2.7 STRAIN THEORY, TRANSFORMATION OF STRAIN, AND PRINCIPAL STRAINS[7]

The theory of stress of a continuous medium rests solely on Newton's laws. As is shown in this section, the theory of strain rests solely on geometric concepts. Both the theories of stress and strain are, therefore, independent of material behavior and, as such, are applicable to the study of all materials. Furthermore, although the theories of stress and strain are based on different physical concepts, mathematically, they are equivalent, as will become evident in the following discussion.

2.7.1 Strain of a Line Element

When a body is deformed, the particle at point $P:(x, y, z)$ passes to the point $P^*:(x^*, y^*, z^*)$ (Figure 2.18). Also, the particle at point $Q:(x + dx, y + dy, z + dz)$ passes to the point $Q^*:(x^* + dx^*, y^* + dy^*, z^* + dz^*)$, and the infinitesimal line element $PQ = ds$ passes into the line element $P^*Q^* = ds^*$. We define the *engineering strain* ϵ_E of the line element $PQ = ds$ as

$$\epsilon_E = \frac{ds^* - ds}{ds} \tag{2.57}$$

Therefore, by this definition, $\epsilon_E > -1$. Equation 2.57 is employed widely in engineering.

The definition of strain given in Eq. 2.57 is not unique. That is, other, equally valid definitions of strain have been proposed. Here we also develop an alternative definition of strain, known as *Green strain*. By Eqs. 2.55, we obtain the total differential

$$dx^* = \frac{\partial x^*}{\partial x}dx + \frac{\partial x^*}{\partial y}dy + \frac{\partial x^*}{\partial z}dz \tag{2.58}$$

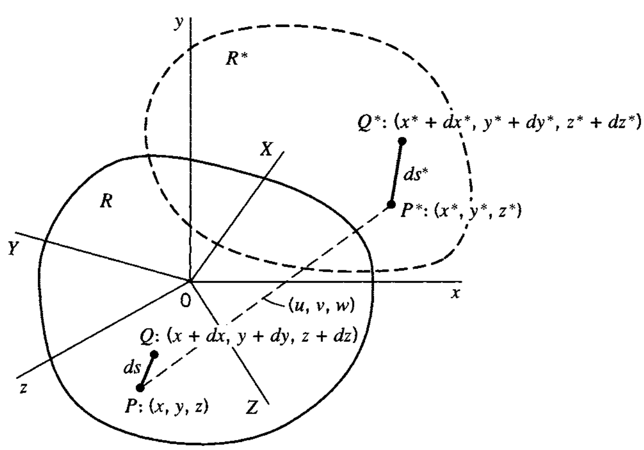

FIGURE 2.18 Line segment *PQ* in undeformed and deformed body.

[7]The theory presented in this section includes quadratic terms in the displacement components (u, v, w) and in the engineering strain ϵ_E. One may discard all quadratic terms in u, v, w, and ϵ_E and directly obtain the theory of strain for small deformations. (See Section 2.8.)

with similar expressions for dy^* and dz^*. Noting that

$$
\begin{aligned}
x^* &= x + u \\
y^* &= y + v \\
z^* &= z + w
\end{aligned}
\tag{2.59}
$$

where (u, v, w) denote the (x, y, z) components of the displacement of P to P^*, and also noting that

$$
\begin{aligned}
(ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\
(ds^*)^2 &= (dx^*)^2 + (dy^*)^2 + (dz^*)^2
\end{aligned}
\tag{2.60}
$$

we find[8] [retaining quadratic terms in derivatives of (u, v, w)]

$$
\begin{aligned}
M = \frac{1}{2}\left[\left(\frac{ds^*}{ds}\right)^2 - 1\right] &= \epsilon_E + \frac{1}{2}\epsilon_E^2 = l^2\epsilon_{xx} + lm\epsilon_{xy} + ln\epsilon_{xz} \\
&\quad + ml\epsilon_{yx} + m^2\epsilon_{yy} + mn\epsilon_{yz} + nl\epsilon_{zx} + nm\epsilon_{zy} + n^2\epsilon_{zz} \\
&= l^2\epsilon_{xx} + m^2\epsilon_{yy} + n^2\epsilon_{zz} + 2lm\epsilon_{xy} + 2ln\epsilon_{xz} + 2mn\epsilon_{yz}
\end{aligned}
\tag{2.61}
$$

where M is called the *magnification factor.* The magnification factor M is a measure of the strain of a line in the body with direction cosines (l, m, n). This quantity is also known as the *total Green strain.* The components of Green strain, from Eq. 2.61, are

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\epsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\
\epsilon_{xy} = \epsilon_{yx} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\epsilon_{xz} = \epsilon_{zx} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
\epsilon_{yz} = \epsilon_{zy} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)
\end{aligned}
\tag{2.62}
$$

where

$$
l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad n = \frac{dz}{ds}
\tag{2.63}
$$

[8] Although one may compute ϵ_E directly from Eq. 2.57, it is mathematically simpler to form the quantity $M = \frac{1}{2}[(ds^*/ds)^2 - 1] = \frac{1}{2}[(1+\epsilon_E)^2 - 1] = \epsilon_E + \frac{1}{2}\epsilon_E^2$. Then one may compute ϵ_E from Eq. 2.61. For small ϵ_E (Section 2.8), $\epsilon_E \approx M$. A more detailed derivation of Eq. 2.61 is given by Boresi and Chong (2000, Section 2-6).

are the direction cosines of line element ds. These are the finite strain–displacement relations. They are valid for any magnitude of displacement (u, v, w) of the body.[9]

We may interpret the quantities ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} physically, by considering line elements ds that lie parallel to the (x, y, z) axes, respectively. For example, let the line element ds (Figure 2.18) lie parallel to the x axis. Then $l = 1$, $m = n = 0$, and Eq. 2.61 yields

$$M_x = \epsilon_{Ex} + \frac{1}{2}\epsilon_{Ex}^2 = \epsilon_{xx} \tag{2.61a}$$

where M_x and ϵ_{Ex} denote the magnification factor and the engineering strain of the element ds (parallel to the x direction). Hence, ϵ_{xx}, physically, is the magnification factor of the line element at P that lies initially in the x direction. In particular, if the engineering strain is small ($\epsilon_{Ex} << 1$), we obtain the result $\epsilon_{xx} \approx \epsilon_{Ex}$: namely, that ϵ_{xx} is approximately equal to the engineering strain for small strains. Similarly, for the cases where initially ds lies parallel to the y axis and then the z axis, we obtain

$$\begin{aligned} M_y &= \epsilon_{Ey} + \frac{1}{2}\epsilon_{Ey}^2 = \epsilon_{yy} \\ M_z &= \epsilon_{Ez} + \frac{1}{2}\epsilon_{Ez}^2 = \epsilon_{zz} \end{aligned} \tag{2.61b}$$

Thus, $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz})$ physically represent the magnification factors for line elements that initially lie parallel to the (x, y, z) axes, respectively.

To obtain a physical interpretation of the components ϵ_{xy}, ϵ_{xz}, ϵ_{yz}, it is necessary to determine the rotation between two line elements initially parallel to the (x, y) axes, (x, z) axes, and (y, z) axes, respectively. To do this, we first determine the final direction of a single line element under the deformation. Then, we use this result to determine the rotation between two line elements.

2.7.2 Final Direction of a Line Element

As a result of the deformation, the line element ds: (dx, dy, dz) deforms into the line element ds^*: (dx^*, dy^*, dz^*). By definition, the direction cosines of ds and ds^* are

$$\begin{aligned} l &= \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad n = \frac{dz}{ds} \\ l^* &= \frac{dx^*}{ds^*}, \quad m^* = \frac{dy^*}{ds^*}, \quad n^* = \frac{dz^*}{ds^*} \end{aligned} \tag{2.64}$$

Alternatively, we may write

$$l^* = \frac{dx^*}{ds}\frac{ds}{ds^*}, \quad m^* = \frac{dy^*}{ds}\frac{ds}{ds^*}, \quad n^* = \frac{dz^*}{ds}\frac{ds}{ds^*} \tag{2.65}$$

By Eqs. 2.58 and 2.59, we find

[9]In small-displacement theory, the quadratic terms in Eqs. 2.62 are neglected. Then, Eqs. 2.62 reduce to Eqs. 2.81.

$$\begin{aligned}
\frac{dx^*}{ds} &= \left(1+\frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n \\
\frac{dy^*}{ds} &= \frac{\partial v}{\partial x}l + \left(1+\frac{\partial v}{\partial y}\right)m + \frac{\partial v}{\partial z}n \\
\frac{dz^*}{ds} &= \frac{\partial w}{\partial x}l + \frac{\partial w}{\partial y}m + \left(1+\frac{\partial w}{\partial z}\right)n
\end{aligned} \tag{2.66}$$

and by Eq. 2.57

$$\frac{ds}{ds^*} = \frac{1}{1+\epsilon_E} \tag{2.67}$$

Hence, Eqs. 2.65–2.67 yield

$$\begin{aligned}
(l+\epsilon_E)l^* &= \left(1+\frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n \\
(l+\epsilon_E)m^* &= \frac{\partial v}{\partial x}l + \left(1+\frac{\partial v}{\partial y}\right)m + \frac{\partial v}{\partial z}n \\
(l+\epsilon_E)n^* &= \frac{\partial w}{\partial x}l + \frac{\partial w}{\partial y}m + \left(1+\frac{\partial w}{\partial z}\right)n
\end{aligned} \tag{2.68}$$

Equations 2.68 represent the final direction cosines of line element ds when it passes into the line element ds^* under the deformation.

2.7.3 Rotation Between Two Line Elements (Definition of Shear Strain)

Next, let us consider two infinitesimal line elements PA and PB of lengths ds_1 and ds_2 emanating from point P. For simplicity, let PA be perpendicular to PB[10] (Figure 2.19). Let the direction cosines of lines PA and PB be (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. As the

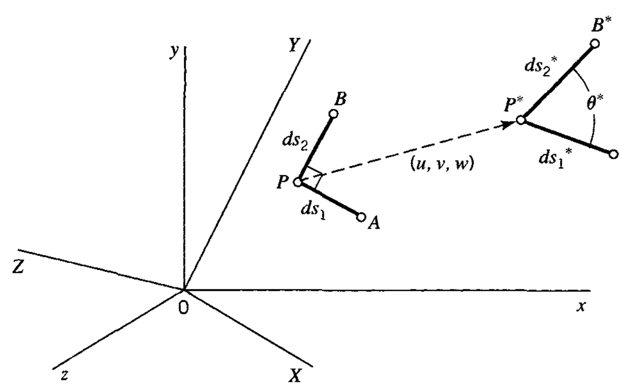

FIGURE 2.19 Line segments *PA* and *PB* before and after deformation.

[10]This restriction is not necessary but is used for simplicity. See Boresi and Chong (2000).

result of the deformation, line elements PA and PB are transformed into line elements P^*A^* and P^*B^*, with direction cosines (l_1^*, m_1^*, n_1^*) and (l_2^*, m_2^*, n_2^*), respectively. Since PA is perpendicular to PB, by the definition of scalar product of vectors

$$\cos\frac{\pi}{2} = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \tag{2.69}$$

Similarly, the angle θ^* between P^*A^* and P^*B^* is defined by

$$\cos\theta^* = l_1^* l_2^* + m_1^* m_2^* + n_1^* n_2^* \tag{2.70}$$

In turn, (l_1^*, m_1^*, n_1^*) and (l_2^*, m_2^*, n_2^*) are expressed in terms of (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively, by means of Eq. 2.68. Hence, by Eqs. 2.68–2.70, we may write with Eqs. 2.62

$$\begin{aligned}\gamma_{12} &= (1+\epsilon_{E1})(1+\epsilon_{E2})\cos\theta^* \\ &= 2l_1 l_2 \epsilon_{xx} + 2m_1 m_2 \epsilon_{yy} + 2n_1 n_2 \epsilon_{zz} + 2(l_1 m_2 + l_2 m_1)\epsilon_{xy} \\ &\quad + 2(m_1 n_2 + m_2 n_1)\epsilon_{yz} + 2(l_1 n_2 + l_2 n_1)\epsilon_{xz}\end{aligned} \tag{2.71}$$

where γ_{12} is defined to be the *engineering shear strain* between line elements PA and PB as they are deformed into P^*A^* and P^*B^* (Figure 2.19).

To obtain a physical interpretation of ϵ_{xy}, we now let PA and PB be oriented initially parallel to axes x and y, respectively. Hence, $l_1 = 1$, $m_1 = n_1 = 0$ and $l_2 = n_2 = 0$, $m_2 = 1$. Then Eq. 2.71 yields the result

$$\gamma_{12} = \gamma_{xy} = 2\epsilon_{xy} \tag{2.72}$$

In other words, $2\epsilon_{xy}$ represents the engineering shear strain between two line elements initially parallel to the x and y axes, respectively. Similarly, we may consider PA and PB to be oriented initially parallel to the y and z axes and then to the x and z axes to obtain similar interpretations for ϵ_{yz} and ϵ_{xz}. Thus,

$$\gamma_{xy} = 2\epsilon_{xy}, \quad \gamma_{yz} = 2\epsilon_{yz}, \quad \gamma_{xz} = 2\epsilon_{xz} \tag{2.73}$$

represent the engineering shear strains between two line elements initially parallel to the (x, y), (y, z), and (x, z) axes, respectively.

If the strains ϵ_{E1}, ϵ_{E2} are small and the rotations are small (e.g., $\theta^* \approx \pi/2$), Eq. 2.71 yields the approximation

$$\gamma_{12} = (1+\epsilon_{E1})(1+\epsilon_{E2})\cos\theta^* \approx \frac{\pi}{2} - \theta^* \tag{2.74}$$

and the engineering shear strain becomes approximately equal to the change in angle between line elements PA and PB.

Other results, which are analogous to those of stress theory (Sections 2.3 and 2.4), also hold. For example, the symmetric array

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} \tag{2.75}$$

is the *strain tensor.* Under a rotation of axes, the components of the strain tensor (ϵ_{xx}, ϵ_{xy}, ϵ_{xz}, ...) transform in exactly the same way as those of the stress tensor (Eqs. 2.15 and 2.17). (Compare Eqs. 2.5 and 2.75. Also compare Eqs. 2.11 and 2.61.)

To show this transformation, consider again axes (x, y, z) and (X, Y, Z), as in Section 2.4, Figure 2.8 (also Figure 2.18), and Table 2.2. The strain components ϵ_{XX}, ϵ_{XY}, $\epsilon_{XZ}, \ldots$, are defined with reference to axes (X, Y, Z) in the same manner as ϵ_{xx}, ϵ_{xy}, $\epsilon_{xz}, \ldots$ are defined relative to axes (x, y, z). Hence, ϵ_{XX} is the extensional strain of a line element at point P (Figure 2.18) that lies in the direction of the X axis, and ϵ_{XY} and ϵ_{XZ} are shear components between pairs of line elements that are parallel to axes (X, Y) and (X, Z), respectively, and so on for ϵ_{YY}, ϵ_{ZZ}, and ϵ_{YZ}. Hence, if we let element ds lie parallel to the X axis, Eq. 2.61, with Table 2.2, yields

$$\epsilon_{XX} = l_1^2\epsilon_{xx} + m_1^2\epsilon_{yy} + n_1^2\epsilon_{zz} + 2l_1m_1\epsilon_{xy} + 2l_1n_1\epsilon_{xz} + 2m_1n_1\epsilon_{yz} \tag{2.76a}$$

For the line elements that lie parallel to axes Y and Z, respectively, we have

$$\epsilon_{YY} = l_2^2\epsilon_{xx} + m_2^2\epsilon_{yy} + n_2^2\epsilon_{zz} + 2l_2m_2\epsilon_{xy} + 2l_2n_2\epsilon_{xz} + 2m_2n_2\epsilon_{yz} \tag{2.76b}$$

$$\epsilon_{ZZ} = l_3^2\epsilon_{xx} + m_3^2\epsilon_{yy} + n_3^2\epsilon_{zz} + 2l_3m_3\epsilon_{xy} + 2l_3n_3\epsilon_{xz} + 2m_3n_3\epsilon_{yz} \tag{2.76c}$$

Similarly, if we take line elements PA and PB parallel, respectively to axes X and Y (Figure 2.19), Eqs. 2.71 and 2.73 yield the result

$$\begin{aligned}\frac{1}{2}\gamma_{XY} = \epsilon_{XY} = {} & l_1l_2\epsilon_{xx} + m_1m_2\epsilon_{yy} + n_1n_2\epsilon_{zz} + (l_1m_2 + l_2m_1)\epsilon_{xy} \\ & + (m_1n_2 + m_2n_1)\epsilon_{yz} + (l_1n_2 + l_2n_1)\epsilon_{xz}\end{aligned} \tag{2.76d}$$

In a similar manner, we find

$$\begin{aligned}\frac{1}{2}\gamma_{YZ} = \epsilon_{YZ} = {} & l_2l_3\epsilon_{xx} + m_2m_3\epsilon_{yy} + n_2n_3\epsilon_{zz} + (l_2m_3 + l_3m_2)\epsilon_{xy} \\ & + (m_2n_3 + m_3n_2)\epsilon_{yz} + (l_2n_3 + l_3n_2)\epsilon_{xz}\end{aligned} \tag{2.76e}$$

$$\begin{aligned}\frac{1}{2}\gamma_{XZ} = \epsilon_{XZ} = {} & l_1l_3\epsilon_{xx} + m_1m_3\epsilon_{yy} + n_1n_3\epsilon_{zz} + (l_1m_3 + l_3m_1)\epsilon_{xy} \\ & + (m_1n_3 + m_3n_1)\epsilon_{yz} + (l_1n_3 + l_3n_1)\epsilon_{xz}\end{aligned} \tag{2.76f}$$

where (l_1, m_1, n_1), (l_2, m_2, n_2), and (l_3, m_3, n_3) are the direction cosines of axes X, Y, and Z, respectively.

Equations 2.76 represent the transformation of the strain tensor $(\epsilon_{xx}, \epsilon_{yy}, \ldots, \epsilon_{yz})$ under a rotation from axes (x, y, z) to axes (X, Y, Z). (See Figures 2.18 and 2.19 and also Figure 2.8.)

2.7.4 Principal Strains

Through any point in an undeformed member, there exist three mutually perpendicular line elements that remain perpendicular under the deformation. The strains of these three line elements are called the *principal strains* at the point. We denote them by $(\epsilon_{E1}, \epsilon_{E2}, \epsilon_{E3})$ and the corresponding principal values of the magnification factor $M = \epsilon_E + \frac{1}{2}\epsilon_E^2$ are denoted by (M_1, M_2, M_3). By analogy with stress theory (Section 2.4), the principal values of the magnification factor are the three roots of the determinantal equation

$$\begin{vmatrix} \epsilon_{xx} - M & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} - M & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} - M \end{vmatrix} = 0 \tag{2.77a}$$

or

$$\begin{aligned} M^3 - \bar{I}_1 M^2 + \bar{I}_2 M - \bar{I}_3 &= 0 \\ M &= \epsilon_E + \frac{1}{2}\epsilon_E^2 \end{aligned} \tag{2.77b}$$

where

$$\begin{aligned} \bar{I}_1 &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\ \bar{I}_2 &= \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{vmatrix} + \begin{vmatrix} \epsilon_{xx} & \epsilon_{xz} \\ \epsilon_{xz} & \epsilon_{zz} \end{vmatrix} + \begin{vmatrix} \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} \\ &= \epsilon_{xx}\epsilon_{yy} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{yy}\epsilon_{zz} - \epsilon_{xy}^2 - \epsilon_{xz}^2 - \epsilon_{yz}^2 \\ \bar{I}_3 &= \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} \end{aligned} \tag{2.78}$$

are the *strain invariants* (see Eqs. 2.19–2.21). Because of the symmetry of the determinant of Eq. 2.77a, the roots M_i, where $i = 1, 2, 3$, are always real. Also since $\epsilon_{Ei} > -1$, then $M_i > -1$.

The three principal strain directions associated with the three principal strains (ϵ_{E1}, ϵ_{E2}, ϵ_{E3}), Eq. 2.77b, are obtained as the solution for (l, m, n) of the equations

$$\begin{aligned} l(\epsilon_{xx} - M) + m\epsilon_{xy} + n\epsilon_{xz} &= 0 \\ l\epsilon_{xy} + m(\epsilon_{yy} - M) + n\epsilon_{yz} &= 0 \\ l\epsilon_{xz} + m\epsilon_{yz} + n(\epsilon_{zz} - M) &= 0 \\ l^2 + m^2 + n^2 &= 1 \end{aligned} \tag{2.79}$$

Recall that only two of the first three of Eqs. 2.79 are independent. The solution $M = M_1$ yields the direction cosines for $\epsilon_E = \epsilon_{E1}$ and so on for $M = M_2(\epsilon_E = \epsilon_{E2})$, $M = M_3(\epsilon_E = \epsilon_{E3})$.

If (x, y, z) axes are principal strain axes, $\epsilon_{xx} = M_1$, $\epsilon_{yy} = M_2$, $\epsilon_{zz} = M_3$, $\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$ and the expressions for the strain invariants $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ reduce to

$$\begin{aligned} \bar{I}_1 &= M_1 + M_2 + M_3 \\ \bar{I}_2 &= M_1 M_2 + M_1 M_3 + M_2 M_3 \\ \bar{I}_3 &= M_1 M_2 M_3 \end{aligned} \tag{2.80}$$

2.8 SMALL-DISPLACEMENT THEORY

The deformation theory developed in Sections 2.6 and 2.7 is purely geometrical and the associated equations are exact. In the small-displacement theory, the quadratic terms in Eqs. 2.62 are discarded. Then

$$
\begin{aligned}
&\epsilon_{xx} \approx \frac{\partial u}{\partial x}, \quad \epsilon_{yy} \approx \frac{\partial v}{\partial y}, \quad \epsilon_{zz} \approx \frac{\partial w}{\partial z} \\
&\epsilon_{xy} \approx \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \epsilon_{xz} \approx \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \epsilon_{yz} \approx \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)
\end{aligned}
\tag{2.81}
$$

are the strain–displacement relations for small-displacement theory. Then, the magnification factor reduces to

$$M \approx \epsilon_{\mathrm{E}} \tag{2.82}$$

Hence, for small-displacement theory, the total Green strain equals the engineering strain of a line element PQ (Figure 2.18).

The above approximations, which are the basis for small-displacement theory, imply that the strains and rotations (excluding rigid-body rotations) are small compared to unity. The latter condition is not necessarily satisfied in the deformation of thin flexible bodies, such as rods, plates, and shells. For these bodies the rotations may be large. Consequently, the small-displacement theory must be used with caution: It is usually applicable for massive (thick) bodies, but it may give results that are seriously in error when applied to thin flexible bodies.

2.8.1 Strain Compatibility Relations

The six strain components are found by Eqs. 2.81 if the three displacement components (u, v, w) are known. However, the three displacement components (u, v, w) cannot be determined by the integration of Eqs. 2.81 if the six strain components are chosen arbitrarily. That is, certain relationships (the *strain compatibility relations*) among the six strain components must exist so that Eqs. 2.81 may be integrated to obtain the three displacement components. To illustrate this point, for simplicity, consider the case of *plane strain* relative to the (x, y) plane. This state of strain is defined by the condition that the displacement components (u, v) are functions of (x, y) only and w = constant. Then Eqs. 2.81 yield

$$
\begin{gathered}
\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0
\end{gathered}
\tag{a}
$$

The strain compatibility condition is obtained by elimination of the two displacement components (u, v) from the three nonzero strain–displacement relations in Eqs. (a). This can be done by differentiation and addition as follows. Note that, by differentiation, Eqs. (a) yield

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x\, \partial y^2}, \quad \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\, \partial y} \tag{b}$$

and

$$\frac{2\partial^2 \epsilon_{xy}}{\partial x\, \partial y} = \frac{\partial^3 u}{\partial x\, \partial y^2} + \frac{\partial^3 v}{\partial x^2\, \partial y} \tag{c}$$

Addition of the right-hand sides of Eqs. (b) shows that the right-hand side of Eq. (c) is obtained. Therefore, the relation

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{2\partial^2 \epsilon_{xy}}{\partial x\, \partial y} \tag{d}$$

among the three strain components exists. This result, valid for small strains, is known as the *strain compatibility relation for plane strain.* In the general case, a similar elimination of (u, v, w) from Eqs. 2.81 yields the results (Boresi and Chong, 2000, Section 2-16)

$$\begin{aligned}
\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial y^2} &= 2\frac{\partial^2 \epsilon_{xy}}{\partial x\, \partial y} \\
\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} &= 2\frac{\partial^2 \epsilon_{xz}}{\partial x\, \partial z} \\
\frac{\partial^2 \epsilon_{zz}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial z^2} &= 2\frac{\partial^2 \epsilon_{yz}}{\partial y\, \partial z} \\
\frac{\partial^2 \epsilon_{zz}}{\partial x\, \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial z^2} &= \frac{\partial^2 \epsilon_{yz}}{\partial z\, \partial x} + \frac{\partial^2 \epsilon_{zx}}{\partial y\, \partial z} \\
\frac{\partial^2 \epsilon_{yy}}{\partial x\, \partial z} + \frac{\partial^2 \epsilon_{xz}}{\partial y^2} &= \frac{\partial^2 \epsilon_{xy}}{\partial y\, \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x\, \partial y} \\
\frac{\partial^2 \epsilon_{xx}}{\partial y\, \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x^2} &= \frac{\partial^2 \epsilon_{xz}}{\partial x\, \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial x\, \partial z}
\end{aligned} \tag{2.83}$$

Equations 2.83 are known as the *strain compatibility equations of small-displacement theory.* It may be shown that if the strain components $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ satisfy Eqs. 2.83, there exist displacement components (u, v, w) that are solutions of Eqs. 2.81. More fully, in the small-displacement theory, the functions $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ are possible components of strain if, and only if, they satisfy Eqs. 2.83. For large displacement theory, the equivalent results are given by Murnahan (1951).

2.8.2 Strain–Displacement Relations for Orthogonal Curvilinear Coordinates

More generally, the strain-displacement relations (Eqs. 2.62) may be written for orthogonal curvilinear coordinates (Figure 2.16). The derivation of the expressions for $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ is a routine problem (Boresi and Chong, 2000). For small-displacement theory, the results are

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left(\frac{\partial u}{\partial x} + \frac{v}{\beta}\frac{\partial \alpha}{\partial y} + \frac{w}{\gamma}\frac{\partial \alpha}{\partial z}\right) \\
\epsilon_{yy} &= \frac{1}{\beta}\left(\frac{\partial v}{\partial y} + \frac{w}{\gamma}\frac{\partial \beta}{\partial z} + \frac{u}{\alpha}\frac{\partial \beta}{\partial x}\right) \\
\epsilon_{zz} &= \frac{1}{\gamma}\left(\frac{\partial w}{\partial z} + \frac{u}{\alpha}\frac{\partial \gamma}{\partial x} + \frac{v}{\beta}\frac{\partial \gamma}{\partial y}\right)
\end{aligned} \tag{2.84}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial u}{\partial y} + \frac{1}{\alpha}\frac{\partial v}{\partial x} - \frac{v}{\alpha\beta}\frac{\partial \beta}{\partial x} - \frac{u}{\alpha\beta}\frac{\partial \alpha}{\partial y}\right)$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial w}{\partial x} + \frac{1}{\gamma}\frac{\partial u}{\partial z} - \frac{u}{\alpha\gamma}\frac{\partial \alpha}{\partial z} - \frac{w}{\alpha\gamma}\frac{\partial \gamma}{\partial x}\right) \qquad (2.84) \text{ continued}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial w}{\partial y} + \frac{1}{\gamma}\frac{\partial v}{\partial z} - \frac{w}{\beta\gamma}\frac{\partial \gamma}{\partial y} - \frac{v}{\beta\gamma}\frac{\partial \beta}{\partial z}\right)$$

where (u, v, w) are the projections of the displacement vector of point (x, y, z) on the tangents to the respective coordinate lines at that point and (α, β, γ) are the metric coefficients of the coordinate system (Eq. 2.47). Equations 2.84 are easily specialized for particular coordinates. For cylindrical coordinates $x = r$, $y = \theta$, $z = z$ and then $\alpha = 1$, $\beta = r$, $\gamma = 1$; for spherical coordinates, $x = r$, $y = \theta =$ colatitude, $z = \phi =$ longitude and then $\alpha = 1$, $\beta = r$, $\gamma = r\sin\theta$ (see Section 2.5), etc.

Thus, we obtain for

Cylindrical Coordinates, where u is the radial displacement (in the r direction), v is the tangential displacement associated with the angular coordinate θ, and w is the longitudinal displacement (in the z direction),

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \\
\gamma_{r\theta} &= 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{rz} = 2\epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\
\gamma_{\theta z} &= 2\epsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned} \qquad (2.85)$$

Spherical Coordinates, where u is the radial displacement (in the r direction), v is the tangential displacement associated with the angular coordinate θ, and w is the tangential displacement associated with the angular coordinate ϕ,

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \epsilon_{\phi\phi} = \frac{u}{r} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} \\
\gamma_{r\theta} &= 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{r\phi} = 2\epsilon_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r} \\
\gamma_{\theta\phi} &= 2\epsilon_{\theta\phi} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta} - w\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}
\end{aligned} \qquad (2.86)$$

Polar Coordinates, where u is the radial displacement (in the r direction) and v is the tangential displacement associated with the angular coordinate θ,

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \qquad (2.87)$$

EXAMPLE 2.8
Three-Dimensional State of Strain

A machine part in the form of a parallelopiped (Figure E2.8) is deformed into the shape indicated by the dashed straight lines (small displacements). The displacements are given by the following relations: $u = C_1xyz$, $v = C_2xyz$, and $w = C_3xyz$.

(a) Determine the state of strain at point E when the coordinates of point E^* for the deformed body are (1.504, 1.002, 1.996).

(b) Determine the normal strain at E in the direction of line EA.

(c) Determine the shear strain at E for the undeformed orthogonal lines EA and EF.

Solution The magnitudes of C_1, C_2, and C_3 are obtained from the fact that the displacements of point E are known as follows: $u_E = 0.004$ m, $v_E = 0.002$ m, and $w_E = -0.004$ m. Thus,

$$u = \frac{0.004}{3}xyz$$

$$v = \frac{0.002}{3}xyz$$

$$w = -\frac{0.004}{3}xyz$$

(a) The strain components for the state of strain at point E are given by Eqs. 2.81. At point E,

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{0.004}{3}yz = 0.00267, \quad \epsilon_{yy} = 0.00200, \quad \epsilon_{zz} = -0.00200$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{0.002}{3}yz + \frac{0.004}{3}xz\right) = 0.00267$$

$$\gamma_{xy} = 2\epsilon_{xy} = 0.00533, \quad \gamma_{xz} = 2\epsilon_{xz} = -0.00067$$

$$\gamma_{yz} = 2\epsilon_{yz} = -0.00300$$

(b) Let the X axis lie along the line from E to A. The direction cosines of EA are $l_1 = 0$, $m_1 = -1/\sqrt{5}$, and $n_1 = -2/\sqrt{5}$. Equations 2.61 and 2.82 give the magnitude for ϵ_{XX}. Thus,

$$\begin{aligned}\epsilon_{XX} &= \epsilon_{yy}m_1^2 + \epsilon_{zz}n_1^2 + 2\epsilon_{yz}m_1n_1 \\ &= \frac{0.00200}{5} - \frac{0.00200(4)}{5} - \frac{0.00300(2)}{5} = -0.00240\end{aligned}$$

(c) Let the Y axis lie along the line from E to F. The direction cosines of EF are $l_2 = -1$, $m_2 = 0$, and $n_2 = 0$. The shear strain $\gamma_{XY} = 2\epsilon_{XY}$ is given by Eq. 2.76d. Thus,

$$\begin{aligned}\gamma_{XY} &= 2\epsilon_{XY} = 2\epsilon_{xy}l_2m_1 + 2\epsilon_{xz}l_2n_1 \\ &= \frac{(0.00533)}{\sqrt{5}} + \frac{(-0.00067)(2)}{\sqrt{5}} = 0.00179\end{aligned}$$

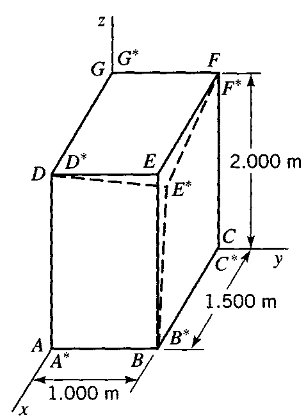

FIGURE E2.8

EXAMPLE 2.9 State of Strain in Torsion-Tension Member

A straight torsion-tension member with a solid circular cross section has a length $L = 6$ m and radius $R = 10$ mm. The member is subjected to tension and torsion loads that produce an elongation $\Delta L = 10$ mm and a rotation of one end of the member with respect to the other end of $\pi/3$ rad. Let the origin of the (r, θ, z) cylindrical coordinate axes lie at the centroid of one end of the member, with the z axis extending along the centroidal axis of the member. The deformations of the member are assumed to occur under conditions of constant volume. The end $z = 0$ is constrained so that only radial displacements are possible there.

(a) Determine the displacements for any point in the member and the state of strain for a point on the outer surface.

(b) Determine the principal strains for the point where the state of strain was determined.

Solution

The change in radius ΔR for the member is obtained from the condition of constant volume. Thus,

$$\begin{aligned} \pi R^2 L &= \pi(R + \Delta R)^2(L + \Delta L) \\ 10^2(6 \times 10^3) &= (10 + \Delta R)^2(6010) \\ \Delta R &= -0.00832 \text{ mm} \end{aligned}$$

(a) The displacements components

$$\begin{aligned} u &= \left(\frac{\Delta R}{R}\right)r = -0.000832r \text{ [mm]} \\ v &= \left(\frac{\pi/3}{L}\right)rz = 0.0001745rz \text{ [mm]} \\ w &= \left(\frac{\Delta L}{L}\right)z = 0.001667z \text{ [mm]} \end{aligned}$$

satisfy the displacement boundary conditions at $z = 0$. The strain components at the outer radius are given by Eqs. 2.85. They are (rounded to six decimal places)

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u}{\partial r} = -0.000832, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = -0.000832 \\ \epsilon_{zz} &= \frac{\partial w}{\partial z} = 0.001667, \quad \gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0 \\ \gamma_{rz} &= 2\epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0, \quad \gamma_{\theta z} = 2\epsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0.001745 \end{aligned}$$

(b) The three principal strains are the three roots of a cubic equation, Eq. 2.77b, where the three invariants of strain are defined by Eqs. 2.78. Choose the (x, y, z) coordinate axes at the point on the outer surface of the member where the strain components have been determined in part (a). Let $x = r$, $y = \theta$, and $z = z$. From Eqs. 2.78,

$$\begin{aligned} \bar{I}_1 &= \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = -0.000832 - 0.000832 + 0.001667 \approx 0 \\ \bar{I}_2 &= \epsilon_{rr}\epsilon_{\theta\theta} + \epsilon_{rr}\epsilon_{zz} + \epsilon_{\theta\theta}\epsilon_{zz} - \epsilon_{r\theta}^2 - \epsilon_{rz}^2 - \epsilon_{\theta z}^2 \\ &= (-0.000832)(-0.000832) + (-0.000832)(0.001667) \\ &\quad + (-0.000832)(0.001667) - \left(\frac{0.001745}{2}\right)^2 \\ &= -2.838 \times 10^{-6} \end{aligned}$$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{vmatrix} = \begin{vmatrix} -0.000832 & 0 & 0 \\ 0 & -0.000832 & \dfrac{0.001745}{2} \\ 0 & \dfrac{0.001745}{2} & 0.001667 \end{vmatrix}$$

$$= 1.785 \times 10^{-9}$$

Substitution of these results into Eq. 2.77b gives the following cubic equation in ϵ $(= M)$:

$$\epsilon^3 - 2.838 \times 10^{-6}\epsilon - 1.785 \times 10^{-9} = 0$$

One principal strain, $\epsilon_{rr} = -0.000832$, is known. Factoring out this root, we find

$$\epsilon^2 - 0.000832\epsilon - 2.146 \times 10^{-6} = 0$$

Solution of this quadratic equation yields the remaining two principal strains. Thus, the three principal strains are

$$\epsilon_1 = 0.001939$$
$$\epsilon_2 = -0.000832$$
$$\epsilon_3 = -0.001107$$

EXAMPLE 2.10
Mohr's Circle for Plane Strain

A state of plane strain ($\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$) at a point in a body is given, with respect to the (x, y, z) axes, as $\epsilon_{xx} = 0.00044$, $\epsilon_{yy} = 0.00016$, and $\epsilon_{xy} = -0.00008$. Determine the principal strains in the (x, y) plane, the orientation of the principal axes of strain, the maximum shear strain, and the strain state on a block rotated by an angle of $\theta' = 25°$ measured counterclockwise with respect to the reference axes.

Solution

Since the components of strain form a symmetric second-order tensor, they are transformed in precisely the same way as stresses. Thus, plane strain states can be represented by Mohr's circle in the same way as plane stress states. By analogy to the development for plane stress, Mohr's circle for plane strain is defined by the equation (see Eq. 2.32)

$$\left[\epsilon_{XX} - \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy})\right]^2 + (\epsilon_{xx} - 0)^2 = \frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2 \tag{a}$$

Equation (a) is the equation of a circle in the $(\epsilon_{XX}, \epsilon_{XY})$ plane with center coordinates

$$\left[\frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}), 0\right] \tag{b}$$

and radius

$$R = \sqrt{\frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{c}$$

The orientation of the principal axes of strain is given by the angle θ, where

$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \tag{d}$$

and θ is measured with respect to the reference x axis, positive in the counterclockwise sense. The principal strains are

$$\epsilon_1 = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} + \sqrt{\frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{e}$$

$$\epsilon_2 = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} - \sqrt{\frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{f}$$

The maximum shear strain is simply the radius of the circle as given by Eq. (c).

For the data given, the state of strain may be expressed as $\epsilon_{xx} = 440\mu$, $\epsilon_{yy} = 160\mu$, and $\epsilon_{xy} = -80\mu$, where $\mu = 10^{-6}$. This representation of strain is known as *microstrain*. The Mohr's circle for these data is shown in Figure E2.10*a*. By Eqs. (b) and (c), the center of the circle is located at point C with coordinates $(300\mu, 0)$ and its radius is $R = 161\mu$. By Eqs. (e) and (f), the principal strains are

$$\epsilon_1 = 300\mu + 161\mu = 461\mu \tag{g}$$

$$\epsilon_2 = 300\mu - 161\mu = 139\mu \tag{h}$$

On Mohr's circle, they correspond to points Q and Q', respectively. The reference strain state is plotted at points $P(\epsilon_{xx}, \epsilon_{xy})$ and $P'(\epsilon_{yy}, -\epsilon_{xy})$. Note that the positive ϵ_{XY} axis is directed downward, as is done with the plane stress case. By Eq. (d) the principal axis corresponding to ϵ_1 in the body is located at an angle of $\theta = -14.87°$ with respect to the x axis. On Mohr's circle, this corresponds to an angle $2\theta = -29.74°$ from line CP to line CQ.

The maximum value of shear strain is $\epsilon_{XY(\max)} = R = 161\mu$. It occurs at an orientation of $\pm 45°$ from the principal axis for ϵ_1 ($\pm 90°$ from line CQ on Mohr's circle). Note that this is the maximum shear strain using the tensor definition of strain. The maximum engineering shear strain is $\gamma_{XY(\max)} = 2\epsilon_{XY(\max)} = 322\mu$ (Eq. 2.73). The strain state on a block at $\theta' = 25°$ ($50°$ on Mohr's circle) is identified by points $S(\epsilon'_{xx}, \epsilon'_{xy})$ and $S'(\epsilon'_{yy}, -\epsilon'_{xy})$. By geometry of the circle, the strain quantities are

$$\begin{aligned}
\epsilon'_{xx} &= OC + R\cos(2\theta' - 2\theta) = 329\mu \\
\epsilon'_{yy} &= OC - R\cos(2\theta' - 2\theta) = 271\mu \\
\epsilon'_{xy} &= -R\sin(2\theta' - 2\theta) = 295\mu
\end{aligned}$$

In Figure E2.10*b*, the deformed shape of an element in the reference orientation is shown. Also illustrated is the deformed shape in the principal orientation, which is at an angle of $\theta = -14.87°$ with respect to the x axis. Notice that in this orientation, the deformed element is not distorted, since the shear strain is zero. Finally, the deformed shape at $\theta' = 25°$ with respect to the reference orientation is shown.

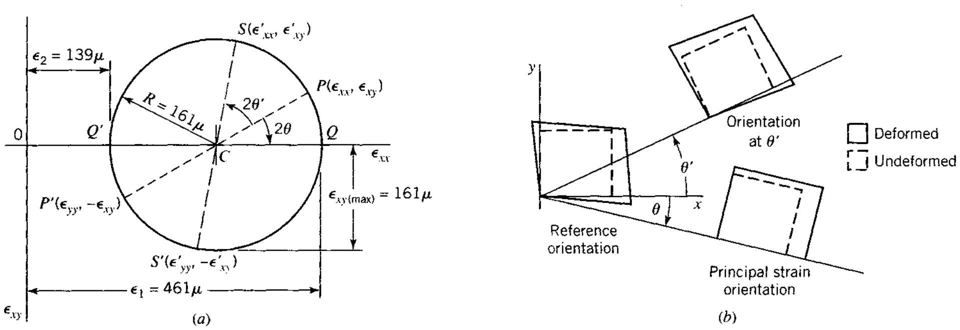

FIGURE E2.10 (*a*) Mohr's circle for plane strain. (*b*) Deformed element in three different orientations.

EXAMPLE 2.11 ***Integration of the Strain–Displacement Relations for Small-Displacement Theory***

For small-displacement plane strain, the strain components in the plate *ABCD* (Figure E2.11), in terms of the coordinate system (x, y), are

$$\epsilon_{xx} = Cy(L-x), \quad \epsilon_{yy} = Dy(L-x), \quad \gamma_{xy} = -(C+D)(A^2-y^2) \tag{a}$$

where A, C, and D are known constants. The displacement components (u, v) at $x = y = 0$ are

$$u(0,0) = 0, \quad v(0,0) = 0 \tag{b}$$

and the slope $\partial u/\partial y$ at $x = y = 0$ is

$$\left.\frac{\partial u}{\partial y}\right|_{x=y=0} = 0 \tag{c}$$

Determine u and v as functions of (x, y).

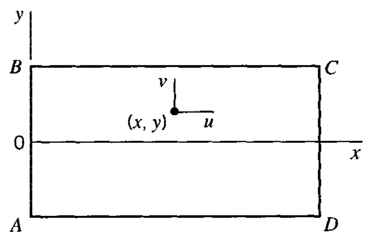

FIGURE E2.11

Solution

By Eqs. 2.81 and (a), we have

$$\epsilon_{xx} = Cy(L-x) = \frac{\partial u}{\partial x} \tag{d}$$

$$\epsilon_{yy} = Dy(L-x) = \frac{\partial v}{\partial y} \tag{e}$$

$$\gamma_{xy} = 2\epsilon_{xy} = -(C+D)(A^2-y^2) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{f}$$

Integration of Eqs. (d) and (e) yields

$$u = Cy\left(Lx - \frac{1}{2}x^2\right) + Y(y) \tag{g}$$

$$v = \frac{1}{2}Dy^2(L-x) + X(x) \tag{h}$$

where X and Y are functions of x and y, respectively. Substituting Eqs. (g) and (h) into Eq. (f) and regrouping terms, we obtain

$$\frac{dX}{dx} + C\left(Lx - \frac{x^2}{2}\right) = -\frac{dY}{dy} + \frac{1}{2}Dy^2 - (C+D)(A^2-y^2) \tag{i}$$

For Eq. (i) to be satisfied, both sides of the equation must be equal to the same constant. That is,

$$\frac{dX}{dx} + C\left(Lx - \frac{1}{2}x^2\right) = E \tag{j}$$

$$-\frac{dY}{dy} + \frac{1}{2}Dy^2 - (C + D)(A^2 - y^2) = E \tag{k}$$

where E is a constant.

Then integrations of Eqs. (j) and (k) yield

$$X(x) = -C\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + Ex + J \tag{l}$$

$$Y(y) = \frac{1}{6}Dy^3 - (C + D)\left(A^2 y - \frac{1}{3}y^3\right) - Ey + K \tag{m}$$

where J and K are constants. By Eqs. (g), (h), (l), and (m), we find

$$u = Cy\left(Lx - \frac{1}{2}x^2\right) + \left(\frac{1}{3}C + \frac{1}{2}D\right)y^3 + \left[E - (C + D)A^2\right]y + F \tag{n}$$

$$v = \frac{1}{2}Dy^2(L - x) - C\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) - Ex + J \tag{o}$$

By Eqs. (b), (n), and (o), $F = 0$ and $J = 0$. Then by Eqs. (c) and (n), E is determined as

$$E = (C + D)A^2$$

Equations (n) and (o) are modified accordingly.

2.9 STRAIN MEASUREMENT AND STRAIN ROSETTES

For members of complex shape subjected to loads, it may be mathematically impractical or impossible to derive analytical load–stress relations. Then, either numerical or experimental methods are used to obtain approximate results. Several experimental methods are used, the most common one being the use of strain gages.

Strain gages are used to measure extensional strains on the free surface of a member or the axial extension/contraction of a bar. They cannot be used to measure the strain at an interior point of a member. To measure interior strains (or stresses), other techniques such as photoelasticity may be used, although this method has been largely superseded by modern numerical techniques. Nevertheless, photoelastic methods are still useful when augmented with modern computer data-acquisition techniques (Kobayashi, 1987). Additional experimental procedures are also available. They include holographic, Moiré, and laser speckle interferometry techniques. These specialized methods lie outside of the scope of this text (see Kobayashi, 1987). We shall discuss only the use of electrical resistance (bonded) strain gages.

Electric strain gages are used to obtain average extensional strain over a given gage length. These gages are made of very fine wire or metal foil and are glued to the surface of the member being tested. When forces are applied to the member, the gage elongates or contracts with the member. The change in length of the gage alters its electrical resistance. The change in resistance can be measured and calibrated to indicate the average extensional strain that occurs over the gage length. To meet various requirements, gages are made in a variety of gage lengths, varying from 4 to 150 mm (approximately 0.15 to 6 in.), and are designed for different environmental conditions.

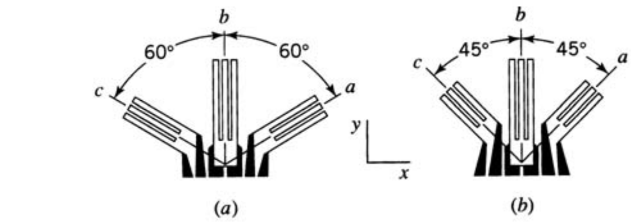

FIGURE 2.20 Rosette strain gages. (*a*) Delta rosette. (*b*) Rectangular rosette.

Three extensional strain measurements in three different directions at a point on the surface of a member are required to determine the average state of strain at that point. Consequently, it is customary to cluster together three gages to form a *strain rosette* that may be cemented to the free surface of a member. Two common forms of rosettes are the *delta rosette* (with three gages spaced at 60° angles) and the *rectangular rosette* (with three gages spaced at 45° angles), as shown in Figure 2.20. From the measurement of extensional strains along the gage arm directions (directions a, b, c in Figure 2.20), one can determine the strain components (ϵ_{xx}, ϵ_{yy}, ϵ_{xy}) at the point, relative to the (x, y) axes. Usually, one of the axes is taken to be aligned with one arm of the rosette, say, the arm a. For instance, we might mount the rosette such that $\epsilon_{xx} = \epsilon_a$, the average extensional strain in the direction a. Then, the components (ϵ_{yy}, ϵ_{xy}) may be expressed in terms of the measured extensional strains ϵ_a, ϵ_b, and ϵ_c in the directions of the three rosette arms a, b, and c, respectively. (See Example 2.12.)

EXAMPLE 2.12
Measurement of Strain on a Surface of a Member

A strain rosette with gages spaced at an angle θ is cemented to the free surface ($\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$) of a member (Figure E2.12). Under a deformation of the member, the extensional strains measured by gages a, b, c are ϵ_a, ϵ_b, ϵ_c, respectively.

(a) Derive equations that determine the strain components ϵ_{xx}, ϵ_{yy}, ϵ_{xy} in terms of ϵ_a, ϵ_b, ϵ_c, and θ.

(b) Specialize the results for the delta rosette ($\theta = 60°$) and rectangular rosette ($\theta = 45°$).

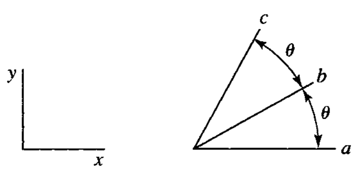

FIGURE E2.12

Solution

(a) The direction cosines of arms a, b, and c are, respectively,

$$(l_a, m_a, n_a) = (1,0,0),\ (l_b, m_b, n_b) = (\cos\theta, \sin\theta, 0),\ (l_c, m_c, n_c) = (\cos 2\theta, \sin 2\theta, 0)$$

The extensional strain of a line element in the direction (l, m, n) is given by Eq. 2.61. Hence, by Eq. 2.61, the extensional strains in the directions of arms a, b, c are

$$\begin{aligned} \epsilon_a &= \epsilon_{xx} \\ \epsilon_b &= \epsilon_{xx}(\cos^2\theta) + \epsilon_{yy}(\sin^2\theta) + 2\epsilon_{xy}(\cos\theta)(\sin\theta) \\ \epsilon_c &= \epsilon_{xx}(\cos^2 2\theta) + \epsilon_{yy}(\sin^2 2\theta) + 2\epsilon_{xy}(\cos 2\theta)(\sin 2\theta) \end{aligned} \tag{a}$$

Equations (a) are three equations that may be solved for ϵ_{xx}, ϵ_{yy}, and ϵ_{xy} in terms of ϵ_a, ϵ_b, and ϵ_c for a given angle θ. The solution is

$$\begin{aligned} \epsilon_{xx} &= \epsilon_a \\ \epsilon_{yy} &= \frac{(\epsilon_a - 2\epsilon_b)\sin 4\theta + 2\epsilon_c \sin 2\theta}{4\sin^2\theta \sin 2\theta} \\ \epsilon_{xy} &= \frac{2\epsilon_a(\sin^2\theta\cos^2 2\theta - \sin^2 2\theta\cos^2\theta) + 2(\epsilon_b \sin^2 2\theta - \epsilon_c \sin^2\theta)}{4\sin^2\theta \sin 2\theta} \end{aligned} \tag{b}$$

(b) For $\theta = 60°$, $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$, $\cos 2\theta = -1/2$, and $\sin 2\theta = \sqrt{3}/2$. Therefore, for $\theta = 60°$, Eqs. (b) yield

$$\epsilon_{xx} = \epsilon_a, \quad \epsilon_{yy} = \frac{2(\epsilon_b + \epsilon_c) - \epsilon_a}{3}, \quad \epsilon_{xy} = \frac{\epsilon_b - \epsilon_c}{\sqrt{3}} \tag{c}$$

For $\theta = 45°$, $\cos\theta = 1/\sqrt{2}$, $\sin\theta = 1/\sqrt{2}$, $\cos 2\theta = 0$, and $\sin 2\theta = 1$. Therefore, for $\theta = 45°$, Eqs. (b) yield

$$\epsilon_{xx} = \epsilon_a, \quad \epsilon_{yy} = \epsilon_c, \quad \epsilon_{xy} = \epsilon_b - \frac{1}{2}(\epsilon_a + \epsilon_c) \tag{d}$$

PROBLEMS

Sections 2.1–2.4

Many of the problems for Sections 2.1–2.4 require the determination of principal stresses and maximum shear stress, as well as the location of the planes on which they act. These quantities are required input in design and failure criteria for structural and mechanical systems.

2.1. The state of stress at a point in a body is given by the following components: σ_{xx} = 50 MPa, σ_{yy} = –30 MPa, σ_{zz} = 20 MPa, σ_{xy} = 5 MPa, σ_{xz} = –30 MPa, and σ_{yz} = 0. Find σ_{Px}, σ_{Py}, σ_{Pz}, and σ_{PS} for point P on a cutting plane Q with normal vector **N**: $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

2.2. The thin, flat plate shown in Figure P2.2 is subjected to uniform shear stress. Find the normal and shear stress acting on the cutting plane A–A.

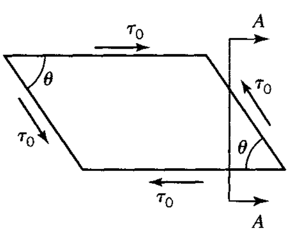

FIGURE P2.2

2.3. At a point in a beam, the stress components are σ_{xx} = 20 MPa, $\sigma_{xz} = 10\sqrt{3}$ MPa, and $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$.

a. Determine the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

b. Determine the direction cosines of the normal to the plane on which σ_1 acts.

2.4. The stress components at a point in a plate are σ_{xx} = –100 MPa, $\sigma_{yy} = \sigma_{zz}$ = 20 MPa, $\sigma_{xy} = \sigma_{yz}$ = 0, and σ_{xz} = –80 MPa.

a. Determine the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

b. Determine the octahedral shear stress.

c. Determine the maximum shear stress and compare it to the octahedral shear stress.

d. Determine the direction cosines of the normal to one of the planes on which the maximum shear stress acts.

2.5. The stress components at a point in a plate are σ_{xx} = 80 MPa, σ_{yy} = 60 MPa, $\sigma_{zz} = \sigma_{xy}$ = 20 MPa, σ_{xz} = 40 MPa, and σ_{yz} = 10 MPa.

a. Determine the stress vector on a plane normal to the vector **i** + 2**j** + **k**.

b. Determine the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

c. Determine the maximum shear stress.

d. Determine the octahedral shear stress.

2.6. The stress array (Eq. 2.5) relative to axes (x, y, z) is given by

$$\mathbf{T} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

where the stress components are in [MPa].

a. Determine the stress invariants of **T**.

b. Consider a rotation of the (x, y) axes by 45° counterclockwise in the (x, y) plane to form axes (X, Y). Let the Z axis and the z axis coincide. Calculate the stress components relative to axes (X, Y, Z). Note that the direction cosines between axes (x, y, z) and (X, Y, Z) are given by Table 2.2, where the rotation of 45° determines $l_1, m_1, n_1, \ldots$.

c. With the results of part (b), determine the stress invariants relative to axes (X, Y, Z) and show that they are the same as the invariants of part (a).

2.7. Relative to axes (x, y, z), body A is loaded so that $\sigma_{xx} = \sigma_0$ and the other stress components are zero. A second body B is loaded so that $\sigma_{xy} = \tau_0$ and the other stress components are zero. The octahedral shear stress τ_{oct} has the same value for both bodies. Determine the ratio σ_0/τ_0.

2.8. The stress components at a point in a flywheel are $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = \tau_0$.

a. In terms of τ_0, calculate the principal stresses.

b. Determine the directions of the principal stress axes, insofar as they are determinate.

2.9. The principal stresses σ_1 and σ_3 are known, and $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Determine the value of σ_2 for which the octahedral shear stress τ_{oct} attains an extreme value.

2.10. The nonzero stress components relative to axes (x, y, z) are $\sigma_{xx} = -90$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 6$ MPa.

a. Determine the principal stresses, $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

b. Calculate the maximum shear stress.

c. Calculate the octahedral shear stress.

d. Determine the angle between the x axis and the X axis, where X is in the direction of the principal stress σ_1.

2.11. Let $(\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z)$ be stress vectors on planes perpendicular, respectively, to the (x, y, z) axes. Show that the sum of the squares of the magnitudes of these stress vectors is an invariant, expressible in terms of the stress invariants I_1 and I_2.

2.12. Determine the principal stresses and their directions for each of the sets of stress components listed in Table P2.12. Also calculate the maximum shear stress and the octahedral shear stress. The units of stress are [MPa].

TABLE P2.12

	σ_{xx}	σ_{yy}	σ_{zz}	σ_{xy}	σ_{xz}	σ_{yz}
(a)	15	−4	10	−3	0	1
(b)	10	−5	0	−5	0	0
(c)	−10	−5	10	2	3	4
(d)	10	−5	−5	2	2	0
(e)	10	0	0	0	0	0

2.13. The state of stress at a point is specified by the following stress components: $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\sigma_{xy} = -75$ MPa, $\sigma_{yz} = 65$ MPa, and $\sigma_{zx} = -55$ MPa. Determine the principal stresses, direction cosines for the three principal stress directions, and maximum shear stress.

2.14. Consider a state of stress in which the nonzero stress components are σ_{xx}, σ_{yy}, σ_{zz}, and σ_{xy}. Note that this is not a state of plane stress since $\sigma_{zz} \neq 0$. Consider another set of coordinate axes (X, Y, Z), with the Z axis coinciding with the z axis and the X axis located counterclockwise through angle θ from the x axis. Show that the transformation equations for this state of stress are identical to Eq. 2.30 or 2.31 for plane stress.

2.15. The state of stress at a point is specified by the following stress components: $\sigma_{xx} = 110$ MPa, $\sigma_{yy} = -86$ MPa, $\sigma_{zz} = 55$ MPa, $\sigma_{xy} = 60$ MPa, and $\sigma_{yz} = \sigma_{zx} = 0$. Determine the principal stresses, direction cosines of the principal stress directions, and maximum shear stress.

2.16. Solve Problem 2.15 using the transformation equations for plane stress (Eq. 2.30 or 2.31).

2.17. The state of plane stress is specified by the following stress components: $\sigma_{xx} = 90$ MPa, $\sigma_{yy} = -10$ MPa, and $\sigma_{xy} = 40$ MPa. Let the X axis lie in the (x, y) plane and be located at $\theta = \pi/6$ clockwise from the x axis. Determine the normal and shear stresses on a plane perpendicular to the X axis; use Eqs. 2.10–2.12.

In Problems 2.18 through 2.21, the Z axis for the transformed axes coincides with the z axis for the volume element on which the known stress components act.

2.18. The nonzero stress components are $\sigma_{xx} = 200$ MPa, $\sigma_{yy} = 100$ MPa, and $\sigma_{xy} = -50$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.19. The nonzero stress components are $\sigma_{xx} = -90$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 60$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.20. The nonzero stress components are $\sigma_{xx} = 80$ MPa, $\sigma_{zz} = -60$ MPa, and $\sigma_{xy} = 30$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.21. The nonzero stress components are $\sigma_{xx} = 150$ MPa, $\sigma_{yy} = 70$ MPa, $\sigma_{zz} = -80$ MPa, and $\sigma_{xy} = -45$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.22. Using transformation equations of plane stress, determine σ_{XX} and σ_{XY} for the X axis located 0.50 rad clockwise from the x axis. The nonzero stress components are given in Problem 2.18.

2.23. Using transformation equations of plane stress, determine σ_{XX} and σ_{XY} for the X axis located 0.15 rad counterclockwise from the x axis. The nonzero stress components are given in Problem 2.19.

2.24. Using transformation equations of plane stress (see Problem 2.14), determine σ_{XX} and σ_{XY} for the X axis located 1.00 rad clockwise from the x axis. The nonzero stress components are given in Problem 2.20.

2.25. Using transformation equations of plane stress (see Problem 2.14), determine σ_{XX} and σ_{XY} for the X axis located 0.70 rad counterclockwise from the x axis. The nonzero stress components are given in Problem 2.21.

2.26. Using Mohr's circle of stress, determine σ_{XX} and σ_{XY} for the X axis located 0.50 rad clockwise from the x axis. The nonzero components of stress are σ_{xx} = 200 MPa, σ_{yy} = 100 MPa, and σ_{xy} = –50 MPa.

2.27. Using Mohr's circle of stress, determine σ_{XX} and σ_{XY} for the X axis located 0.15 rad counterclockwise from the x axis. The nonzero components of stress are σ_{xx} = –90 MPa, σ_{yy} = 50 MPa, and σ_{xy} = 60 MPa.

2.28. Using Mohr's circle of stress, determine σ_{XX} and σ_{XY} for the X axis located 1.00 rad clockwise from the x axis. The nonzero components of stress are σ_{xx} = 80 MPa, σ_{zz} = –60 MPa, and σ_{xy} = 30 MPa.

2.29. Using Mohr's circle of stress, determine σ_{XX} and σ_{XY} for the X axis located 0.70 rad counterclockwise from the x axis. The nonzero components of stress are σ_{xx} = 150 MPa, σ_{yy} = 70 MPa, σ_{zz} = –80 MPa, and σ_{xy} = –45 MPa.

2.30. A volume element at the free surface is shown in Figure P2.30. The state of stress is plane stress with σ_{xx} = 100 MPa. Determine the other stress components.

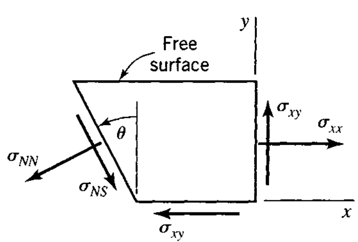

FIGURE P2.30

2.31. Determine the unknown stress components for the volume element in Figure P2.31.

2.32. Determine the unknown stress components for the volume element in Figure P2.32.

2.33. Determine the unknown stress components for the volume element in Figure P2.33.

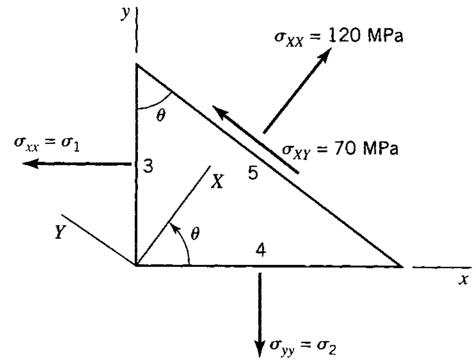

FIGURE P2.31

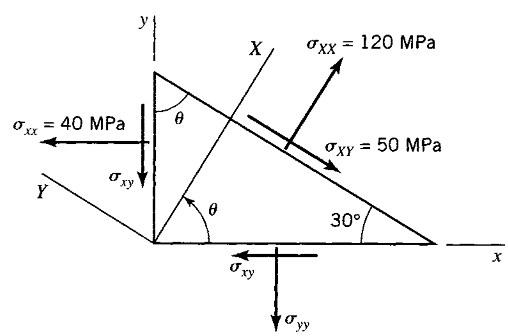

FIGURE P2.32

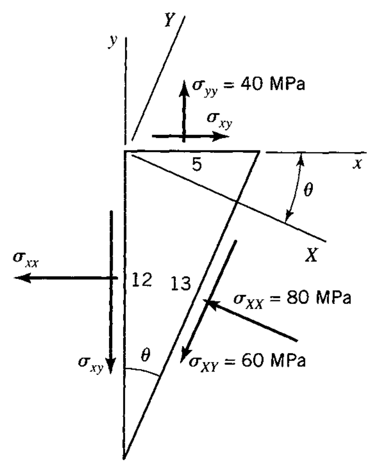

FIGURE P2.33

In Problems 2.34 through 2.38, determine the principal stresses, maximum shear stress, and octahedral shear stress.

2.34. The nonzero stress components are σ_{xx} = −100 MPa, σ_{yy} = 60 MPa, and σ_{xy} = −50 MPa.

2.35. The nonzero stress components are σ_{xx} = 180 MPa, σ_{yy} = 90 MPa, and σ_{xy} = 50 MPa.

2.36. The nonzero stress components are σ_{xx} = −150 MPa, σ_{yy} = −70 MPa, σ_{zz} = 40 MPa, and σ_{xy} = −60 MPa.

2.37. The nonzero stress components are σ_{xx} = 80 MPa, σ_{yy} = −35 MPa, σ_{zz} = −50 MPa, and σ_{xy} = 45 MPa.

2.38. The nonzero stress components are σ_{xx} = 95 MPa, σ_{yy} = 0, σ_{zz} = 60 MPa, and σ_{xy} = −55 MPa.

2.39. The state of stress at a point is given by σ_{xx} = −120 MPa, σ_{yy} = 140 MPa, σ_{zz} = 66 MPa, σ_{xy} = 45 MPa, σ_{yz} = −65 MPa, and σ_{zx} = 25 MPa. Determine the three principal stresses and directions associated with the three principal stresses.

2.40. The state of stress at a point is given by σ_{xx} = 0, σ_{yy} = 100 MPa, σ_{zz} = 0, σ_{xy} = −60 MPa, σ_{yz} = 35 MPa, and σ_{zx} = 50 MPa. Determine the three principal stresses.

2.41. The state of stress at a point is given by σ_{xx} = 120 MPa, σ_{yy} = −55 MPa, σ_{zz} = −85 MPa, σ_{xy} = −55 MPa, σ_{yz} = 33 MPa, and σ_{zx} = −75 MPa. Determine the three principal stresses and maximum shear stress.

2.42. The state of stress at a point is given by σ_{xx} = −90 MPa, σ_{yy} = −60 MPa, σ_{zz} = 40 MPa, σ_{xy} = 70 MPa, σ_{yz} = −40 MPa, and σ_{zx} = −55 MPa. Determine the three principal stresses and maximum shear stress.

2.43. The state of stress at a point is given by σ_{xx} = −150 MPa, σ_{yy} = 0, σ_{zz} = 80 MPa, σ_{xy} = −40 MPa, σ_{yz} = 0, and σ_{zx} = 50 MPa. Determine the three principal stresses and maximum shear stress.

2.44. a. Solve Example 2.1 using Mohr's circle and show the orientation of the volume element on which the principal stresses act.

b. Determine the maximum shear stress and show the orientation of the volume element on which it acts.

2.45. At a point on the flat surface of a member, load–stress relations give the following stress components relative to the (x, y, z) axes, where the z axis is perpendicular to the surface: σ_{xx} = 240 MPa, σ_{yy} = 100 MPa, σ_{xy} = −80 MPa, and $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$.

a. Determine the principal stresses using Eq. 2.20 and then again using Eqs. 2.36 and 2.37.

b. Determine the principal stresses using Mohr's circle and show the orientation of the volume element on which these principal stresses act.

c. Determine the maximum shear stress and maximum octahedral shear stress.

Sections 2.5–2.8

The problems for Sections 2.5–2.8 involve displacements, deformations, and strain states at a point in a structural or machine member. These quantities, as with their stress counterparts, are important in design and failure criteria.

2.46. Verify the reduction of Eqs. 2.46 to Eqs. 2.50.

2.47. Verify the reduction of Eqs. 2.46 to Eqs. 2.53.

2.48. Verify the reduction of Eqs. 2.50 to Eqs. 2.54.

2.49. Show that Eqs. 2.65–2.67 yield Eqs. 2.68.

2.50. By the procedure outlined in the text, derive Eq. 2.71.

2.51. By the procedure outlined in the text, derive Eq. 2.76d.

2.52. By the procedure outlined in the text, derive Eqs. 2.76e and 2.76f.

2.53. The tension member in Figure P2.53 has the following dimensions: L = 5 m, b = 100 mm, and h = 200 mm. The (x, y, z) coordinate axes are parallel to the edges of the member, with origin 0 located at the centroid of the left end. Under the deformation produced by load P, the origin 0 remains located at the centroid of the left end and the coordinate axes remain parallel to the edges of the deformed member. Under the action of load P, the bar elongates 20 mm. Assume that the volume of the bar remains constant with $\epsilon_{xx} = \epsilon_{yy}$.

a. Determine the displacements for the member and the state of strain at point Q, assuming that the small-displacement theory holds.

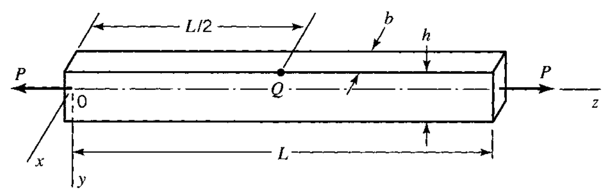

FIGURE P2.53

b. Determine ϵ_{zz} at point Q based on the assumption that displacements are not small.

2.54. In many practical engineering problems, the state of strain is approximated by the condition that the normal and shear strains for some direction, say, the z direction, are zero; that is, $\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy}$ = 0 (plane strain). Assume that ϵ_{xx}, ϵ_{yy}, and ϵ_{xy} for the (x, y) coordinate axes shown in Figure P2.54 are known. Let the (X, Y) coordinate axes be defined by a counterclockwise rotation through angle

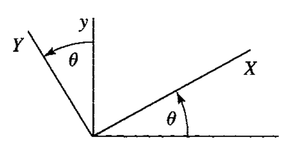

FIGURE P2.54

θ as indicated in Figure P2.54. Analogous to the transformation for plane stress, show that the transformation equations of plane strain are

$$\epsilon_{XX} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\epsilon_{xy}\sin\theta\cos\theta$$

$$\epsilon_{YY} = \epsilon_{xx}\sin^2\theta + \epsilon_{yy}\cos^2\theta - 2\epsilon_{xy}\sin\theta\cos\theta$$

$$\epsilon_{XY} = -\epsilon_{xx}\sin\theta\cos\theta + \epsilon_{yy}\sin\theta\cos\theta + \epsilon_{xy}(\cos^2\theta - \sin^2\theta)$$

(See Eq. 2.30.)

2.55. A square panel in the side of a ship (Figure P2.55) is loaded so that the panel is in a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).

a. Determine the displacements for the panel given the deformations shown and the strain components for the (x, y) coordinate axes.

b. Determine the strain components for the (X, Y) axes.

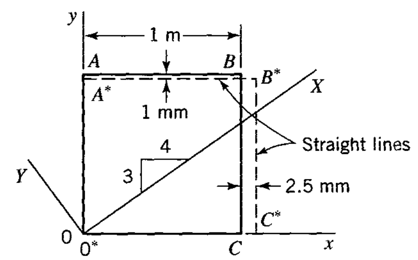

FIGURE P2.55

2.56. A square glass block in the side of a skyscraper (Figure P2.56) is loaded so that the block is in a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).

a. Determine the displacements for the block for the deformations shown and the strain components for the (x, y) coordinate axes.

b. Determine the strain components for the (X, Y) axes.

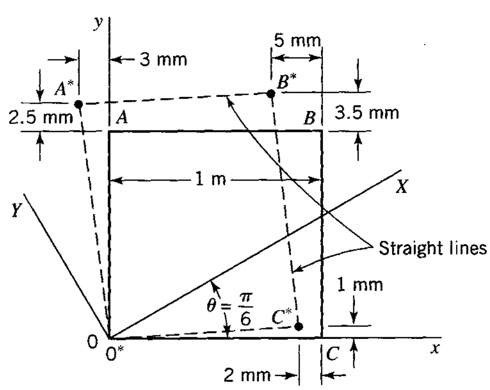

FIGURE P2.56

2.57. Determine the orientation of the (X, Y) coordinate axes for principal directions in Problem 2.56. What are the principal strains?

2.58. A rectangular airplane wing spar (Figure P2.58) is loaded so that a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$) exists.

a. Determine the displacements for the spar for the deformations shown and the strain components at point B.

b. Let the X axis extend from point 0 through point B. Determine ϵ_{XX} at point B.

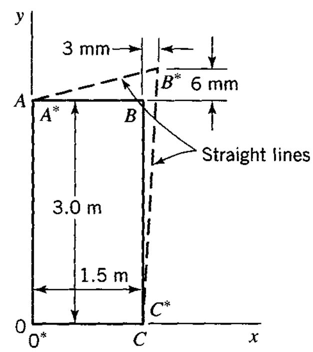

FIGURE P2.58

2.59. The nonzero strain components at a point in a machine member are $\epsilon_{xx} = 0.00180$, $\epsilon_{yy} = -0.00108$, and $\gamma_{xy} = 2\epsilon_{xy} = -0.00220$. Using the transformation equations for plane strain (see Problem 2.54), determine the principal strain directions and principal strains.

2.60. Solve for the principal strains in Problem 2.59 by using Eqs. 2.77b and 2.78.

2.61. Determine the principal strains at point E for the deformed parallelepiped in Example 2.8.

2.62. When solid circular torsion members are used to obtain material properties for finite strain applications, an expression for the engineering shear strain γ_{zx} is needed, where the (x, z) plane is a tangent plane and the z axis is parallel to the axis of the member as indicated in Figure P2.62. Consider an element $ABCD$ in Figure P2.62 for the undeformed member. Assume that the member deforms such that the volume remains constant and the diameter remains unchanged. (This is an approximation of the real behavior of many metals.) Thus, for the deformed element $A^*B^*C^*D^*$, $A^*B^* = AB$, $C^*D^* = CD$, and the distance along the z axis of the member between the parallel curved

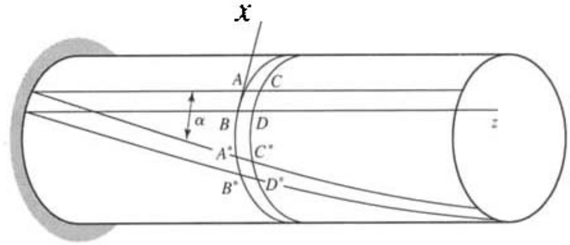

FIGURE P2.62

lines A^*B^* and C^*D^* remains unchanged. Show that Eq. 2.71 gives the result $\gamma_{zx} = \tan\alpha$, where α is the angle between AC and A^*C^*, where $\gamma_{zx} = 2\epsilon_{zx}$ is defined to be the engineering shear strain.

2.63. A state of plane strain exists at a point in a beam, with the nonzero strain components $\epsilon_{xx} = -2000\ \mu$, $\epsilon_{yy} = 400\ \mu$, and $\epsilon_{xy} = -900\ \mu$.

a. Determine the principal strains in the (x, y) plane and the orientation of the rectangular element on which they act. (See Example 2.10.)

b. Determine the maximum shear strain in the (x, y) plane and the orientation of the rectangular element on which it acts.

c. Show schematically the deformed shape of a rectangular element in the reference orientation, along with the original undeformed element. (See Example 2.10.)

2.64. A square plate, 1 m long on a side, is loaded in a state of plane strain and is deformed as shown in Figure P2.64.

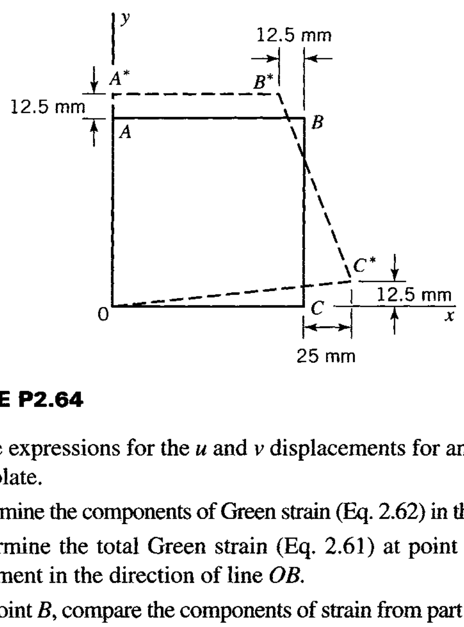

FIGURE P2.64

a. Write expressions for the u and v displacements for any point on the plate.

b. Determine the components of Green strain (Eq. 2.62) in the plate.

c. Determine the total Green strain (Eq. 2.61) at point B for a line element in the direction of line OB.

d. For point B, compare the components of strain from part b to the components of strain for small-displacement theory (Eq. 2.81).

e. Compare the strain determined in part c to the corresponding strain using small-displacement theory.

2.65. Determine whether the following statements are *true* or *false*.

a. Strain theory depends on the material being considered.

b. Stress theory depends on strain theory.

c. The mathematical theories of stress and strain are equivalent.

d. The correct strains of a strained material must be compatible.

2.66. Show that the strain components of Example 2.11 are compatible.

2.67. For a problem of small-displacement plane strain, the strain components in a machine part, relative to axes (x, y, z), are

$$\epsilon_{xx} = A(L-x), \quad \epsilon_{yy} = B(L-x), \quad \epsilon_{xy} = 0 \tag{a}$$

Determine the (x, y) displacement components (u, v), for the case where the displacement components (u, v) vanish at $x = y = 0$ and the slopes $(\partial u/\partial y, \partial v/\partial x)$ are equal at $x = y = 0$. That is,

$$u|_{x=y=0} = v|_{x=y=0} = 0 \tag{b}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x=y=0} = \left.\frac{\partial v}{\partial x}\right|_{x=y=0} \tag{c}$$

2.68. Show that Eqs. 2.84 reduce to Eqs. 2.85 for cylindrical coordinates.

2.69. Show that Eqs. 2.84 reduce to Eqs. 2.86 for spherical coordinates.

2.70. Show that Eqs. 2.84 reduce to Eqs. 2.87 for plane polar coordinates.

2.71. Assume that the machine part shown in Figure E2.8 undergoes the (x, y, z) displacements

$$u = c_1xz, \quad v = c_2yz, \quad w = c_3z$$

where the meter is the unit of length for (u, v, w) and (x, y, z). Use small-displacement theory to:

a. Determine the components of strain at point E, in terms of c_1, c_2, and c_3.

b. Determine the normal strain at E in the direction of the line EC, in terms of c_1, c_2, and c_3.

c. Determine the shear strain at E for the lines EF and ED, in terms of c_1, c_2, and c_3.

d. Obtain numerical values for parts a, b, and c, for $c_1 = 0.002\ \mathrm{m}^{-1}$, $c_2 = 0.004\ \mathrm{m}^{-1}$, and $c_3 = -0.004\ \mathrm{m}^{-1}$.

2.72. The small-displacement strain components in a cam lobe are

$$\epsilon_{xx} = Az^3, \quad \epsilon_{yy} = Bx^2, \quad \epsilon_{zz} = Cx^2$$
$$\epsilon_{xy} = Dxy, \quad \epsilon_{xz} = Exz^2, \quad \epsilon_{yz} = Fxz$$

where A, B, C, D, E, and F are small constants. Determine whether or not these strain components can be compatible.

Section 2.9

The problems for Section 2.9 involve the determination of principal strains and maximum shear strain from measurements by strain rosettes.

2.73. Show that for Example 2.12, when $\theta = 45°$, the principal strains are given by

$$\epsilon_1 = \frac{1}{2}(\epsilon_a + \epsilon_c) + R$$
$$\epsilon_2 = \frac{1}{2}(\epsilon_a + \epsilon_c) - R$$
$$R = \frac{1}{2}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]^{1/2}$$

and the maximum strain ϵ_1 is located at angle ϕ, counterclockwise from the x axis, where

$$\tan 2\phi = \frac{2\epsilon_b - \epsilon_a - \epsilon_c}{\epsilon_a - \epsilon_c}$$

2.74. Show that for Example 2.12, when $\theta = 60°$, the principal strains are given by

$$\epsilon_1 = \frac{1}{3}(\epsilon_a + \epsilon_b + \epsilon_c) + R$$
$$\epsilon_2 = \frac{1}{3}(\epsilon_a + \epsilon_b + \epsilon_c) - R$$
$$R = \frac{1}{3}\left[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2\right]^{1/2}$$

and the maximum strain ϵ_1 is located at angle ϕ, counterclockwise from the x axis, where

$$\tan 2\phi = \frac{\sqrt{3}(\epsilon_b - \epsilon_c)}{2\epsilon_a - \epsilon_b - \epsilon_c}$$

2.75. For the rectangular strain rosette (Figure 2.20*b*), let arm *a* be directed along the positive x axis of axes (x, y).

a. Show that the maximum principal strain is located at angle θ, counterclockwise to the x axis, where

$$\tan 2\theta = \frac{2\epsilon_b - \epsilon_a - \epsilon_c}{\epsilon_a - \epsilon_c}$$

b. Show that the two principal surface strains ϵ_1 and ϵ_2 are given by

$$\epsilon_1 = \frac{\epsilon_a + \epsilon_c}{2} + R, \quad \epsilon_2 = \frac{\epsilon_a + \epsilon_c}{2} - R$$

where

$$R = \frac{1}{2}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]^{1/2}$$

c. Construct the corresponding Mohr's circle for the rectangular rosette.

2.76. For the delta strain rosette (Figure 2.20*a*), let arm *a* be directed along the positive x axis of axes (x, y).

a. Show that the maximum principal strain is located at angle θ, counterclockwise to the x axis, where

$$\tan 2\theta = \frac{\sqrt{3}(\epsilon_b - \epsilon_c)}{2\epsilon_a - \epsilon_b - \epsilon_c}$$

b. Show that the two principal surface strains ϵ_1 and ϵ_2 are given by

$$\epsilon_1 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} + R, \quad \epsilon_2 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} - R$$

where

$$R = \frac{1}{3}\left[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2\right]^{1/2}$$

c. Construct the corresponding Mohr's circle for the delta rosette.

2.77. Let the arm *a* of a delta rosette (Figure 2.20*a*) be directed along the positive x axis of axes (x, y). From measurements, $\epsilon_a = 2450\ \mu$, $\epsilon_b = 1360\ \mu$, and $\epsilon_c = -1310\ \mu$. Determine the two principal surface strains, the direction of the principal axes, and the associated maximum shear strain ϵ_{xy}.

2.78. Let the arm *a* of a rectangular rosette (Figure 2.20*b*) be directed along the positive x axis of axes (x, y). Using Mohr's circle of strain, show that $2\epsilon_{xy} = \gamma_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c$.

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