CHAPTER 5

APPLICATIONS OF ENERGY METHODS

Energy methods are used widely to obtain solutions to elasticity problems and determine deflections of structures and machines. Since energy is a scalar quantity, energy methods are sometimes called scalar methods. In this chapter, energy methods are employed to obtain elastic deflections of statically determinate structures and to determine redundant reactions and deflections of statically indeterminate structures. The applications of energy methods in this book are limited mainly to linearly elastic material behavior and small displacements. However, in Sections 5.1 and 5.2, energy methods are applied to two nonlinear problems to demonstrate their generality.

For the determination of the deflections of structures, two energy principles are presented: 1. the principle of stationary potential energy and 2. Castigliano's theorem on deflections.

5.1 PRINCIPLE OF STATIONARY POTENTIAL ENERGY

We employ the concept of generalized coordinates $(x_1, x_2, \ldots, x_n)$ to describe the shape of a structure in equilibrium (Langhaar, 1989; Section 1.2). Since plane cross sections of the members are assumed to remain plane, the changes of the generalized coordinates denote the translation and rotation of the cross section of the member.

In this chapter, we consider applications in which a finite number of degrees of freedom, equal to the number of generalized coordinates, specifies the configuration of the system.

Consider a system with a finite number of degrees of freedom that has the equilibrium configuration $(x_1, x_2, \ldots, x_n)$. A virtual (imagined) displacement is imposed such that the new configuration is $(x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n)$, where $(\delta x_1, \delta x_2, \ldots, \delta x_n)$ is the virtual displacement.[1] The virtual work δW corresponding to the virtual displacement is given by

$$\delta W = Q_1 \delta x_1 + Q_2 \delta x_2 + \cdots + Q_i \delta x_i + \cdots + Q_n \delta x_n \tag{a}$$

where $(Q_1, Q_2, \ldots, Q_i, \ldots, Q_n)$ are components of the generalized load. They are functions of the generalized coordinates. Let Q_i be defined for a given cross section of the structure;

[1]Note that the virtual displacement must not violate the essential boundary conditions (support conditions) for the structure.

Q_i is a force if δx_i is a translation of the cross section, and Q_i is a moment (or torque) if δx_i is a rotation of the cross section.

For a deformable body the virtual work δW corresponding to virtual displacement of a mechanical system may be separated into the sum

$$\delta W = \delta W_e + \delta W_i \tag{b}$$

where δW_e is the virtual work of the external forces and δW_i is the virtual work of the internal forces.

Analogous to the expression for δW in Eq. (a), under a virtual displacement (δx_1, δx_2, ..., δx_n), we have

$$\delta W_e = P_1 \delta x_1 + P_2 \delta x_2 + \cdots + P_n \delta x_n \tag{c}$$

where ($P_1, P_2, \ldots, P_n$) are functions of the generalized coordinates ($x_1, x_2, \ldots, x_n$). By analogy to the Q_i in Eq. (a), the functions ($P_1, P_2, \ldots, P_n$) are called the components of generalized *external* load. If the generalized coordinates (x_1, x_2, ..., x_n) denote displacements and rotations that occur in a system, the variables ($P_1, P_2, \ldots, P_n$) may be identified as the components of the prescribed external forces and couples that act on the system.

Now imagine that the virtual displacement takes the system completely around any closed path. At the end of the closed path, we have $\delta x_1 = \delta x_2 = \cdots = \delta x_n = 0$. Hence, by Eq. (c), $\delta W_e = 0$. In our applications, we consider only systems that undergo elastic behavior. Then the virtual work δW_i of the internal forces is equal to the negative of the virtual change in the elastic strain energy δU, that is,

$$\delta W_i = -\delta U \tag{d}$$

where $U = U(x_1, x_2, \ldots, x_n)$ is the total strain energy of the system. Since the system travels around a closed path, it returns to its initial state and, hence, $\delta U = 0$. Consequently, by Eq. (d), $\delta W_i = 0$. Accordingly, the total virtual work δW [Eq. (b)] also vanishes around a closed path. The condition $\delta W = 0$ for virtual displacements that carry the system around a closed path indicates that the system is *conservative*. The condition $\delta W = 0$ is known as the *principle of stationary potential energy.*

For a conservative system (e.g., elastic structure loaded by conservative external forces), the virtual change in strain energy δU of the structure under the virtual displacement (δx_1, δx_2, ..., δx_n) is

$$\delta U = \frac{\partial U}{\partial x_1}\delta x_1 + \frac{\partial U}{\partial x_2}\delta x_2 + \cdots + \frac{\partial U}{\partial x_n}\delta x_n \tag{e}$$

Then, Eqs. (a) through (e) yield the result

$$\begin{aligned} Q_1 \delta x_1 + Q_2 \delta x_2 + \cdots + Q_n \delta x_n &= P_1 \delta x_1 + P_2 \delta x_2 + \cdots + P_n \delta x_n \\ &\quad - \frac{\partial U}{\partial x_1}\delta x_1 - \frac{\partial U}{\partial x_2}\delta x_2 - \cdots - \frac{\partial U}{\partial x_n}\delta x_n \end{aligned}$$

or

$$Q_i = P_i - \frac{\partial U}{\partial x_i}, \quad i = 1, 2, \ldots, n \tag{f}$$

For any system with finite degrees of freedom, if the components Q_i of the generalized force vanish, then the system is in equilibrium. Therefore, by Eq. (f), an elastic system with n degrees of freedom is in equilibrium if (Langhaar, 1989; Section 1.9)

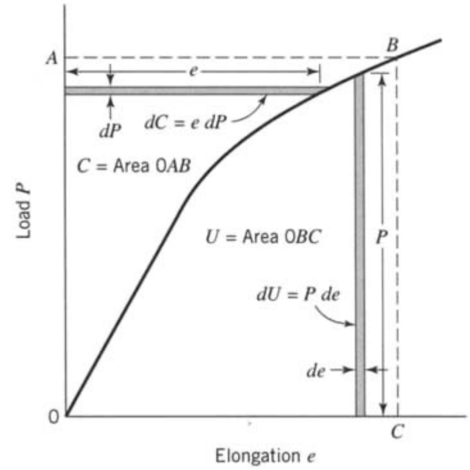

FIGURE 5.1 Nonlinear elastic load–elongation curve.

$$P_i = \frac{\partial U}{\partial x_i}, \quad i = 1, 2, \ldots, n \tag{5.1}$$

The relation given in Eq. 5.1 is sometimes referred to as *Castigliano's first theorem*. For a structure, the strain energy U is obtained as the sum of the strain energies of the members of the structure. Note the similarity between Eqs. 5.1 and Eqs. 3.11.

As a simple example, consider a uniform bar loaded at its ends by an axial load P. Let the bar be made of a nonlinear elastic material with the load–elongation curve indicated in Figure 5.1. The area below the curve represents the total strain energy U stored in the bar, that is, $U = \int P\, de$; then by Eq. 5.1, $P = \partial U/\partial e$, where P is the generalized external force and e the generalized coordinate. If the load–elongation data for the bar are plotted as a stress–strain curve, the area below the curve is the strain-energy density U_0 stored in the bar (see Figure 3.1). Then, $U_0 = \int \sigma\, d\epsilon$ and, by Eqs. 3.11, $\sigma = \partial U_0/\partial \epsilon$.

Equation 5.1 is valid for nonlinear elastic (conservative) problems in which the nonlinearity is due either to finite geometry changes or material behavior, or both. The equation is also valid for systems with inelastic materials as long as the loading is monotonic and proportional. The following example problem indicates the application of Eq. 5.1 for finite geometry changes.

EXAMPLE 5.1
Equilibrium of a Linear Elastic Two-Bar System

Two bars AB and CB of lengths L_1 and L_2, respectively, are attached to a rigid foundation at points A and C, as shown in Figure E5.1a. The cross-sectional area of bar AB is A_1 and that of bar CB is A_2. The corresponding moduli of elasticity are E_1 and E_2. Under the action of horizontal and vertical forces P and Q, pin B undergoes finite horizontal and vertical displacement with components u and v, respectively (Figure E5.1a). The bars AB and CB remain linearly elastic.

(a) Derive formulas for P and Q in terms of u and v.

(b) Let $E_1A_1/L_1 = K_1 = 2.00$ N/mm and $E_2A_2/L_2 = K_2 = 3.00$ N/mm, and let $b_1 = h = 400$ mm and $b_2 = 300$ mm. For $u = 30$ mm and $v = 40$ mm, determine the values of P and Q using the formulas derived in part (a).

(c) Consider the equilibrium of the pin B in the displaced position B^* and verify the results of part (b).

(d) For small displacement components u and v (u, $v << L_1$, L_2), linearize the formulas for P and Q derived in part (a).

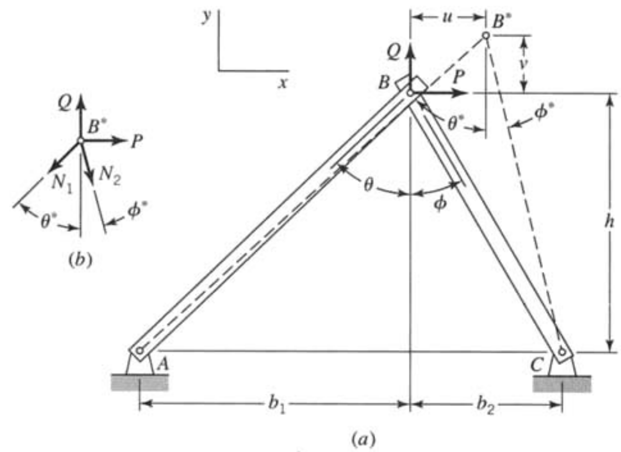

FIGURE E5.1

Solution

(a) For this problem the generalized external forces are $P_1 = P$ and $P_2 = Q$ and the generalized coordinates are $x_1 = u$ and $x_2 = v$. For the geometry of Figure E5.1*a*, the elongations e_1 and e_2 of bars 1 (bar *AB* with length L_1) and 2 (bar *CB* with length L_2) can be obtained in terms of *u* and *v* as follows:

$$\begin{aligned}(L_1 + e_1)^2 &= (b_1 + u)^2 + (h + v)^2, \quad L_1^2 = b_1^2 + h^2 \\ (L_2 + e_2)^2 &= (b_2 - u)^2 + (h + v)^2, \quad L_2^2 = b_2^2 + h^2\end{aligned} \tag{a}$$

Solving for (e_1, e_2), we obtain

$$\begin{aligned}e_1 &= \sqrt{(b_1 + u)^2 + (h + v)^2} - L_1 \\ e_2 &= \sqrt{(b_2 - u)^2 + (h + v)^2} - L_2\end{aligned} \tag{b}$$

Since each bar remains linearly elastic, the strain energies U_1 and U_2 of bars *AB* and *CB* are

$$\begin{aligned}U_1 &= \frac{1}{2}N_1 e_1 = \frac{E_1 A_1}{2L_1} e_1^2 \\ U_2 &= \frac{1}{2}N_2 e_2 = \frac{E_2 A_2}{2L_2} e_2^2\end{aligned} \tag{c}$$

where N_1 and N_2 are the tension forces in the two bars. The elongations of the two bars are given by the relation $e_i = N_i L_i / E_i A_i$. The total strain energy *U* for the structure is equal to the sum $U_1 + U_2$ of the strain energies of the two bars; therefore by Eqs. (c),

$$U = \frac{E_1 A_1}{2L_1} e_1^2 + \frac{E_2 A_2}{2L_2} e_2^2 \tag{d}$$

The magnitudes of *P* and *Q* are obtained by differentiation of Eq. (d) with respect to *u* and *v*, respectively (see Eq. 5.1). Thus,

$$\begin{aligned}P &= \frac{\partial U}{\partial u} = \frac{E_1 A_1 e_1}{L_1} \frac{\partial e_1}{\partial u} + \frac{E_2 A_2 e_2}{L_2} \frac{\partial e_2}{\partial u} \\ Q &= \frac{\partial U}{\partial v} = \frac{E_1 A_1 e_1}{L_1} \frac{\partial e_1}{\partial v} + \frac{E_2 A_2 e_2}{L_2} \frac{\partial e_2}{\partial v}\end{aligned} \tag{e}$$

The partial derivatives of e_1 and e_2 with respect to *u* and *v* are obtained from Eqs. (b). Taking the derivatives and substituting in Eqs. (e), we find

$$
\begin{aligned}
P &= \frac{E_1A_1(b_1+u)}{L_1}\frac{\sqrt{(b_1+u)^2+(h+v)^2}-L_1}{\sqrt{(b_1+u)^2+(h+v)^2}} \\
&\quad - \frac{E_2A_2(b_2-u)}{L_2}\frac{\sqrt{(b_2-u)^2+(h+v)^2}-L_2}{\sqrt{(b_2-u)^2+(h+v)^2}} \\
Q &= \frac{E_1A_1(h+v)}{L_1}\frac{\sqrt{(b_1+u)^2+(h+v)^2}-L_1}{\sqrt{(b_1+u)^2+(h+v)^2}} \\
&\quad + \frac{E_2A_2(h+v)}{L_2}\frac{\sqrt{(b_2-u)^2+(h+v)^2}-L_2}{\sqrt{(b_2-u)^2+(h+v)^2}}
\end{aligned} \tag{f}
$$

(b) Substitution of the values K_1, K_2, b_1, b_2, h, L_1, L_2, u, and v into Eqs. (f) gives

$$
\begin{aligned}
P &= 43.8 \text{ N} \\
Q &= 112.4 \text{ N}
\end{aligned} \tag{g}
$$

(c) The values of P and Q may be verified by determining the tension forces N_1 and N_2 in the two bars, determining directions of the axes of the two bars for the deformed configuration, and applying equations of equilibrium to a free-body diagram of pin B^*. Elongations $e_1 = 49.54$ mm and $e_2 = 16.24$ mm are given by Eqs. (b). The tension forces N_1 and N_2 are

$$
\begin{aligned}
N_1 &= e_1K_1 = 99.08 \text{ N} \\
N_2 &= e_2K_2 = 48.72 \text{ N}
\end{aligned}
$$

Angles θ^* and ϕ^* for the directions of the axes of the two bars for the deformed configurations are found to be 0.7739 and 0.5504 rad, respectively. The free-body diagram of pin B^* is shown in Figure E5.1*b*. The equations of equilibrium are

$$
\begin{aligned}
\sum F_x &= 0 = P - N_1\sin\theta^* + N_2\sin\phi^*; \quad \text{hence, } P = 43.8 \text{ N} \\
\sum F_y &= 0 = Q - N_1\cos\theta^* - N_2\cos\phi^*; \quad \text{hence, } Q = 112.4 \text{ N}
\end{aligned}
$$

These values of P and Q agree with those of Eqs. (g).

(d) If displacements u and v are very small compared to b_1 and b_2, and, hence, with respect to L_1 and L_2, simple approximate expressions for P and Q may be obtained. For example, we find by the binomial expansion to linear terms in u and v that

$$
\begin{aligned}
\sqrt{(b_1+u)^2+(h+v)^2} &= L_1 + \frac{b_1u}{L_1} + \frac{hv}{L_1} \\
\sqrt{(b_2-u)^2+(h+v)^2} &= L_2 - \frac{b_2u}{L_2} + \frac{hv}{L_2}
\end{aligned}
$$

With these approximations, Eqs. (f) yield the linear relations

$$
\begin{aligned}
P &= \frac{E_1A_1b_1}{L_1^3}(b_1u+hv) + \frac{E_2A_2b_2}{L_2^3}(b_2u-hv) \\
Q &= \frac{E_1A_1h}{L_1^3}(b_1u+hv) + \frac{E_2A_2h}{L_2^3}(-b_2u+hv)
\end{aligned}
$$

If these equations are solved for the displacements u and v, the resulting relations are identical to those derived by means of Castigliano's theorem on deflections for linearly elastic materials (Sections 5.3 and 5.4).

5.2 CASTIGLIANO'S THEOREM ON DEFLECTIONS

The derivation of Castigliano's theorem on deflections is based on the concept of complementary energy C of the system. Consequently, the theorem is sometimes called the "principle of complementary energy." The complementary energy C is equal to the strain energy U in the case of linear material response. However, for nonlinear material response, complementary energy and strain energy are not equal (see Figure 5.1 and also Figure 3.1).

In the derivation of Castigliano's theorem, the complementary energy C is regarded as a function of generalized forces $(F_1, F_2, \ldots, F_p)$ that act on a system that is mounted on rigid supports (say the beam in Figure 5.2). The complementary energy C depends also on distributed loads that act on the beam, as well as the weight of the beam. However, these distributed forces do not enter explicitly into consideration in the derivation. In addition, the beam may be subjected to temperature effects (e.g., thermal strains; see Boresi and Chong, 2000; Chapter 4), which are not considered here.

Castigliano's theorem may be stated generally as follows (Langhaar, 1989; Section 4.10):

If an elastic system is supported so that rigid-body displacements of the system are prevented, and if certain concentrated forces of magnitudes $F_1, F_2, \ldots, F_p$ act on the system, in addition to distributed loads and thermal strains, the displacement component q_i of the point of application of the force F_i, is determined by the equation

$$q_i = \frac{\partial C}{\partial F_i}, \quad i = 1, 2, \ldots, p \tag{5.2}$$

Note the similarity of Eqs. 5.2 and 3.19. The relation given by Eq. 5.2 is sometimes referred to as Castigliano's second theorem. With reference to Figure 5.2, the displacement q_1 at the location of F_1 in the direction of F_1 is given by the relation $q_1 = \partial C/\partial F_1$.

The derivation of Eq. 5.2 is based on the assumption of small displacements; therefore, Castigliano's theorem is restricted to small displacements of the structure. The complementary energy C of a structure composed of m members may be expressed by the relation

$$C = \sum_{i=1}^{m} C_i$$

where C_i denotes the complementary energy of the ith member (Langhaar, 1989; Section 4.10).

Castigliano's theorem on deflections may be extended to compute the rotation of line elements in a system subjected to couples. For example, consider again a beam that is supported on rigid supports and subjected to external concentrated forces of magnitudes $F_1, F_2, \ldots, F_p$ (Figure 5.3). Let two of the concentrated forces (F_1, F_2) be parallel, lie in a principal plane of the cross section, have opposite senses, and act perpendicular to the

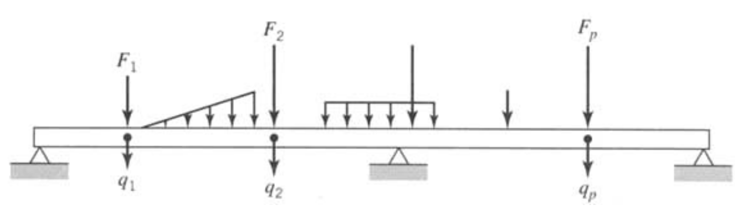

FIGURE 5.2 Beam on rigid supports.

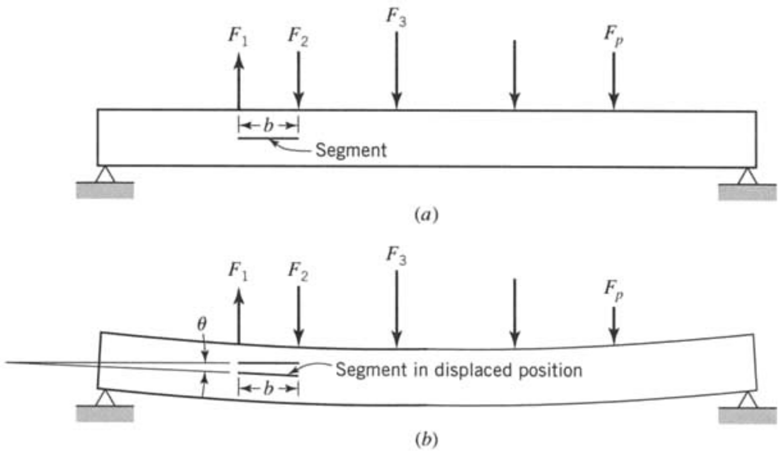

FIGURE 5.3 (*a*) Beam before deformation. (*b*) Beam after deformation.

ends of a line element of length b in the beam (Figure 5.3*a*). Then, Eq. 5.2 shows that the rotation θ (Figure 5.3*b*) of the line segment resulting from the deformations is given by the relation

$$\theta = \frac{1}{b}\frac{\partial C}{\partial F_1} + \frac{1}{b}\frac{\partial C}{\partial F_2} \tag{a}$$

where we have employed the condition of small displacements. To interpret this result, we employ the chain rule of partial differentiation of the complementary energy function C with respect to a scalar variable S. Considering the magnitudes of F_1 and F_2 to be functions of S, we have by the chain rule

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial F_1}\frac{\partial F_1}{\partial S} + \frac{\partial C}{\partial F_2}\frac{\partial F_2}{\partial S} \tag{b}$$

In particular, we take the variable S equal to F_1 and F_2, that is, $S = F_1 = F_2 = F$, where F denotes the magnitudes of F_1 and F_2. Then, $\partial F_1/\partial S = \partial F_2/\partial S = 1$, and we obtain by Eq. (b)

$$\frac{\partial C}{\partial F} = \frac{\partial C}{\partial F_1} + \frac{\partial C}{\partial F_2} \tag{c}$$

Consequently, Eqs. (a) and (c) yield

$$\theta = \frac{1}{b}\frac{\partial C}{\partial F} \tag{d}$$

and since the equal and opposite forces F_1, F_2 constitute a couple of magnitude $M = bF$, Eq. (d) may be written in the form $\theta = \partial C/\partial M$. More generally, for couples M_i and rotations θ_i, we may write

$$\theta_i = \frac{\partial C}{\partial M_i}, \quad i = 1, 2, \ldots, s \tag{5.3}$$

Hence, Eq. 5.3 determines the angular displacement θ_i of the arm of a couple of magnitude M_i that acts on an elastic structure. The sense of θ_i is the same as that of the couple M_i.

Whereas Eqs. 5.2 and 5.3 are restricted to small displacements, they may be applied to structures that possess nonlinear elastic material behavior (Langhaar, 1989). The following example problem indicates the application of Eq. 5.2 for nonlinear elastic material behavior.

EXAMPLE 5.2 Equilibrium of a Nonlinear Elastic Two-Bar System

Let the two bars in Figure E5.1 be made of a nonlinear elastic material whose stress–strain diagram is approximated by the relation $\epsilon = \epsilon_0 \sinh(\sigma/\sigma_0)$, where ϵ_0 and σ_0 are material constants (Figure E5.2). The system is subjected to known loads P and Q. By means of Castigliano's theorem on deflections, determine the small displacement components u and v. Let $P = 10.0$ kN, $Q = 30.0$ kN, $\sigma_0 = 70.0$ MPa, $\epsilon_0 = 0.001$, $b_1 = h = 400$ mm, $b_2 = 300$ mm, and $A_1 = A_2 = 300$ mm^2. Show that the values for u and v agree with those obtained by a direct application of equations of equilibrium and the consideration of the geometry of the deformed bars.

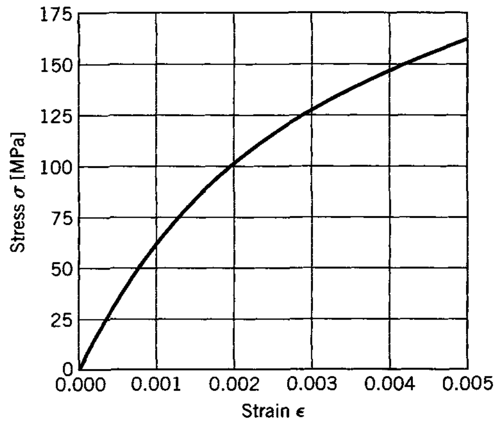

FIGURE E5.2

Solution

Let N_1 and N_2 be the tensions in bars AB and CB. From the equilibrium conditions for pin B, we find

$$N_1 = \frac{L_1(Qb_2 + Ph)}{h(b_1 + b_2)}$$
$$N_2 = \frac{L_2(Qb_2 - Ph)}{h(b_1 + b_2)} \tag{a}$$

The complementary energy C for the system is equal to the sum of the complementary energies for the two bars. Thus,

$$C = C_1 + C_2 = \int_0^{N_1} e_1 dN_1 + \int_0^{N_2} e_2 dN_2 \tag{b}$$

With $e_1 = \epsilon_1 L_1$ and $e_2 = \epsilon_2 L_2$, Eq. (b) becomes

$$C = \int_0^{N_1} L_1\epsilon_0 \sinh\frac{N_1}{A_1\sigma_0} dN_1 + \int_0^{N_2} L_2\epsilon_0 \sinh\frac{N_2}{A_2\sigma_0} dN_2 \tag{c}$$

The displacement components u and v are obtained by substitution of Eq. (c) into Eq. 5.2. Thus, we find

$$u = q_P = \frac{\partial C}{\partial P} = L_1\epsilon_0\left(\sinh\frac{N_1}{A_1\sigma_0}\right)\frac{\partial N_1}{\partial P} + L_2\epsilon_0\left(\sinh\frac{N_2}{A_2\sigma_0}\right)\frac{\partial N_2}{\partial P}$$
$$v = q_Q = \frac{\partial C}{\partial Q} = L_1\epsilon_0\left(\sinh\frac{N_1}{A_1\sigma_0}\right)\frac{\partial N_1}{\partial Q} + L_2\epsilon_0\left(\sinh\frac{N_2}{A_2\sigma_0}\right)\frac{\partial N_2}{\partial Q} \tag{d}$$

The partial derivatives of N_1 and N_2 with respect to P and Q are obtained by means of Eqs. (a). Taking derivatives and substituting into Eqs. (d), we obtain

$$u = \frac{L_1^2\epsilon_0}{b_1+b_2}\sinh\frac{L_1(Qb_2+Ph)}{A_1\sigma_0 h(b_1+b_2)} - \frac{L_2^2\epsilon_0}{b_1+b_2}\sinh\frac{L_2(Qb_1-Ph)}{A_2\sigma_0 h(b_1+b_2)}$$
$$v = \frac{L_1^2 b_2\epsilon_0}{h(b_1+b_2)}\sinh\frac{L_1(Qb_2+Ph)}{A_1\sigma_0 h(b_1+b_2)} + \frac{L_2^2 b_1\epsilon_0}{h(b_1+b_2)}\sinh\frac{L_2(Qb_1-Ph)}{A_2\sigma_0 h(b_1+b_2)} \quad \text{(e)}$$

Substitution of values for P, Q, σ_0, ϵ_0, b_1, b_2, h, A_1, and A_2 gives, with Eq. (a) of Example 5.1,

$$\begin{aligned} u &= 0.4709 \text{ mm} \\ v &= 0.8119 \text{ mm} \end{aligned} \quad \text{(f)}$$

An alternate method of calculating u and v is as follows: Determine tensions N_1 and N_2 in the two bars by Eqs. (a); next, determine elongations e_1 and e_2 for the two bars and use these values of e_1 and e_2 along with geometric relations to calculate values for u and v. Equations (a) give $N_1 = 26.268$ kN and $N_2 = 14.286$ kN. Elongations e_1 and e_2 are given by the relations

$$e_1 = L_1\epsilon_0\sinh\frac{N_1}{A_1\sigma_0} = 565.68(0.001)\sinh\frac{26{,}268}{300(70)} = 0.9071 \text{ mm}$$

$$e_2 = L_2\epsilon_0\sinh\frac{N_2}{A_2\sigma_0} = 500.00(0.001)\sinh\frac{14{,}286}{300(70)} = 0.3670 \text{ mm}$$

With e_1 and e_2 known, values of u and v are given by the following geometric relations:

$$u = \frac{e_1\cos\phi - e_2\cos\theta}{\sin\theta\cos\phi + \cos\theta\sin\phi} = 0.4709 \text{ mm}$$
$$v = \frac{e_1\sin\phi + e_2\cos\theta}{\sin\theta\cos\phi + \cos\theta\sin\phi} = 0.8119 \text{ mm}$$

These values of u and v agree with those of Eqs. (f). Thus, Eq. 5.2 gives the correct values of u and v for this problem of nonlinear material behavior.

5.3 CASTIGLIANO'S THEOREM ON DEFLECTIONS FOR LINEAR LOAD–DEFLECTION RELATIONS

In the remainder of this chapter we limit our consideration mainly to linear elastic material behavior and small displacements. Then, the resulting load–deflection relation for either a member or structure is linear, the strain energy U is equal to the complementary energy C, and the principle of superposition applies. Then, Eqs. 5.2 and 5.3 may be written

$$q_i = \frac{\partial U}{\partial F_i}, \quad i = 1, 2, \ldots, p \tag{5.4}$$

$$\theta_i = \frac{\partial U}{\partial M_i}, \quad i = 1, 2, \ldots, s \tag{5.5}$$

where $U = U(F_1, F_2, \circ, F_p, M_1, M_2, \circ, M_s)$.

The strain energy U is

$$U = \int U_0\, dV \tag{5.6}$$

where U_0 is the strain-energy density. In the remainder of this chapter we restrict ourselves to linear elastic, isotropic, homogeneous materials for which the strain-energy density is (see Eq. 3.33)

$$U_0 = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2G}(\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2) \tag{5.7}$$

With load–stress formulas derived for the members of the structure, U_0 may be expressed in terms of the loads that act on the structure. Then, Eq. 5.6 gives U as a function of the loads. Equations 5.4 and 5.5 can then be used to obtain displacements at the points of application of the concentrated forces or the rotations in the direction of the concentrated moments. Three types of loads are considered in this chapter for the various members of a structure: 1. axial loading, 2. bending of beams, and 3. torsion. In practice, it is convenient to obtain the strain energy for each type of load acting alone and then add together these strain energies to obtain the total strain energy U, instead of using load–stress formulas and Eqs. 5.6 and 5.7 to obtain U.

5.3.1 Strain Energy U_N for Axial Loading

The equation for the total strain energy U_N resulting from axial loading is derived for the tension members shown in Figures 5.4*a* and 5.4*d*. In general, the cross-sectional area A of the tension member may vary *slowly* with axial coordinate z. The line of action of the loads (the z axis) passes through the centroid of every cross section of the tension member. Consider two sections BC and DF of the tension member in Figure 5.4*a* at distance dz apart. After the loads are applied, these sections are displaced to B^*C^* and D^*F^* (shown by the enlarged free-body diagram in Figure 5.4*b*) and the original length dz has elongated an amount de_z. For linear elastic material behavior, de_z varies linearly with N as indicated in Figure 5.4*c*. The shaded area below the straight line is equal to the strain energy dU_N for the segment dz of the tension member. The total strain energy U_N for the tension member becomes

$$U_N = \int dU_N = \int \frac{1}{2} N \, de_z$$

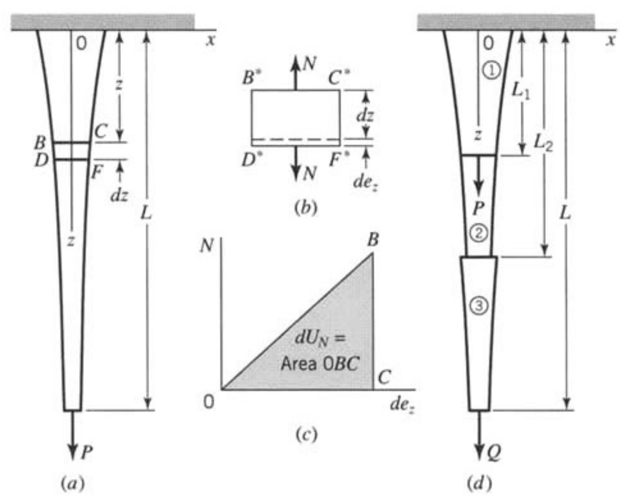

FIGURE 5.4 Strain energy resulting from axial loading of member.

Noting that $de_z = \epsilon_{zz} dz$ and assuming that the cross-sectional area varies slowly, we have $\epsilon_{zz} = \sigma_{zz}/E$ and $\sigma_{zz} = N/A$, where A is the cross-sectional area of the member at section z. Then, we write U_N in the form

$$U_N = \int_0^L \frac{N^2}{2EA} dz \tag{5.8}$$

At abrupt changes in material properties, load, and cross section, the values of E, N, and A change abruptly (see Figure 5.4*d*). Then, we must account approximately for these changes by writing U_N in the form

$$U_N = \int_0^{L_1} \frac{N_1^2}{2E_1A_1} dz + \int_{L_1}^{L_2} \frac{N_2^2}{2E_2A_2} dz + \int_{L_2}^{L} \frac{N_3^2}{2E_3A_3} dz \tag{5.9}$$

where an abrupt change in load occurs at L_1 and an abrupt change in cross-sectional area occurs at L_2. The subscripts 1, 2, 3 refer to properties in parts 1, 2, 3 of the member (Figure 5.4*d*).

Strain Energy for Axially Loaded Springs

Consider an elastic spring subjected to an axial force Q (Figure 5.5*a*). The load–deflection curve for the spring is not necessarily linear (Figure 5.5*b*). The external force Q is applied slowly so that it remains in equilibrium with the internal tension force F in the spring. The potential energy U of the axially loaded spring that undergoes an axial elongation δ is defined as the work that Q performs under the elongation δ. Thus, by Figure 5.5*b*,

$$U = \int dU = \int_0^{\delta} Q\, dx \tag{5.10}$$

Similarly, by Figure 5.5*b*, the complementary strain energy C of the spring is

$$C = \int dC = \int_0^{P} x\, dQ \tag{5.11}$$

For a linear spring (Figure 5.5*c*), $C = U$. Hence, we may write

$$U = C = \int_0^{\delta} Q\, dx$$

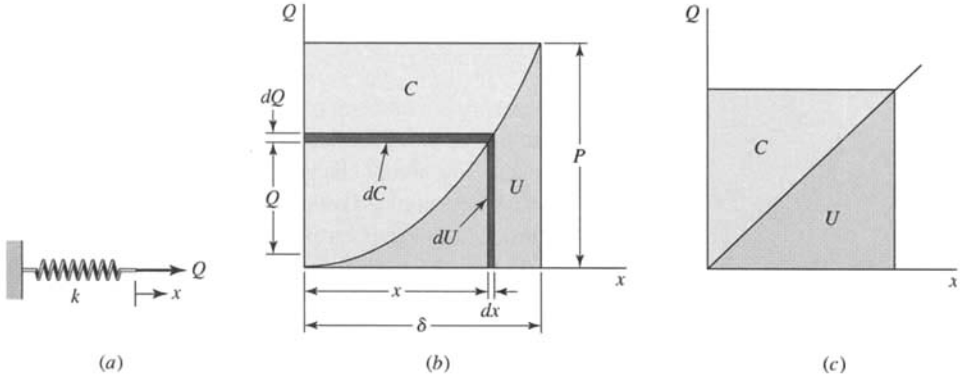

FIGURE 5.5 (*a*) Axially loaded spring. (*b*) Nonlinear load–elongation response. (*c*) Linear load–elongation response.

Also, for a linear spring, the internal tension force is $F = kx$, where k is a spring constant [F/L] and x is the elongation of the spring from its initially unstretched length L_0. Since for equilibrium, $F = Q$,

$$U = C = \int_0^{\delta} F\,dx = \int_0^{\delta} kx\,dx = \frac{1}{2}k\delta^2 \tag{5.12}$$

The magnitude of the energy that a spring can absorb is a function of its elongation δ, and it is important in certain mechanical designs. For example, springs are commonly used to absorb energy in vehicles (automobiles and aircraft) when they are subjected to impact loads (rough roads and landings).

5.3.2 Strain Energies U_M and U_S for Beams

Consider a beam of uniform (or slowly varying) cross section, as in Figure 5.6*a*. We take the (x, y, z) axes with origin at the centroid of the cross section and with the z axis along the axis of the beam, the y axis down, and the x axis normal to the plane of the paper. The (x, y) axes are assumed to be principal axes for each cross section of the beam (see Appendix B). The loads P, Q, and R are assumed to lie in the (y, z) plane. The flexure formula

$$\sigma_{zz} = \frac{M_x y}{I_x}$$

is assumed to hold, where M_x is the internal bending moment with respect to the principal x axis, I_x is the moment of inertia of the cross section at z about the x axis, and y is measured from the (x, z) plane. Consider two sections BC and DF of the unloaded beam at distance dz apart. After the loads are applied to the beam, plane sections BC and DF are displaced to B^*C^* and D^*F^* and are assumed to remain plane. An enlarged free-body diagram of the deformed beam segment is shown in Figure 5.6*b*. M_x causes plane D^*F^* to rotate through angle $d\phi$ with respect to B^*C^*. For linear elastic material behavior, $d\phi$ varies linearly with M_x as indicated in Figure 5.6*c*. The shaded area below the inclined straight line is equal to the strain energy dU_M resulting from bending of the beam segment dz. An additional strain energy dU_S caused by the shear V_y is considered later. The strain energy U_M for the beam caused by M_x becomes

$$U_M = \int dU_M = \int \frac{1}{2}M_x\,d\phi$$

Noting that $d\phi = de_z/y$ and $de_z = \epsilon_{zz}dz$, and assuming that $\epsilon_{zz} = \sigma_{zz}/E$ and $\sigma_{zz} = M_x y/I_x$, we may write U_M in the form

$$U_M = \int \frac{M_x^2}{2EI_x}\,dz \tag{5.13}$$

where in general M_x is a function of z. Equation 5.13 represents the strain energy resulting from bending about the x axis. A similar relation is valid for bending about the y axis for loads lying in the (x, z) plane. For abrupt changes in material E, moment M_x, or moment of inertia I_x, the value of U_M may be computed following the same procedure as for U_N (Eq. 5.9).

Equation 5.13 is exact for pure bending but is only approximate for shear loading as indicated in Figure 5.6*a*. However, more exact solutions and experimental data indicate that Eq. 5.13 is fairly accurate, except for relatively short beams.

An exact expression for the strain energy U_S resulting from shear loading of a beam is difficult to obtain. Consequently, an approximate expression for U_S is often used. When corrected by an appropriate coefficient, the use of this approximate expression often leads to fairly reliable results. The correction coefficients are discussed later.

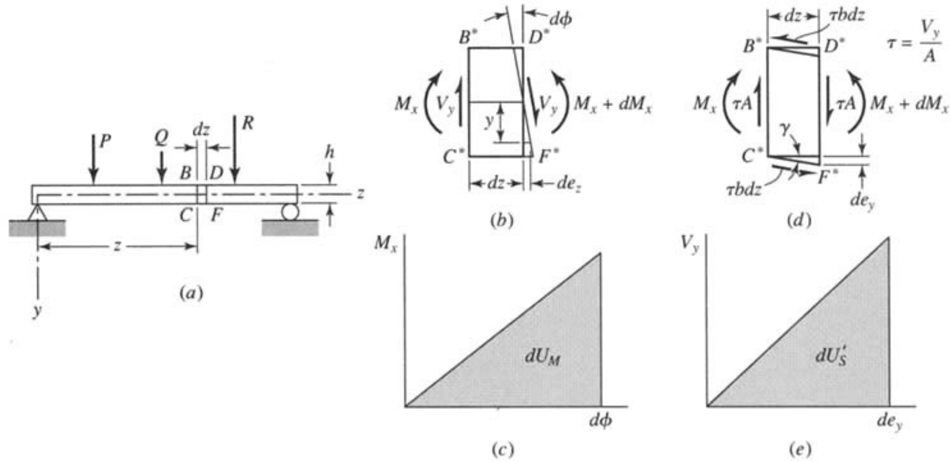

FIGURE 5.6 Strain energy resulting from bending and shear.

Because of the shear V_y (Figure 5.6*b*), shear stresses σ_{zy} are developed in each cross section; the magnitude of σ_{zy} is zero at both the top and bottom of the beam since the beam is not subjected to shear loads on the top or bottom surfaces. We define an average value of σ_{zy} as $\tau = V_y/A$. We assume that this average shear stress acts over the entire beam cross section (Figure 5.6*d*) and, for convenience, assume that the beam cross section is rectangular with thickness b. Because of the shear, the displacement of face D^*F^* with respect to face B^*C^* is de_y. For linear elastic material behavior, de_y varies linearly with V_y, as indicated in Figure 5.6*e*. The shaded area below the inclined straight line is equal to the strain energy dU_S' for the beam segment dz. A correction coefficient k is now defined such that the exact expression for the shear strain energy dU_S of the element is equal to kdU_S'. Then, the shear strain energy U_S for the beam resulting from shear V_y is

$$U_S = \int k\, dU_S' = \int \frac{k}{2} V_y\, de_y$$

Noting that $de_y = \gamma dz$ and assuming that $\gamma = \tau/G$ and $\tau = V_y/A$, we may write U_S in the form

$$U_S = \int \frac{kV_y^2}{2GA}\, dz \tag{5.14}$$

Equation 5.14 represents the strain energy for shear loading of a beam. The value of V_y is generally a function of z. Also, the cross-sectional area A may vary slowly with z. For abrupt changes in material E, shear V_y, or cross-sectional area A, the value of U_S may be computed following the same procedure as for U_N (Eq. 5.9).

An exact expression of U_S may be obtained, provided the exact shear stress distribution σ_{zy} is known. Then substitution of σ_{zy} into Eq. 5.7 to obtain U_0 (with the other stress components being zero) and then substitution into Eq. 5.6 yields U_S. However, the exact distribution of σ_{zy} is often difficult to obtain, and approximate distributions are used. For example, consider a segment dy of thickness b of a beam cross section. In the engineering theory of beams, the stress component σ_{zy} is assumed to be uniform over thickness b. With this assumption (see Section 1.1)

$$\sigma_{zy} = \frac{V_y Q}{I_x b}$$

TABLE 5.1 Correction Coefficients for Strain Energy Due to Shear

Beam cross section	k
Thin rectangle[a]	1.20
Solid circular[b]	1.33
Thin-wall circular[b]	2.00
I-section, channel, box-section[c]	1.00

[a]Exact value.
[b]Calculated as the ratio of the shear stress at the neutral surface to the average shear stress.
[c]The area A for the I-section, channel, or box-section is the area of the web hb, where h is the section depth and b the web thickness. The load is applied perpendicular to the axis of the beam and in the plane of the web.

where Q is the first moment about the x axis of the area above the line of length b with ordinate y. Generally, σ_{zy} is not uniform over thickness b. Nevertheless, if for a beam of rectangular cross section, one assumes that σ_{zy} is uniform over b, it may be shown that $k = 1.20$.

Exact values of k are not generally available. Fortunately, in practical problems, the shear strain energy U_S is often small compared to U_M. Hence, for practical problems, the need for exact values of U_S is not critical. Consequently, as an expedient approximation, the correction coefficient k in Eq. 5.14 may be obtained as the ratio of the shear stress at the neutral surface of the beam calculated using V_yQ/I_xb to the average shear stress V_y/A. For example, by this procedure, the magnitude of k for the rectangular cross section is

$$k = \frac{V_yQ}{I_xb}\frac{A}{V_y} = \frac{Qbh}{I_xb} = \frac{(bh/2)(h/4)h}{\frac{1}{12}bh^3} = 1.50 \tag{5.15}$$

This value is larger and hence more conservative than the more exact value 1.20. Nevertheless, since the more precise value is known, we recommend that $k = 1.20$ be used for rectangular cross sections. Values of k are listed in Table 5.1 for several beam cross sections.

5.3.3 Strain Energy U_T for Torsion

The strain energy U_T for a torsion member with circular cross section (Figure 5.7a) may be derived as follows: Let the z axis lie along the centroidal axis of the torsion member. Before torsional loads T_1 and T_2 are applied, sections BC and DF are a distance dz apart. After the torsional loads are applied, these sections become sections B^*C^* and D^*F^*, with section

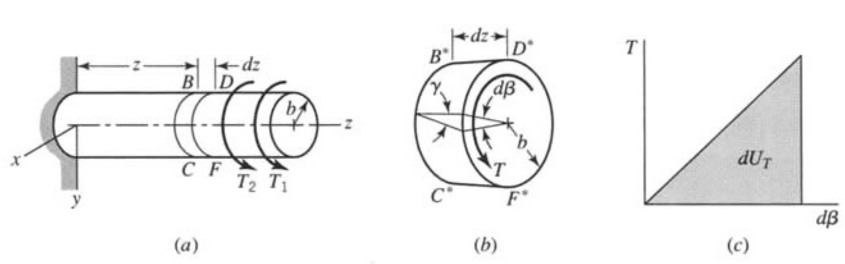

FIGURE 5.7 Strain energy resulting from torsion.

D^*F^* rotated relative to section B^*C^* through the angle $d\beta$, as shown in the enlarged free-body diagram of the element of length dz (Figure 5.7*b*). For linear elastic material behavior, $d\beta$ varies linearly with T (Figure 5.7*c*). The shaded area below the inclined straight line is equal to the torsional strain energy dU_T for the segment dz of the torsion member. Hence, the total torsional strain energy U_T for the torsional member becomes

$$U_T = \int dU_T = \int \frac{1}{2} T\, d\beta$$

Noting that $b\, d\beta = \gamma dz$ and assuming that $\gamma = \tau/G$ and $\tau = Tb/J$ (where b is the radius and J is the polar moment of inertia of the cross section), we may write U_T in the form

$$U_T = \int \frac{T^2}{2GJ} dz \tag{5.16}$$

Equation 5.16 represents the strain energy for a torsion member with circular cross section. The unit angle of twist θ for a torsion member of circular cross section is given by $\theta = T/GJ$. Torsion of noncircular cross sections is treated in Chapter 6. Equation 5.16 is valid for other cross sections if the unit angle of twist θ for a given cross section replaces T/GJ in Eq. 5.16. For abrupt changes in material E, torsional load T, or polar moment of inertia J, the value of U_T may be computed following the same procedure as for U_N (Eq. 5.9).

EXAMPLE 5.3
Linear Spring–Weight System

Consider weights W_1 and W_2 supported by the linear springs shown in Figure E5.3. The spring constants are k_1 and k_2. Determine the displacements q_1 and q_2 of the weights as functions of W_1, W_2, k_1, and k_2. Assume that the weights are applied slowly so that the system is always in equilibrium as the springs are stretched from their initially unstretched lengths.

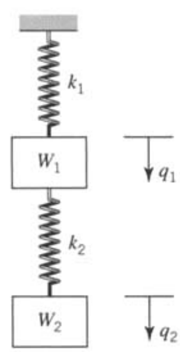

FIGURE E5.3

Solution

Under the displacements q_1 and q_2, spring 1 undergoes an elongation q_1 and spring 2 undergoes an elongation $q_2 - q_1$. Hence, by Eq. 5.12, the potential energy stored in the springs is

$$U = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 \tag{a}$$

The internal tension forces in springs 1 and 2 are

$$F_1 = k_1 q_1, \quad F_2 = k_2(q_2 - q_1) \tag{b}$$

But by equilibrium,

$$F_1 = W_1 + W_2, \quad F_2 = W_2 \tag{c}$$

Then, by Eqs. (a)–(c), the potential energy in terms of W_1, W_2, k_1, and k_2 is

$$U = \frac{1}{2}\frac{(W_1 + W_2)^2}{k_1} + \frac{1}{2}\frac{W_2^2}{k_2} \tag{d}$$

By Eqs. 5.4 and (d), we obtain

$$\begin{aligned} q_1 &= \frac{\partial U}{\partial W_1} = \frac{W_1 + W_2}{k_1} \\ q_2 &= \frac{\partial U}{\partial W_2} = \frac{W_1 + W_2}{k_1} + \frac{W_2}{k_2} \end{aligned} \tag{e}$$

Equation (e) agrees with a direct solution of Eqs. (b) and (c), without consideration of potential energy.

EXAMPLE 5.4
Nonlinear Spring System

Assume that the force–elongation relation for the springs of Example 5.3 is

$$F = kx^2 \tag{a}$$

Determine the displacements q_1 and q_2 of the weights as functions of W_1, W_2, k_1, and k_2.

Solution

By Eq. (a),

$$x = \left(\frac{F}{k}\right)^{1/2} \tag{b}$$

Hence, by Eqs. 5.11 and (b), the complementary strain energy of the system is

$$C = \int_0^{F_1} \left(\frac{F}{k_1}\right)^{1/2} dF + \int_0^{F_2} \left(\frac{F}{k_2}\right)^{1/2} dF$$

or

$$C = \frac{2}{3}\left(\frac{F_1^{3/2}}{k_1^{1/2}} + \frac{F_2^{3/2}}{k_2^{1/2}}\right) \tag{c}$$

By equilibrium, as in Example 5.3,

$$F_1 = W_1 + W_2, \quad F_2 = W_2 \tag{d}$$

So, Eqs. (c) and (d) yield

$$C = \frac{2}{3}\left(\frac{(W_1 + W_2)^{3/2}}{k_1^{1/2}} + \frac{W_2^{3/2}}{k_2^{1/2}}\right) \tag{e}$$

Then, by Eqs. 5.2 and (e), we obtain

$$\begin{aligned} q_1 &= \frac{\partial C}{\partial W_1} = \left(\frac{W_1 + W_2}{k_1}\right)^{1/2} \\ q_2 &= \frac{\partial C}{\partial W_2} = \left(\frac{W_1 + W_2}{k_1}\right)^{1/2} + \left(\frac{W_2}{k_2}\right)^{1/2} \end{aligned}$$

5.4 DEFLECTIONS OF STATICALLY DETERMINATE STRUCTURES

In the analysis of many engineering structures, the equations of static equilibrium are both necessary and sufficient to solve for unknown reactions and for internal actions in the members of the structure. For example, the simple structure shown in Figure E5.1 is such a structure; the equations of static equilibrium are sufficient to solve for the tensions N_1 and N_2 in members AB and CB, respectively. Structures for which the equations of static equilibrium are sufficient to determine the unknown tensions, shears, etc., uniquely are said to be *statically determinate structures*. Implied in the expression "statically determinate" is the condition that the deflections caused by the loads are so small that the geometry of the initially unloaded structure remains essentially unchanged and the angles between members are essentially constant. If these conditions were not true, the internal tensions, etc., could not be determined without including the effects of the deformation and, hence, they could not be determined solely from the equations of equilibrium.

The truss shown in Figure 5.8 is a statically determinate truss. A physical characteristic of a statically determinate structure is that every member is essential for the proper functioning of the structure under the various loads to which it is subjected. For example, if member AC were to be removed from the truss of Figure 5.8, the truss would collapse.

Often, additional members are added to structures to stiffen the structure (reduce deflections), to strengthen the structure (increase its load-carrying capacity), or to provide alternate load paths (in the event of failure of one or more members). For example, for such purposes an additional diagonal member BD may be added to the truss of Figure 5.8; see Figure 5.9. Since the equations of static equilibrium are just sufficient for the analysis of the truss of Figure 5.8, they are not adequate for the analysis of the truss of Figure 5.9. Accordingly, the truss of Figure 5.9 is said to be a *statically indeterminate structure*. The analysis of statically indeterminate structures requires additional information (additional equations) beyond that obtained from the equations of static equilibrium.

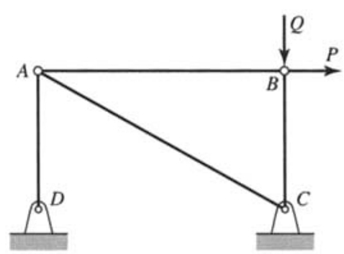

FIGURE 5.8 Statically determinate truss.

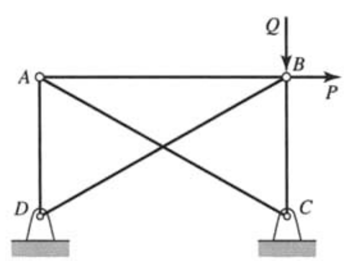

FIGURE 5.9 Statically indeterminate truss.

In this section, the analysis of statically determinate structures is discussed. The analysis of statically indeterminate structures is presented in Section 5.5.

The strain energy U for a structure is equal to the sum of the strain energies of its members. The loading for the jth member of the structure is assumed to be such that the strain energy U_j for that member is

$$U_j = U_{Nj} + U_{Mj} + U_{Sj} + U_{Tj}$$

where U_{Nj}, U_{Mj}, U_{Sj}, and U_{Tj} are given by Eqs. 5.8, 5.13, 5.14, and 5.16, respectively. In the remainder of this chapter the limitations placed on the member cross section in the derivation of each of the components of the strain energy are assumed to apply. For instance, each beam is assumed to undergo bending about a principal axis of the beam cross section (see Appendix B and Chapter 7); Eqs. 5.13 and 5.14 are valid only for bending about a principal axis. For simplicity, we consider bending about the x axis (taken to be a principal axis) and let $M_x = M$ and $V_y = V$.

With the total strain energy U of the structure known, the deflection q_i of the structure at the location of a concentrated force F_i in the direction of F_i is (see Eq. 5.4)

$$\begin{aligned} q_i &= \frac{\partial U}{\partial F_i} \\ &= \sum_{j=1}^{m}\left(\int \frac{N_j}{E_j A_j}\frac{\partial N_j}{\partial F_i}\,dz + \int \frac{k_j V_j}{G_j A_j}\frac{\partial V_j}{\partial F_i}\,dz \right. \\ &\quad \left. + \int \frac{M_j}{E_j I_j}\frac{\partial M_j}{\partial F_i}\,dz + \int \frac{T_j}{G_j J_j}\frac{\partial T_j}{\partial F_i}\,dz\right) \end{aligned} \tag{5.17}$$

and the angle (slope) change θ_i of the structure at the location of a concentrated moment M_i in the direction of M_i is (see Eq. 5.5)

$$\begin{aligned} \theta_i &= \frac{\partial U}{\partial M_i} \\ &= \sum_{j=1}^{m}\left(\int \frac{N_j}{E_j A_j}\frac{\partial N_j}{\partial M_i}\,dz + \int \frac{k_j V_j}{G_j A_j}\frac{\partial V_j}{\partial M_i}\,dz \right. \\ &\quad \left. + \int \frac{M_j}{E_j I_j}\frac{\partial M_j}{\partial M_i}\,dz + \int \frac{T_j}{G_j J_j}\frac{\partial T_j}{\partial M_i}\,dz\right) \end{aligned} \tag{5.18}$$

where m is the number of members in the structure. Use of Castigliano's theorem on deflections, as expressed in Eqs. 5.17 and 5.18, to determine deflections or rotations at the location of a concentrated force or moment is outlined in the following procedure:

1. Write an expression for each of the internal actions (axial force, shear, moment, and torque) in each member of the structure in terms of the applied external loads.
2. To determine the deflection q_i of the structure at the location of a concentrated force F_i and in the directed sense of F_i, differentiate each of the internal action expressions with respect to F_i. Similarly, to determine the rotation θ_i of the structure at the location of a concentrated moment M_i and in the directed sense of M_i, differentiate each of the internal action expressions with respect to M_i.

3. Substitute the expressions for internal actions obtained in Step 1 and the derivatives obtained in Step 2 into Eq. 5.17 or 5.18 and perform the integration. The result is a relationship between the deflection q_i (or rotation θ_i) and the externally applied loads.
4. Substitute the magnitudes of the external loads into the result obtained in Step 3 to obtain a numerical value for the displacement q_i or rotation θ_i.

5.4.1 Curved Beams Treated as Straight Beams

The strain energy resulting from bending (see Eq. 5.13) was derived by assuming that the beam is straight. The magnitude of U_M for curved beams is derived in Chapter 9, where it is shown that the error in using Eq. 5.13 to determine U_M is negligible as long as the radius of curvature of the beam is more than twice its depth. Consider the curved beam in Figure 5.10 whose strain energy is the sum of U_N, U_S, and U_M, each of which is caused by the same load P. If the radius of curvature R of the curved beam is large compared to the beam depth, the magnitudes of U_N and U_S will be small compared to U_M and can be neglected. We assume that U_N and U_S can be neglected when the ratio of length to depth is greater than 10. The resulting error is often less than 1% and will seldom exceed 5%. Numerical results are obtained in Examples 5.9 and 5.10.

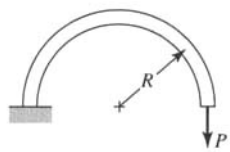

FIGURE 5.10 Curved cantilever beam.

EXAMPLE 5.5
End Load on a Cantilever Beam

Determine the deflection under load P of the cantilever beam shown in Figure E5.5. Assume that the beam length L is more than five times the beam depth h.

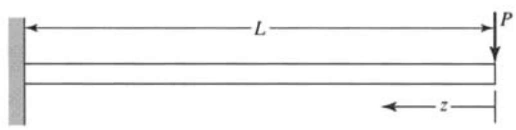

FIGURE E5.5 End-loaded cantilever beam.

Solution

Since $L > 5h$, the strain energy U_S is small and will be neglected. Therefore, the total strain energy is (Eq. 5.13)

$$U = U_M = \int_0^L \frac{M_x^2}{2EI_x} dz \tag{a}$$

By Castigliano's theorem, the deflection q_P is (Eq. 5.2)

$$q_P = \frac{\partial U}{\partial P} = \int_0^L \frac{M_x}{EI_x} \frac{\partial M_x}{\partial P} dz \tag{b}$$

By Figure E5.5, $M_x = Pz$. Therefore, $\partial M_x/\partial P = z$, and by Eq. (b), we find

$$q_P = \int_0^L \frac{Pz^2}{EI_x} dz = \frac{PL^3}{3EI_x} \tag{c}$$

This result agrees with elementary beam theory.

EXAMPLE 5.6 *Cantilever Beam Loaded in Its Plane*

The cantilever beam in Figure E5.6 has a rectangular cross section and is subjected to equal loads P at the free end and at the center as shown.

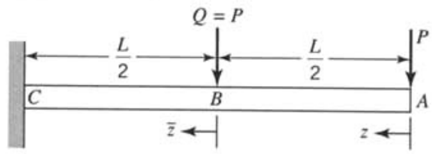

FIGURE E5.6

(a) Determine the deflection of the free end of the beam.

(b) What is the error in neglecting the strain energy resulting from shear if the beam length L is five times the beam depth h? Assume that the beam is made of steel ($E = 200$ GPa and $G = 77.5$ GPa).

Solution

(a) To determine the dependencies of the shear V and moment M on the end load P, it is necessary to distinguish between the loads at A and B. Let the load at B be designated by Q. The moment and shear functions are continuous from A to B and from B to C. From A to B, we have

$$V = P, \quad \therefore \frac{\partial V}{\partial P} = 1$$

$$M = Pz, \quad \therefore \frac{\partial M}{\partial P} = z$$

From B to C, we have

$$V = P + Q, \quad \therefore \frac{\partial V}{\partial P} = 1$$

$$M = P\left(\bar{z} + \frac{L}{2}\right) + Q\bar{z}, \quad \therefore \frac{\partial M}{\partial P} = \bar{z} + \frac{L}{2}$$

where we have chosen point B as the origin of coordinate $\bar{z}$ for the length from B to C. Equation 5.17 gives (with $Q = P$ and $k = 1.2$)

$$q_p = \int_0^{L/2} \frac{1.2P}{GA}(1)\,dz + \int_0^{L/2} \frac{Pz}{EI}(z)\,dz + \int_0^{L/2} \frac{2.4P}{GA}(1)\,d\bar{z}$$
$$+ \int_0^{L/2} \frac{P(2\bar{z} + L/2)}{EI}\left(\bar{z} + \frac{L}{2}\right)d\bar{z} = \frac{1.8PL}{GA} + \frac{7PL^3}{16EI} \tag{a}$$

(b) Since the beam has a rectangular section, $A = bh$ and $I = bh^3/12$. Equation (a) can be rewritten as follows:

$$\frac{Ebq_P}{P} = \frac{1.8LE}{Gh} + \frac{7(12)L^3}{16h^3}$$
$$= \frac{1.8(5)(200)}{77.5} + \frac{7(12)(5^3)}{16}$$
$$= 23.23 + 656.25$$
$$= 679.48$$

Therefore, the error in neglecting shear term is $\frac{23.23(100)}{679.48} = 3.42\%$

Alternatively, one could have used the approximate value $k = 1.50$ (Eq. 5.15). Then the estimate of shear contribution would have been increased by the ratio $1.50/1.20 = 1.25$. Overall the shear contribution would still remain small.

EXAMPLE 5.7
A Shaft–Beam Mechanism

A shaft AB is attached to the beam $CDFH$; see Figure E5.7. A torque of 2.00 kN • m is applied to the end B of the shaft. Determine the rotation of section B. The shaft and beam are made of an aluminum alloy for which $E = 72.0$ GPa and $G = 27.0$ GPa. Assume that the hub DF is rigid.

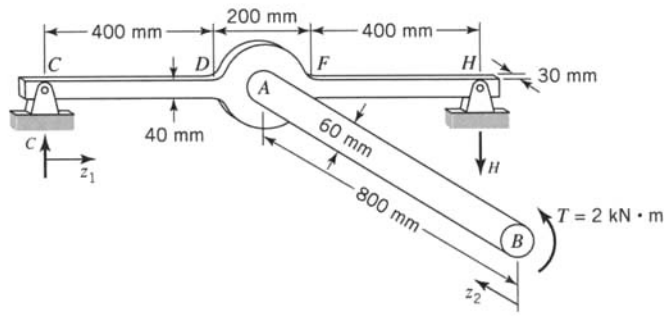

FIGURE E5.7 Shaft–beam mechanism.

Solution

Since the beam is slender, the strain energy resulting from shear can be neglected. Hence, the total strain energy of the mechanism is

$$U = U_M + U_T = 2\int_0^{400} \frac{M_x^2}{2EI_x}\,dz + \int_0^{800} \frac{T^2}{2GJ}\,dz \tag{a}$$

The pin reactions at C and H have the same magnitude but opposite sense. Therefore, moment equilibrium yields the result

$$H = C = T/1000 \tag{b}$$

By Castigliano's theorem, the angular rotation at B is

$$\theta_B = \frac{\partial U}{\partial T} = 2\int_0^{400} \frac{M_x}{EI_x}\frac{\partial M_x}{\partial T}\,dz_1 + \int_0^{800} \frac{T}{GJ}\,dz_2 \tag{c}$$

By Figure E5.7 and Eq. (b), we have

$$T = 2{,}000{,}000\ \text{N} \bullet \text{mm}, \quad M_x = Cz_1 = 0.001Tz_1\ [\text{N} \bullet \text{mm}] \tag{d}$$

With $I_x = (30)(40)^3/12 = 160{,}000$ mm^4 and $J = \pi(60)^4/32 = 1.272 \times 10^6$ mm^4, Eqs. (c) and (d) yield $\theta_B = 0.054$ rad.

EXAMPLE 5.8
Uniformly Loaded Cantilever Beam

The cantilever beam in Figure E5.8a is subjected to a uniformly distributed load w. Determine the deflection of the free end. Neglect the shear strain energy U_S.

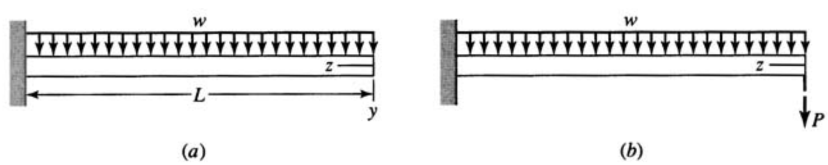

FIGURE E5.8 Uniformly loaded cantilever beam.

Solution

Since there is no force acting at the free end, we introduce the fictitious force P (Figure E5.8*b*). (See the discussion in Section 5.4.2.) Hence, the deflection q_P at the free end is given by (see Eq. 5.2)

$$q_P = \int_0^L \frac{M_x}{EI_x} \frac{\partial M_x}{\partial P}\bigg|_{P=0} dz \tag{a}$$

The bending moment M_x caused by P and w is

$$M_x = Pz + \frac{1}{2}wz^2 \tag{b}$$

and

$$\frac{\partial M_x}{\partial P} = z \tag{c}$$

Substitution of Eqs. (b) and (c) into Eq. (a) yields

$$q_P = \frac{1}{EI_x}\int_0^L \left(Pz + \frac{1}{2}wz^2\right)\bigg|_{P=0} z\, dz = \frac{wL^4}{8EI_x} \tag{d}$$

This result agrees with elementary beam theory.

EXAMPLE 5.9 *Curved Beam Loaded in Its Plane*

The curved beam in Figure E5.9 has a 30-mm square cross section and radius of curvature $R = 65$ mm. The beam is made of a steel for which $E = 200$ GPa and $\nu = 0.29$.

(a) If $P = 6.00$ kN, determine the deflection of the free end of the curved beam in the direction of P.

(b) What is the error in the deflection if U_N and U_S are neglected?

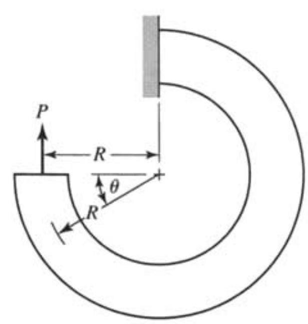

FIGURE E5.9

Solution

The shear modulus for the steel is $G = E/[2(1 + \nu)] = 77.5$ GPa.

(a) It is convenient to use polar coordinates. For a cross section of the curved beam located at angle θ from the section on which P is applied (Figure E5.9),

$$\begin{aligned} N &= P\cos\theta, & \frac{\partial N}{\partial P} &= \cos\theta \\ V &= P\sin\theta, & \frac{\partial V}{\partial P} &= \sin\theta \\ M &= PR(1-\cos\theta), & \frac{\partial M}{\partial P} &= R(1-\cos\theta) \end{aligned} \tag{a}$$

Substitution into Eq. 5.17 gives

$$q_P = \int_0^{3\pi/2} \frac{P\cos\theta}{EA}(\cos\theta)R\,d\theta + \int_0^{3\pi/2} \frac{kP\sin\theta}{GA}(\sin\theta)R\,d\theta + \int_0^{3\pi/2} \frac{PR(1-\cos\theta)}{EI}R(1-\cos\theta)R\,d\theta \tag{b}$$

Using the trigonometric identities $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ and $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$, we find that

$$q_P = \frac{PR}{EA}\int_0^{3\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta + \frac{1.2PR}{GA}\int_0^{3\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta + \frac{PR^3}{EI}\int_0^{3\pi/2}\left(1-2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$q_P = \frac{3\pi PR}{4EA} + \frac{1.2(3\pi)PR}{4GA} + \left(\frac{9\pi}{4}+2\right)\left(\frac{PR^3}{EI}\right)$$

$$= \frac{3\pi(65)(6000)}{4(200\times 10^3)(30)^2} + \frac{1.2(3\pi)(65)(6000)}{4(77{,}500)(30)^2} + \left(\frac{9\pi}{4}+2\right)\frac{(65)^3(6000)(12)}{(200\times 10^3)(30)^4}$$

$$= 0.0051 + 0.0158 + 1.1069 = 1.1278$$

(b) If U_N and U_S are neglected,

$$q_P = 1.1069 \text{ mm}$$

and the percentage error in the deflection calculation is

$$\text{error} = \frac{(1.1278 - 1.1069)100}{1.1278} = 1.85\%$$

This error is small enough to be neglected for most engineering applications. The ratio of length to depth for this beam is $3\pi(65)/[2(30)] = 10.2$ (see Section 5.4.1).

EXAMPLE 5.10 *Semicircular Cantilever Beam*

The semicircular cantilever beam in Figure E5.10 has a radius of curvature R and a circular cross section of diameter d. It is subjected to loads of magnitude P at points B and C.

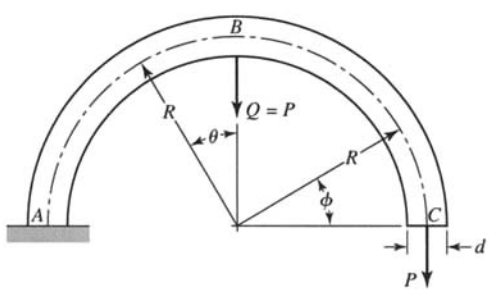

FIGURE E5.10

(a) Determine the vertical deflection at C in terms of P, modulus of elasticity E, shear modulus G, radius of curvature R, cross-sectional area A, and moment of inertia of the cross section I.

(b) Let $P = 150$ N, $R = 200$ mm, $d = 20.0$ mm, $E = 200$ GPa, and $G = 77.5$ GPa. Determine the effect of neglecting the strain energy U_S due to shear.

Solution **(a)** Since we wish to determine the vertical deflection at C, the contribution to the total strain energy of the load at C must be distinguished from the contribution of the load at B. Therefore, as in Example 5.6, we denote the load P at B by Q (= P in magnitude) (Figure E5.10). For section BC, we have

$$\begin{aligned} N &= P\cos\phi, & \frac{\partial N}{\partial P} &= \cos\phi \\ V &= P\sin\phi, & \frac{\partial V}{\partial P} &= \sin\phi \\ M &= PR(1-\cos\phi), & \frac{\partial M}{\partial P} &= R(1-\cos\phi) \end{aligned} \tag{a}$$

For section AB, we have

$$\begin{aligned} N &= (P+Q)\sin\theta, & \frac{\partial N}{\partial P} &= \sin\theta \\ V &= (P+Q)\cos\theta, & \frac{\partial V}{\partial P} &= \cos\theta \\ M &= PR(1+\sin\theta)+QR\sin\theta, & \frac{\partial M}{\partial P} &= R(1+\sin\theta) \end{aligned} \tag{b}$$

Substitution of Eqs. (a) and (b) into Eq. 5.17 yields, with $k = 1.33$,

$$\begin{aligned} q_C = &\int_0^{\pi/2} \frac{P\cos^2\phi}{EA} R\,d\phi + \int_0^{\pi/2} \frac{1.33P\sin^2\phi}{GA} R\,d\phi \\ &+ \int_0^{\pi/2} \frac{PR^2(1-\cos\phi)^2}{EI} R\,d\phi + \int_0^{\pi/2} \frac{(P+Q)\sin^2\theta}{EA} R\,d\theta \\ &+ \int_0^{\pi/2} \frac{1.33(P+Q)\cos^2\theta}{GA} R\,d\theta \\ &+ \int_0^{\pi/2} \frac{PR(1+\sin\theta)+QR\sin\theta}{EI} R\,(1+\sin\theta)\,R\,d\theta \end{aligned} \tag{c}$$

Integration yields, if we note that $Q = P$,

$$q_C = \frac{3\pi}{4}\frac{PR}{EA} + \frac{3\pi}{4}\frac{1.33PR}{GA} + \left(\frac{7\pi}{4}+1\right)\frac{PR^3}{EI} \tag{d}$$

The three terms on the right-hand side of Eq. (d) are the contributions of the axial force, shear, and moment, respectively, to the displacement q_C.

(b) For $P = 150$ N, $R = 200$ mm, $d = 20.0$ mm (hence, $A = 314$ mm^2 and $I = 7850$ mm^4), $E = 200$ GPa, and $G = 77.5$ GPa, Eq. (d) yields

$$q_C = 0.0011 + 0.0039 + 4.9666 = 4.97 \text{ mm}$$

where 0.0011 is due to the axial force, 0.0039 is due to shear, and 4.9666 is due to moment. The contributions of axial load and shear are negligible. Since $R/d = 200/20 = 10$, this result confirms the statement at the end of Section 5.4.1.

5.4.2 Dummy Load Method and Dummy Unit Load Method

As illustrated in the preceding examples, Castigliano's theorem on deflections, as expressed in Eqs. 5.17 and 5.18, is useful for the determination of deflections and rotations at the locations of concentrated forces and moments. Frequently, it is necessary to determine the deflection or rotation at a location that has no corresponding external load (see Example 5.8). For example, we might want to know the rotation at the free end of a cantilever beam that is subjected to concentrated loads at midspan and at the free end but has no concentrated moment at the free end (as is the case in Example 5.6). Castigliano's theorem on deflections can be applied in these situations as well. The modified procedure, known as the *dummy load method*, is as follows:

1. Apply a fictitious force F_i (or fictitious moment M_i) at the location and in the direction of the displacement q_i (or rotation θ_i) to be determined.
2. Write an expression for each of the internal actions (axial force, shear, moment, and torque) in each member of the structure in terms of the applied external forces and moments, including the fictitious force (or moment).
3. To determine the deflection q_i of the structure at the location of a fictitious force F_i and in the sense of F_i, differentiate each of the internal action expressions with respect to F_i. Similarly, to determine the rotation θ_i of the structure at the location of a fictitious moment M_i and in the sense of M_i, differentiate each of the internal action expressions with respect to M_i.
4. Substitute the expressions for the internal actions obtained in Step 2 and the derivatives obtained in Step 3 into Eq. 5.17 or 5.18 and perform the integration. The result is a relationship between the deflection q_i (or rotation θ_i) and the externally applied loads, including the fictitious force F_i (or moment M_i).
5. Since, in fact, the fictitious force (or moment) does not act on the structure, set its value to zero in the relation obtained in Step 4. Then substitute the numerical values of the external loads into this result to obtain the numerical value for the displacement q_i (or rotation θ_i).

The name *dummy load method* derives from the procedure. A fictitious (or *dummy*) load is applied, its effect on internal actions is determined, and then it is removed.

If the procedure is limited to small deflections of linear elastic structures (consisting of tension members, compression members, beams, and torsion bars), then the derivatives of the internal actions with respect to the fictitious loads are equivalent to the internal actions that result from a *unit force* (or *unit moment*) applied at the point of interest. When the method is used in this manner, it is referred to as the *dummy unit load method*. The net effect of this procedure is that it eliminates the differentiation in Eqs. 5.17 and 5.18. The internal actions (axial, shear, moment, and torque, respectively) in member j resulting from a *unit force* at location i may be represented as

$$n_{ji}^F = \frac{\partial N_j}{\partial F_i}, \quad v_{ji}^F = \frac{\partial V_j}{\partial F_i}, \quad m_{ji}^F = \frac{\partial M_j}{\partial F_i}, \quad t_{ji}^F = \frac{\partial T_j}{\partial F_i} \tag{5.19a}$$

Similarly, the internal actions in member j resulting from a *unit moment* at location i may be represented as

$$n_{ji}^M = \frac{\partial N_j}{\partial M_i}, \quad v_{ji}^M = \frac{\partial V_j}{\partial M_i}, \quad m_{ji}^M = \frac{\partial M_j}{\partial M_i}, \quad t_{ji}^M = \frac{\partial T_j}{\partial M_i} \tag{5.19b}$$

In the dummy unit load approach, Eqs. 5.17 and 5.18 take the form

$$q_i = \sum_{j=1}^{m} \left(\int \frac{N_j n_{ji}^F}{E_j A_j} dz + \int \frac{k_j V_j v_{ji}^F}{G_j A_j} dz + \int \frac{M_j m_{ji}^F}{E_j I_j} dz + \int \frac{T_j t_{ji}^F}{G_j J_j} dz \right) \tag{5.20a}$$