.12*a*). Pass a section through some point H of member BE and apply equal and opposite tensions N'_{BE} and N''_{BE}, as indicated in Figure 5.12*c*. Since the component of the deflection of point H along member BE is not zero, it is not obvious that

$$\frac{\partial U}{\partial N_{BE}} = 0 \tag{a}$$

To prove that Eq. (a) is valid, it is necessary to distinguish between tensions N'_{BE} and N''_{BE}. The displacement of point H in the direction of N'_{BE} is given by (see Eq. 5.17)

$$q_{N'_{BE}} = \frac{\partial U}{\partial N'_{BE}} \tag{b}$$

and in the direction of N''_{BE}, the displacement is given by

$$q_{N''_{BE}} = \frac{\partial U}{\partial N''_{BE}} \tag{c}$$

These displacements $q_{N'_{BE}}$ and $q_{N''_{BE}}$ are collinear, have equal magnitudes, but have opposite senses. Hence, by Eqs. (b) and (c) we have

$$\frac{\partial U}{\partial N'_{BE}} + \frac{\partial U}{\partial N''_{BE}} = 0 \tag{d}$$

The reduction of Eq. (d) to Eq. (a) then follows by the same technique employed in the reduction of Eq. (a) of Section 5.2 to Eq. (d) of Section 5.2, since $N'_{BE} = N''_{BE} = N_{BE}$. In a similar manner, it may be shown for the structure in Figure 5.12*b* that

$$\frac{\partial U}{\partial N} = 0, \quad \frac{\partial U}{\partial V} = 0, \quad \frac{\partial U}{\partial M} = 0 \tag{5.23}$$

where N, V, and M are the internal member forces for any given section of member ABC.

Note: In the application of Eqs. 5.21 and 5.22 to the system with redundant supports or redundant members, it is assumed that the unloaded system is stress-free (see Figure 5.11). Consequently, redundant supports exert no force on the structure initially. However, in certain applications, these conditions do not hold. For example, consider the beam in Figure 5.13.

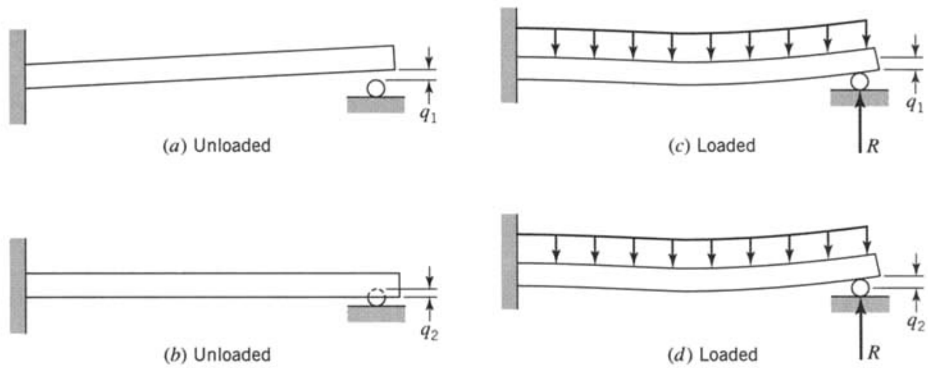

FIGURE 5.13 Effect of support settlement or thermal expansion or contraction on redundant supports of loaded beams. (*a*), (*b*) Unloaded. (*c*), (*d*) Loaded.

Initially, the right end of the beam may be lifted off the support, or the end support may exert a force on the beam because of either support settlement or thermal expansion or contraction. As a result, the end of the beam (in the absence of the redundant support) may be raised a distance q_1 above the location of the support before the beam is loaded (Figure 5.13*a*) or it may be a distance q_2 below the support location (Figure 5.13*b*).

If the displacement magnitudes q_1 or q_2 of the end of the beam (in the absence of the support) are known, we may compute the reaction R for the loaded beam (Figures 5.13*c* and *d*) by the relations

$$q_1 = -\frac{\partial U}{\partial R} \quad \text{or} \quad q_2 = \frac{\partial U}{\partial R} \tag{5.24}$$

where the minus sign indicates that displacement q_1 and force R have opposite senses.

5.5.1 Deflections of Statically Indeterminate Structures

A structure is not altered if we remove the redundant members or redundant supports and replace them by external forces and moments that are identical to the forces and moments exerted by the deleted parts. These forces and moments are initially unknown, but we may denote them by $R_1, R_2, \ldots$. Then we may derive formulas for the displacements and rotations of the various parts of the simplified statically determinate structure that carries the prescribed loading and the statically indeterminate reactions $R_1, R_2, \ldots$. By setting the deflections at the redundant supports to known values, we obtain equations that determine $R_1, R_2, \ldots$. The procedure is illustrated in the following examples.

EXAMPLE 5.15
Uniformly Loaded Propped Cantilever Beam

Consider the reaction R at the right end of the beam shown in Figure E5.15*a* as the redundant. The right-hand support is conceived to be removed, and the external force R is applied (Figure E5.15*b*).

(a) Determine the reaction R, enforcing the condition that $q_R = 0$.

(b) Determine the midspan displacement of the beam.

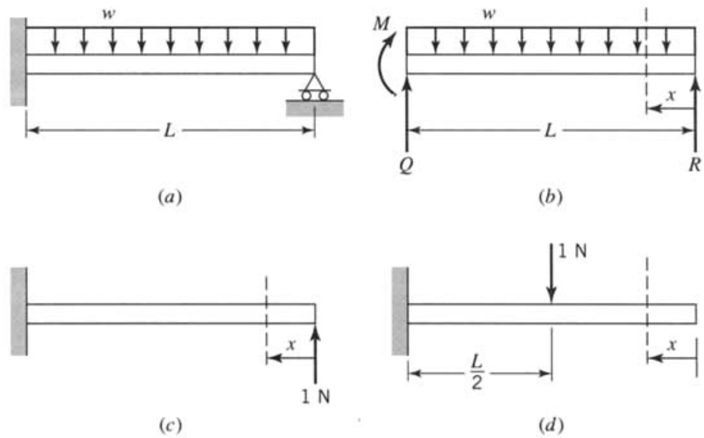

FIGURE E5.15

Solution **(a)** A free-body diagram of the beam is shown in Figure E5.15*b*. The bending moment in the beam at section x is

$$M = Rx - \frac{1}{2}wx^2$$

Let us use the dummy unit load method. Hence, we apply a 1-N load at the free end of the beam (Figure E5.15*c*). The moment at section x caused by this load is

$$m = x$$

Neglecting the effect of shear, we have, by Eq. 5.20a,

$$q_R = \int_0^L \frac{Mm}{EI}\,dx = \frac{1}{EI}\int_0^L \left(Rx^2 - \frac{1}{2}wx^3\right)dx$$

or

$$q_R = \frac{RL^3}{3EI} - \frac{wL^4}{8EI}$$

Since R is the reaction of the fixed support, $q_R = 0$. Therefore,

$$R = \frac{3}{8}wL$$

(b) To determine the midspan deflection of the beam, apply a (downward) unit load at $x = L/2$ (Figure E5.15*d*).

for $0 \le x \le \frac{L}{2}$,

$$M = \frac{3}{8}wLx - \frac{1}{2}wx^2$$
$$m = 0$$

for $\frac{L}{2} \le x \le L$,

$$M = \frac{3}{8}wLx - \frac{1}{2}wx^2$$
$$m = \frac{L}{2} - x$$

Hence, by Eq. 5.20a,

$$q_{L/2} = \frac{1}{EI}\int_{L/2}^{L}\left(\frac{3}{8}wLx - \frac{1}{2}wx^2\right)\left(\frac{L}{2} - x\right)dx$$

Integration yields

$$q_{L/2} = \frac{wL^4}{192EI}$$

EXAMPLE 5.16 *Statically Indeterminate System*

Determine the reactions at C for member ABC in Figure E5.16a and the deflection of point B in the direction of P. Assume U_N and U_S are so small that they can be neglected.

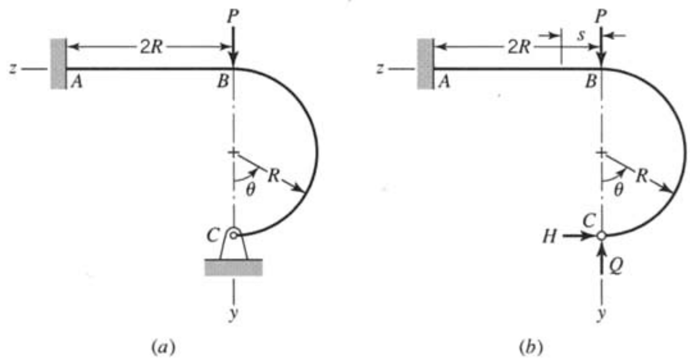

FIGURE E5.16

Solution

The support at C allows rotation but prevents displacements. Our first problem is to determine the redundant reactions Q and H (Figure E5.16b) at C. Since the y displacement at C is zero, Eq. 5.21 gives

$$\frac{\partial U}{\partial Q} = 0 = \int_0^{\pi}\frac{[QR\sin\theta - HR(1-\cos\theta)]}{EI}R(\sin\theta)\,Rd\theta + \int_0^{2R}\frac{[(Q-P)s + 2HR]}{EI}s\,ds$$

or

$$Q\left(\frac{\pi}{2} + \frac{8}{3}\right) + 2H - \frac{8P}{3} = 0 \qquad \text{(a)}$$

Since the z displacement at C is zero, Eq. 5.21 gives

$$\frac{\partial U}{\partial H} = 0 = \int_0^{\pi}\frac{[QR\sin\theta - HR(1-\cos\theta)]}{EI}[-R(1-\cos\theta)]\,R\,d\theta + \int_0^{2R}\frac{[(Q-P)s + 2HR]}{EI}2R\,ds$$

or

$$2Q + H\left(\frac{3\pi}{2} + 8\right) - 4P = 0 \qquad \text{(b)}$$

Simultaneous solution of Eqs. (a) and (b) gives

$$Q = 0.5193P$$
$$H = 0.2329P$$

In the application of Castigliano's theorem, the quantities H, Q, and P are considered to be independent. Then since the moment in the curved part BC [namely, $QR \sin\theta - HR(1 - \cos\theta)$] is independent of P, we need only consider the strain energy of part AB. Thus, with Eq. (c), we obtain

$$\begin{aligned} q_P &= \frac{\partial U}{\partial P} = \int_0^{2R} \frac{[(Q-P)s + 2HR]}{EI}(-s)\,ds \\ &= \frac{1}{EI}\left(\frac{8}{3}PR^3 - \frac{8}{3}QR^3 - 4HR^3\right) \end{aligned} \qquad \text{(c)}$$

or

$$q_P = 0.3503\frac{PR^3}{EI}$$

Alternatively, we may consider H and Q to be functions of P. Then, by the chain rule, with $U = U[P, H(P), Q(P)]$,

$$\begin{aligned} q_P &= \frac{\partial U[P, H(P), Q(P)]}{\partial P} \\ &= \frac{\partial U}{\partial P} + \frac{\partial U}{\partial H}\frac{\partial H}{\partial P} + \frac{\partial U}{\partial Q}\frac{\partial Q}{\partial P} \end{aligned} \qquad \text{(d)}$$

However, the boundary conditions at C require that $\partial U/\partial H = 0$ and $\partial U/\partial Q = 0$. Equation (d) is simplified accordingly, and we again obtain Eq. (c).

Notice that the above argument is applicable to indeterminate structures in general. That is, the boundary condition requirement that $\partial U/\partial H = 0$ and $\partial U/\partial Q = 0$ can be used to simplify the expression for strain energy in the structure. Only the strain energy in the released structure resulting from the applied load needs to be considered.

EXAMPLE 5.17 *Statically Indeterminate Truss*

The inverted king post truss in Figure E5.17 is constructed of a 160-mm-deep by 60-mm-wide rectangular steel beam ABC (E_{AC} = 200 GPa and Y_{AC} = 240 MPa), a 15-mm diameter steel rod ADC (E_{DC} = 200 GPa and Y_{DC} = 500 MPa), and a 40 mm by 40 mm white oak compression strut BD (E_{BD} = 12.4 GPa and Y_{BD} = 29.6 MPa). Determine the magnitude of the load P that can be applied to the king post truss if all parts are designed using a factor of safety SF = 2.00 against yielding. Neglect stress concentrations.

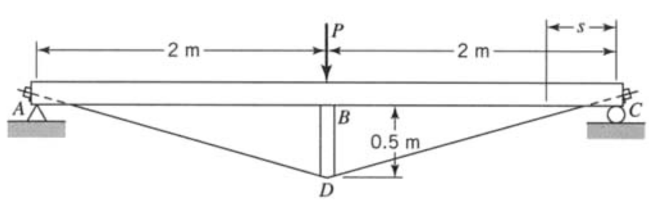

FIGURE E5.17

Solution Let member BD be the redundant member of the king post truss. We will include strain energy U_N for both strut BD and rod ADC; however, U_N and U_S for the beam are so small compared to U_M that they can be neglected. Let the compression load in strut BD be N_{BD}. Equations of equilibrium at joint D give

$$N_{DC} = \sqrt{4.25}N_{BD}$$

The bending moment in the beam at distance s from either C or A is

$$M = \left(\frac{P}{2} - \frac{N_{BD}}{2}\right)s$$

Equation (5.21) gives

$$\begin{aligned}\frac{\partial U}{\partial N_{BD}} = 0 &= \frac{N_{BD}L_{BD}}{E_{BD}A_{BD}} + 2\frac{N_{DC}L_{DC}}{E_{DC}A_{DC}}\frac{\partial N_{DC}}{\partial N_{BD}} \\ &\quad + 2\int_0^{L_{BC}} \frac{M}{E_{AC}I_{AC}}\frac{\partial M}{\partial N_{BD}}\,ds \\ &= \frac{500N_{BD}}{E_{BD}A_{BD}} + \frac{2(\sqrt{4.25}\times 10^3)(4.25)N_{BD}}{E_{DC}A_{DC}} \\ &\quad + 2\int_0^{2000} \frac{\left(\frac{P}{2} - \frac{N_{BD}}{2}\right)s}{E_{AC}I_{AC}}\left(-\frac{s}{2}\right)ds\end{aligned}$$

which can be simplified to give

$$P = N_{BD}\left[1 + \frac{500}{E_{BD}A_{BD}}\frac{3E_{AC}I_{AC}}{4\times 10^9} + \frac{\sqrt{4.25}(8.5\times 10^3)}{E_{DC}A_{DC}}\frac{3E_{AC}I_{AC}}{4\times 10^9}\right] \qquad \text{(a)}$$

But $A_{BD} = 40(40) = 1600\ \text{mm}^2$, $A_{DC} = \pi(15)^2/4 = 176.7\ \text{mm}^2$, and

$$I_{AC} = 60(160)^3/12 = 20.48\times 10^6\text{mm}^4$$

These along with other given values when substituted in Eq. (a) give

$$P = 2.601N_{BD}$$

The axial loads in strut BD and rod ADC and the maximum moment in beam ABC can now be written as functions of P.

$$\begin{aligned}N_{BD} &= 0.384P\ [\text{N}] \\ N_{DC} &= 0.793P\ [\text{N}] \\ M_{\max} &= 616P\ [\text{N}\bullet\text{mm}]\end{aligned}$$

Since the working stress for each member is half the yield stress for the member, a limiting value of P is obtained for each member. For compression strut BD

$$\frac{Y_{BD}}{2} = \frac{29.6}{2} = \frac{N_{BD}}{A_{BD}} = \frac{0.384P}{1600}$$

$$P = 61{,}700\ \text{N}$$

For rod ADC

$$\frac{Y_{DC}}{2} = \frac{500}{2} = \frac{N_{DC}}{A_{DC}} = \frac{0.797P}{176.7}$$

$$P = 55{,}700 \text{ N}$$

For beam ABC

$$\frac{Y_{AC}}{2} = \frac{240}{2} = \frac{M_{\max}c}{I_{AC}} = \frac{616P(80)}{20.48 \times 10^6}$$

$$P = 49{,}900 \text{ N}$$

Thus, the design load for the king post truss is 49.9 kN.

EXAMPLE 5.18 *Spring-Supported I-Beam*

An aluminum alloy I-beam (depth = 100 mm, $I = 2.45 \times 10^6$ mm^4, and $E = 72.0$ GPa) has a length of 6.8 m and is supported by seven springs ($K = 110$ N/mm) spaced at distance $l = 1.10$ m center to center along the beam (Figure E5.18*a*). A load $P = 12.0$ kN is applied at the center of the beam over the center spring. Determine the load carried by each spring, the deflection of the beam under the load, the maximum bending moment, and the maximum bending stress in the beam.

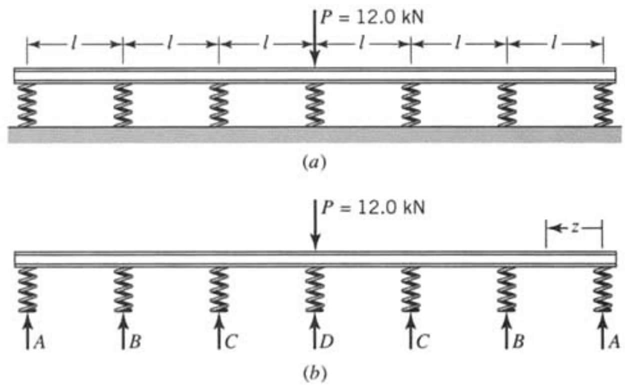

FIGURE E5.18

Solution

It is assumed that the springs are attached to the beam so that the springs can develop tensile as well as compressive forces. Because of symmetry, there are only four unknown spring forces: A, B, C, and D. A free-body diagram of the beam with springs attached is shown in Figure E5.18*b*. Let the loads B, C, and D carried by the springs be redundant reactions. The magnitudes of these reactions are obtained using Eq. 5.21 as

$$\frac{\partial U}{\partial B} = 0, \quad \frac{\partial U}{\partial C} = 0, \quad \frac{\partial U}{\partial D} = 0 \tag{a}$$

The strain energy U for the beam and springs (if we neglect U_S for the beam) is given by the relation

$$U = 2\int_0^l \frac{M^2}{2EI}dz + 2\int_l^{2l} \frac{M^2}{2EI}dz + 2\int_{2l}^{3l} \frac{M^2}{2EI}dz + 2\left(\frac{A^2}{2K} + \frac{B^2}{2K} + \frac{C^2}{2K}\right) + \frac{D^2}{2K} \tag{b}$$

The moments in the three integrals are functions of the reaction A, which can be eliminated from Eq. (b) by the equilibrium force equation for the y direction:

$$A = \frac{P}{2} - B - C - \frac{D}{2} \tag{c}$$

The moments for the three segments of the beam are

$$\begin{aligned}
&0 \le z \le l \\
&\qquad M = Az \qquad\qquad\qquad\qquad = \frac{P}{2}z - Bz - Cz - \frac{D}{2}z \\
&l \le z \le 2l \\
&\qquad M = Az + B(z-l) \qquad\qquad = \frac{P}{2}z - Bl - Cz - \frac{D}{2}z \\
&2l \le z \le 3l \\
&\qquad M = Az + B(z-l) + C(z-2l) = \frac{P}{2}z - Bl - 2Cl - \frac{D}{2}z
\end{aligned} \tag{d}$$

Substitution of Eqs. (b)–(d) into the first of Eqs. (a) gives

$$\begin{aligned}
\frac{\partial U}{\partial B} = 0 = {} & \frac{2}{EI}\int_0^l \left(\frac{P}{2}z - Bz - Cz - \frac{D}{2}z\right)(-z)\,dz \\
& + \frac{2}{EI}\int_l^{2l} \left(\frac{P}{2}z - Bl - Cz - \frac{D}{2}z\right)(-l)\,dz \\
& + \frac{2}{EI}\int_{2l}^{3l} \left(\frac{P}{2}z - Bl - 2Cl - \frac{D}{2}z\right)(-l)\,dz \\
& + \frac{2}{K}\left(\frac{P}{2} - B - C - \frac{D}{2}\right)(-1) + \frac{2B}{K}
\end{aligned}$$

which can be simplified to give

$$0 = 12BEI + 6CEI + 3DEI - 3PEI - 13PKl^3 + 14BKl^3 + 23CKl^3 + 13DKl^3 \tag{e}$$

Substitution of Eqs. (b)–(d) into the second and third of Eqs. (a) gives, after simplification,

$$0 = 6BEI + 12CEI + 3DEI - 3PEI - 23PKl^3 + 23BKl^3 + 40CKl^3 + 23DKl^3 \tag{f}$$

$$0 = 6BEI + 6CEI + 9DEI - 3PEI - 27PKl^3 + 26BKl^3 + 46CKl^3 + 27DKl^3 \tag{g}$$

Equations (e)–(g) are three simultaneous equations in the three unknowns, B, C, and D. Their magnitudes depend on the magnitudes of E, I, and K. Using the values specified in the problem, we have

$$\begin{aligned}
0 &= B + 1.0622C + 0.5838D - 0.5838P \\
0 &= B + 1.8015C + 0.8804D - 0.8804P \\
0 &= B + 1.6019C + 1.1389D - 0.9213P
\end{aligned} \tag{h}$$

The solution of Eqs. (h) and (c) is

$$\begin{aligned}
A &= -0.0379P = -455 \text{ N} \\
B &= 0.1014P = 1217 \text{ N} \\
C &= 0.2578P = 3094 \text{ N} \\
D &= 0.3573P = 4288 \text{ N}
\end{aligned}$$

The maximum deflection of the beam is the deflection under the load P, which is equal to the deflection of the spring at D: