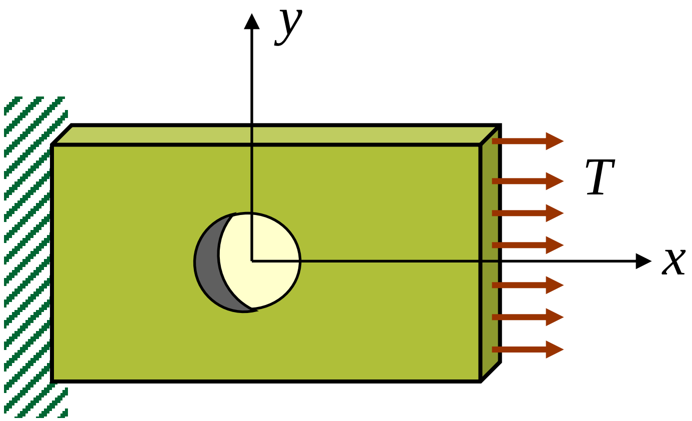
Two-Dimensional Problems Using CST Elements

(Initial notes are designed by Dr. Nazri Kamsah)

8-1 Introduction

A thin plate of thickness t, with a hole in the middle, is subjected to a uniform traction load, T as shown. This 3-D plate can be analyzed as a **two-dimensional** problem.

2-D problems generally fall into two categories: *plane stress* and *plane strain*.



A *plane stress* problem

a) Plane Stress

The thin plate can be analyzed as a *plane stress* problem, where the normal and shear stresses perpendicular to the x-y plane are *assumed* to be zero, i.e.

$$\sigma_z = 0; \ \ \tau_{xz} = 0; \ \ \tau_{yz} = 0$$

The *nonzero* stress components are

$$\sigma_x \neq 0; \ \ \sigma_y \neq 0; \ \ \tau_{xy} \neq 0$$

b) Plane Strain

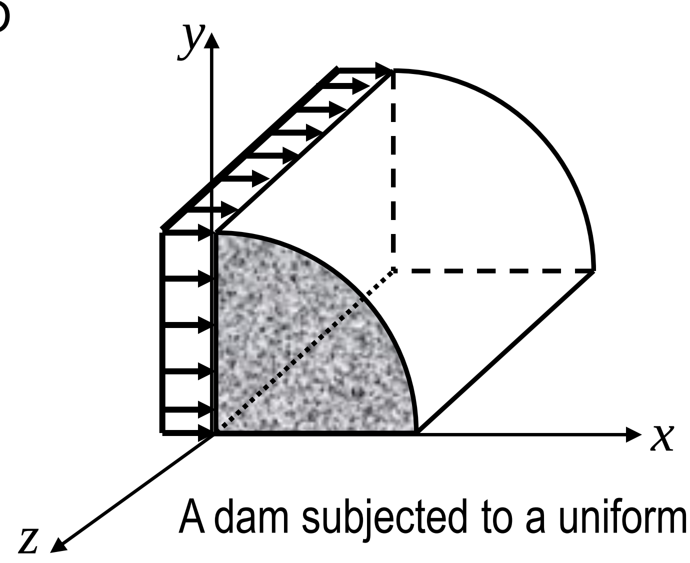
A dam subjected to uniform pressure and a pipe under a uniform internal pressure can be analyzed in two-dimension as *plain strain* problems.

The strain components perpendicular to the x-y plane are assumed to be zero, i.e.

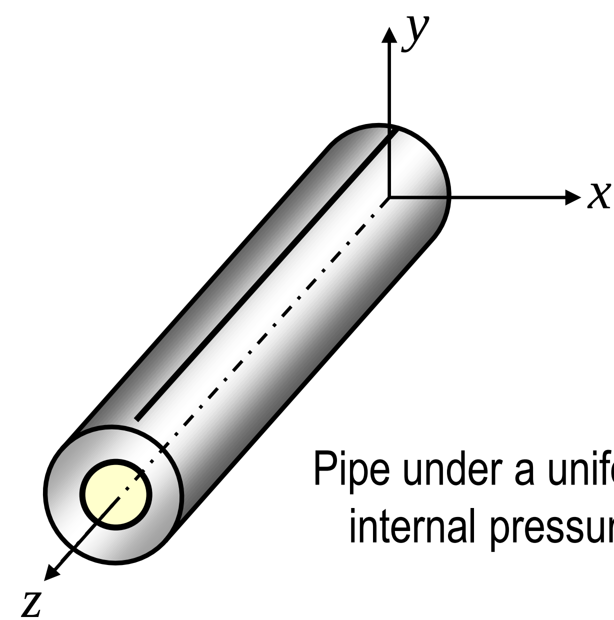
$$\varepsilon_z = 0; \ \gamma_{xz} = 0; \ \gamma_{yz} = 0$$

Thus, the *nonzero* strain components are ε_x , ε_y , and γ_{xy}.

$$\varepsilon_x \neq 0; \ \varepsilon_y \neq 0; \ \gamma_{xy} \neq 0$$



A dam subjected to a uniform pressure



Pipe under a uniform internal pressure

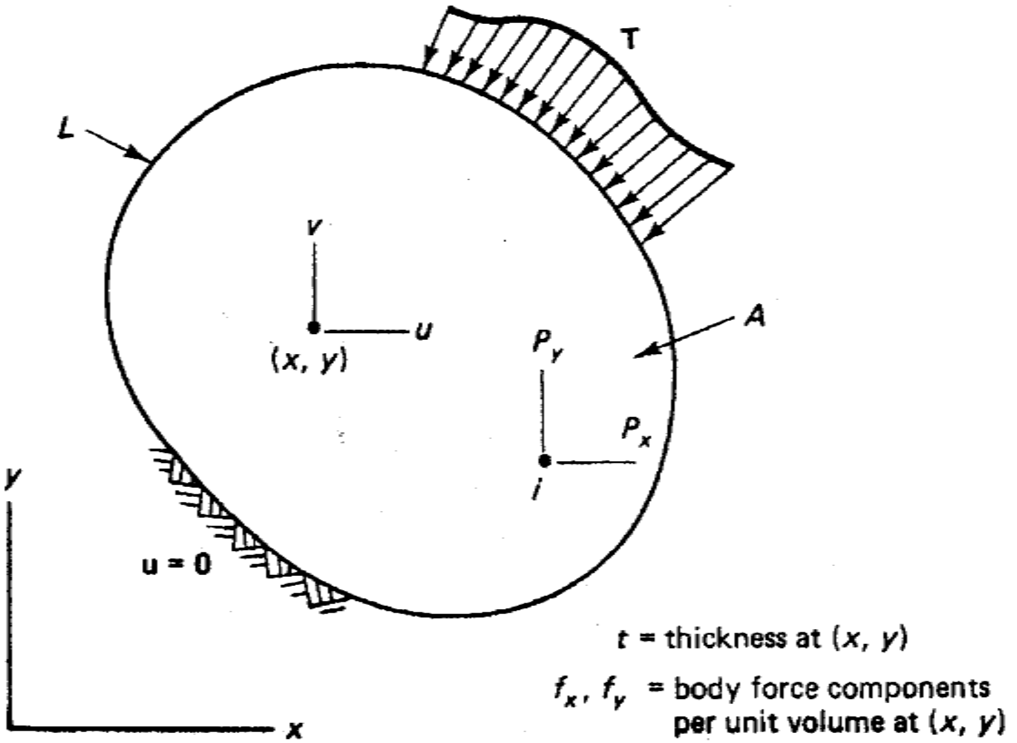
8-2 General Loading Condition

A two-dimensional body can be subjected to **three** types of forces:

a) Concentrated forces, P_x & P_y at a point, i;

b) Body forces, $f_{b,x}$ & $f_{b,y}$ acting at its *centroid*;

c) Traction force, T (i.e. force per unit length), acting along a *perimeter*.

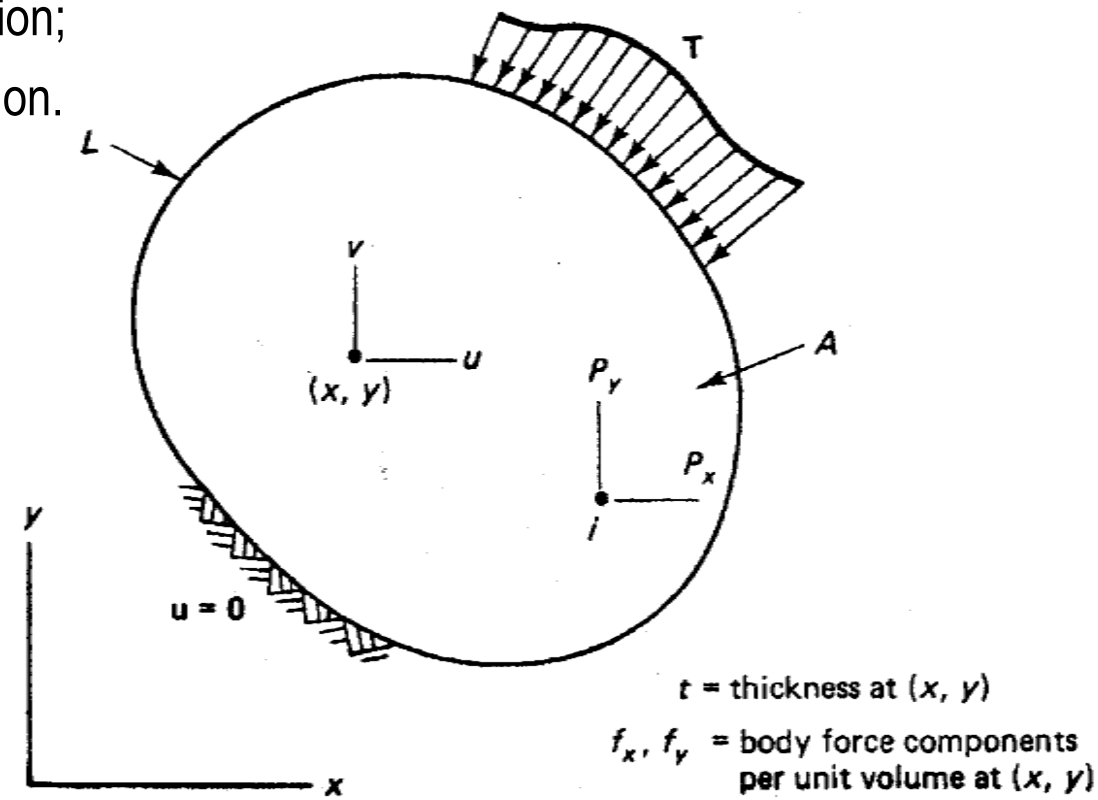


The 2-dimensional body experiences a deformation due to the applied loads.

At any point in the body, there are two components of displacement, i.e.

u = displacement in x-direction;

v = displacement in y-direction.



Stress-Strain Relation

Recall, at any point in the body, there are three components of strains, i.e.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

The corresponding stress components at that point are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

The stresses and strains are related through,

$$\{\sigma\} = [D]\{\varepsilon\}$$

where [D] is called the *material matrix*, given by

$$[D] = \frac{E}{1-v^2} \cdot \begin{Bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{Bmatrix}$$

for *plane stress* problems and

$$[D] = \frac{E}{(1+v)(1-2v)} \cdot \begin{Bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2}-v \end{Bmatrix}$$

for *plane strain* problems.

8-3 Finite Element Modeling

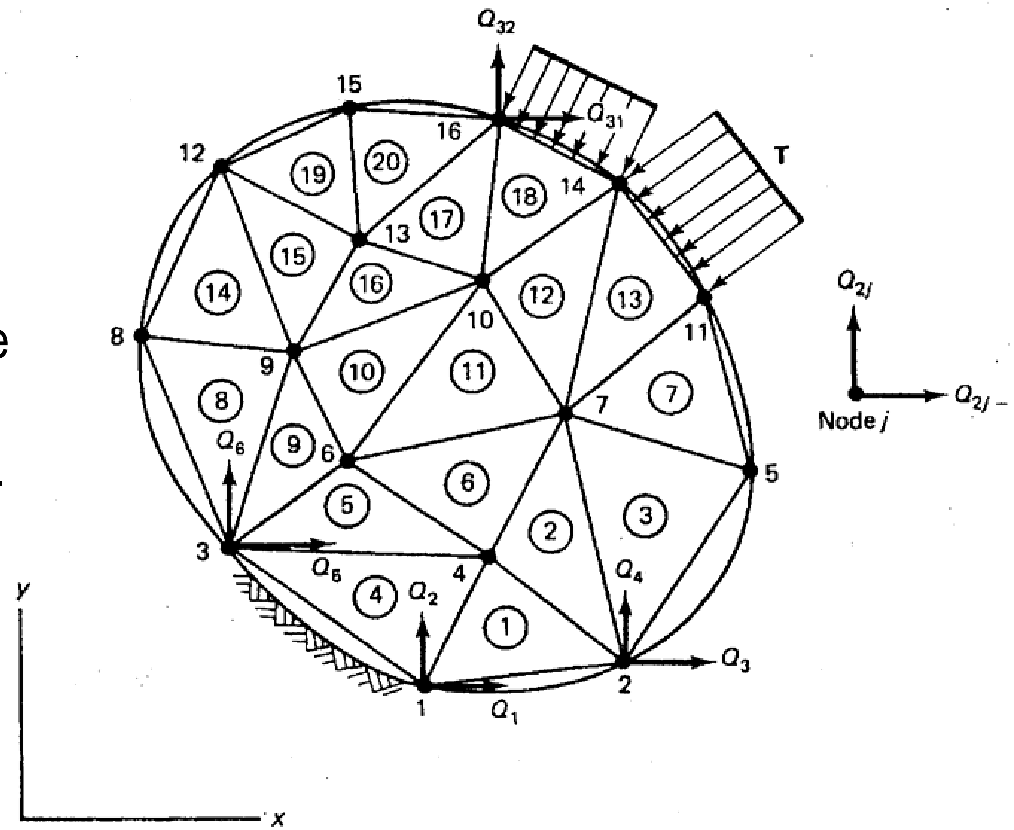
The two-dimensional body is transformed into finite element model by subdividing it using triangular elements.

Note:

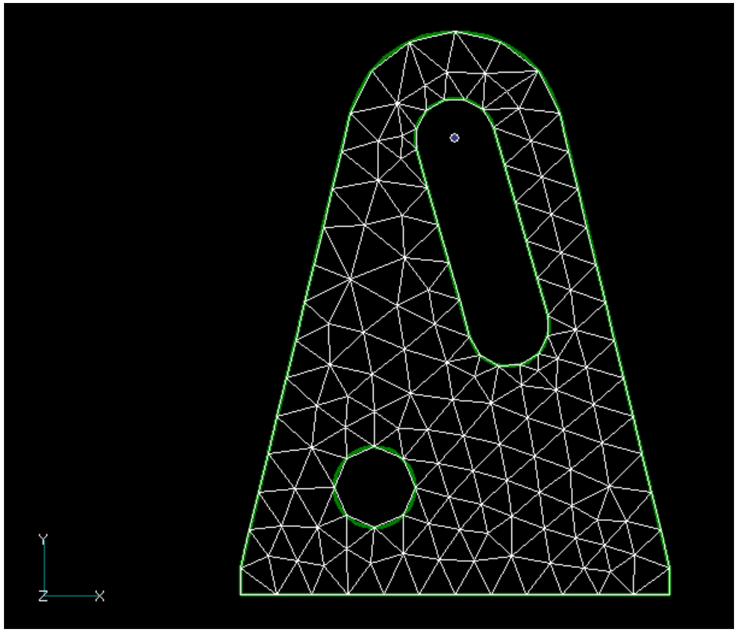
1. *Unfilled* region exists for curved boundaries, affecting accuracy of the solution. The accuracy can be improved by using smaller elements.

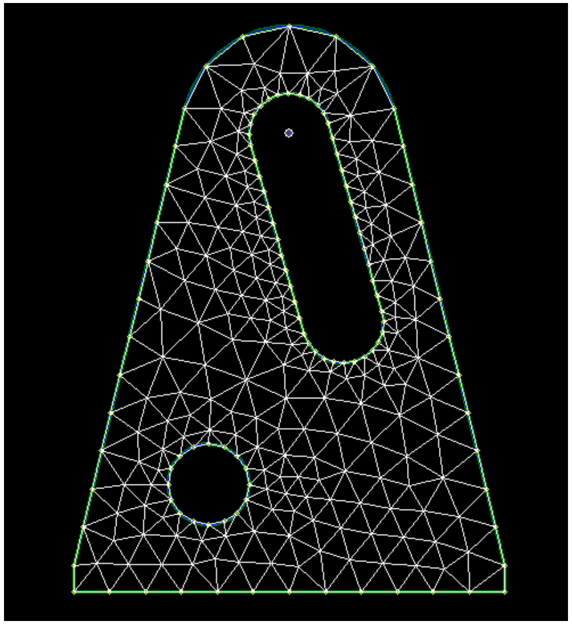
2. There are **two** displacement components at a node. Thus, at a node j, the displacements are:

$$Q_{2j-1} \quad \text{in } x\text{-direction}$$
$$Q_{2j} \quad \text{in } y\text{-direction}$$



SME 3033 FINITE ELEMENT METHOD





Finite element model of a *bracket*.

8-4 Constant-Strain Triangle (CST)

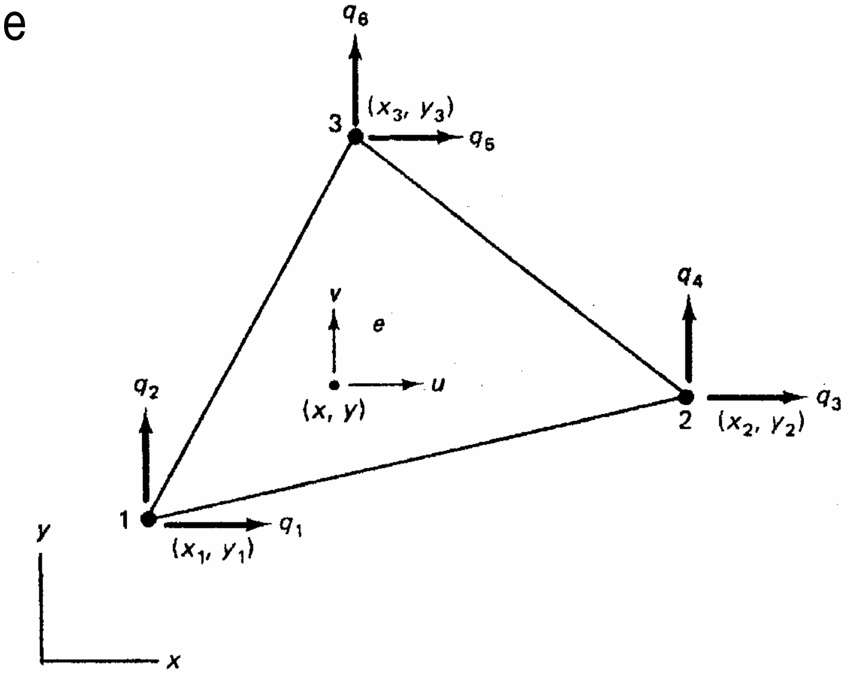
Consider a single triangular element as shown.

The **local** node numbers are assigned in the *counterclockwise* order.

The local nodal displacement vector for a single element is given by,

$$\{q\} = [q_1, \quad q_2, \quad \ldots, \quad q_6]^T$$

Within the element, displacement at any point of coordinate (x, y), is represented by **two** components, i.e. u in the x-direction and v in the y-direction.



Note: We need to express u and v in terms of the nodal displacement components, i.e. $q_1, q_2, \ldots, q_6$.

8-5 Displacement Functions

Displacement components u and v at any point (x, y) within the element are related to the nodal displacement components through

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 \quad \text{(i)}$$

where N_1, N_2 and N_3 are the *linear* shape functions, given by

$$N_1 = \xi; \qquad N_2 = \eta; \qquad N_3 = 1 - \xi - \eta \quad \text{(ii)}$$

in which ξ and η are the *natural coordinates* for the triangular element.

Substituting Eq.(ii) into Eq.(i) and simplifying, we obtain alternative expressions for the *displacement functions*, i.e.

$$u = (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5$$
$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6 \quad \text{(iii)}$$

Eq.(i) can be written in a matrix form as,

$$\{u\} = [N]\{q\}$$

where

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$\{q\} = [q_1, \quad q_2, \quad ..., \quad q_6]^T$$

For the triangular element, the coordinates (x, y) of any point within the element can be expressed in terms of the nodal coordinates, using the **same** shape functions N_1, N_2 and N_3. We have,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

This is called an *isoparametric* representation.

Substituting for N_i using eq. (ii), we get

$$x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3$$
$$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$$

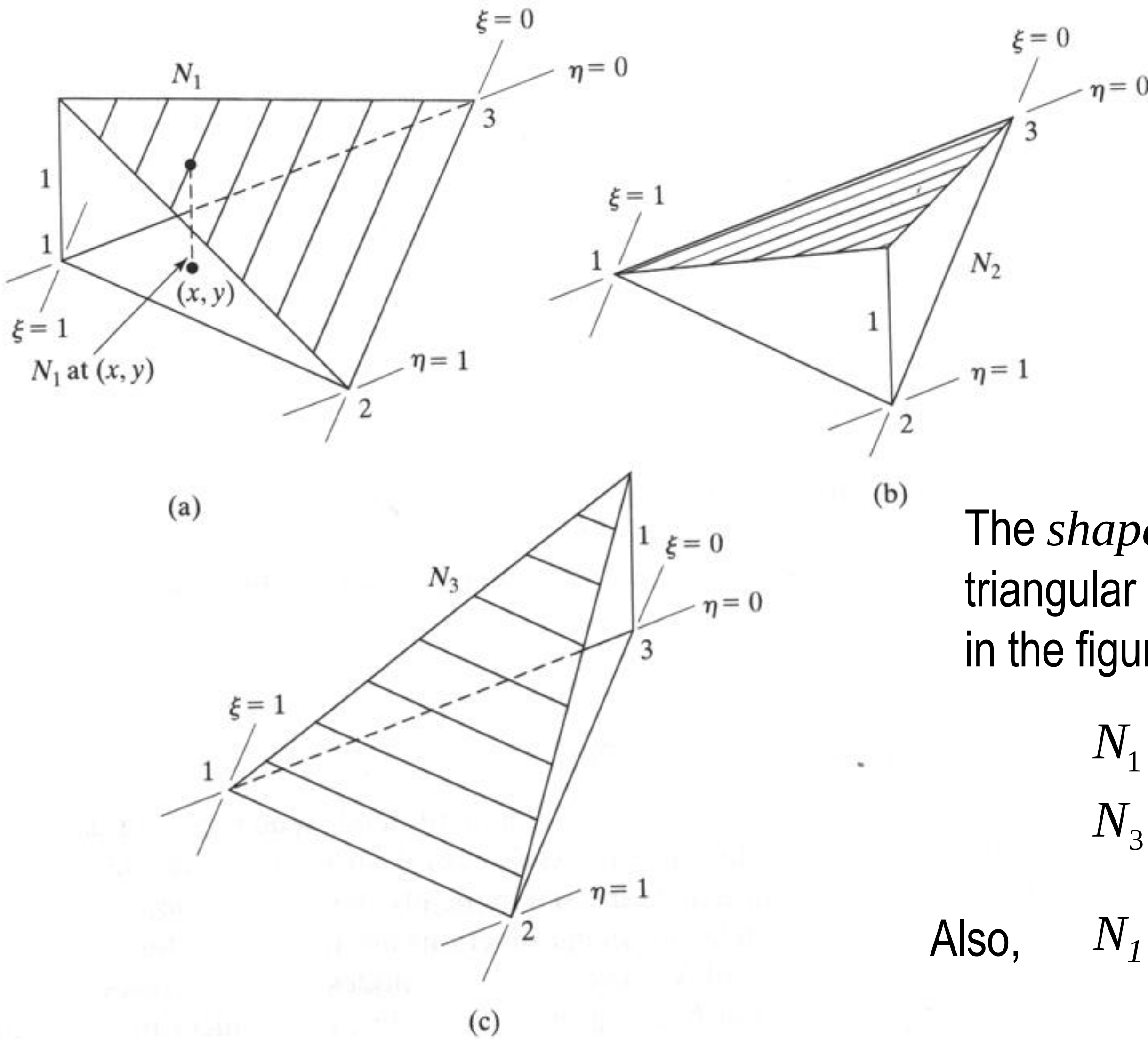
Using the notation, $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$, the above equations can then be written as

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$

Note: The above equations relate the x- and y-coordinates to the ξ- and η-coordinates (the natural coordinates). We observe that,

$$x_{13} = x_1 - x_3$$
$$y_{23} = y_2 - y_3$$

8-6 The Shape Functions



The *shape functions* for the triangular element are illustrated in the figures. Recall, we have

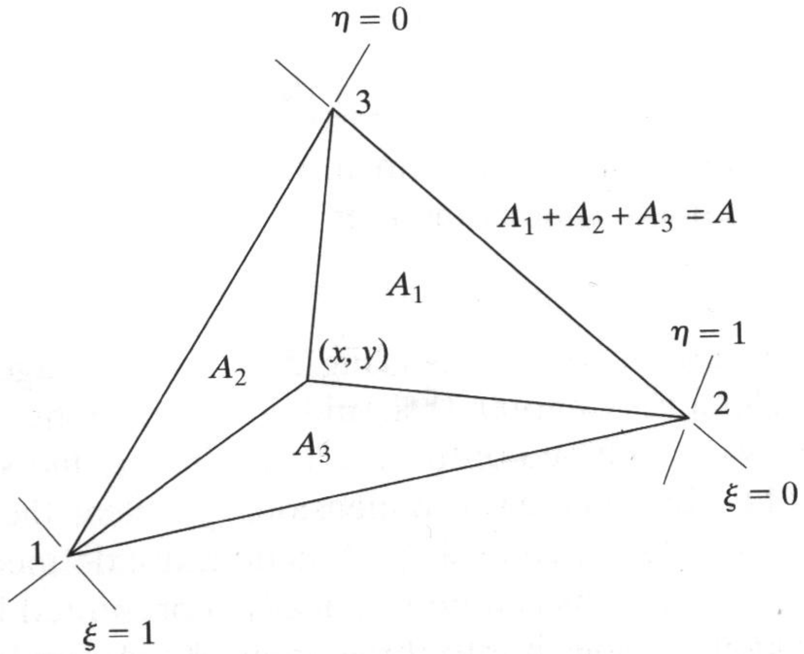
$$N_1 = \xi; \quad N_2 = \eta;$$

$$N_3 = 1 - \xi - \eta$$

Also, $N_1 + N_2 + N_3 = 1$

Area Coordinate Representation

The shape functions can be physically represented by *area coordinates*,



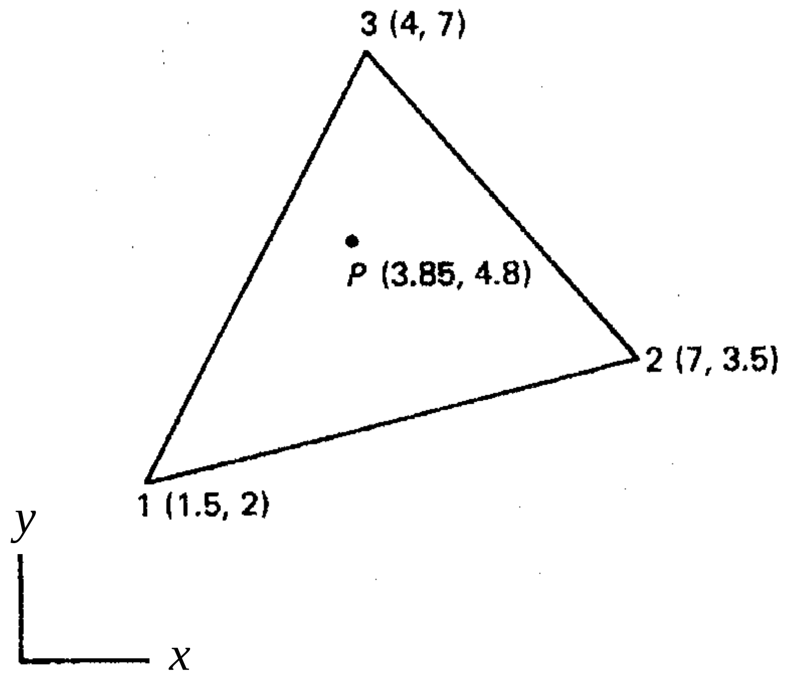
$$N_1 = \frac{A_1}{A}; \quad N_2 = \frac{A_2}{A}; \quad N_3 = \frac{A_3}{A}$$

where A is the area of the triangular element, i.e.

$$A = A_1 + A_2 + A_3$$

Exercise 8-1

Consider a triangular element shown below. Evaluate the *shape functions* N_1, N_2, and N_3 at an interior point P.



The triangular element for solution.

Solution

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 = 1.5N_1 + 7N_2 + 4N_3 = 3.85$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 = 2N_1 + 3.5N_2 + 7N_3 = 4.8$$

Using the notation, $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$, the above become

$$x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 = -2.5\xi + 3\eta + 4 = 3.85$$

$$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3 = -5\xi - 3.5\eta + 7 = 4.8$$

Simplifying the equations yields,

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

Solving the equations simultaneously, we obtain ξ = 0.3 and h = 0.2. Thus, the shape functions for the triangular element are,

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

8-7 Area of the Triangular Element

The area, A of any arbitrarily oriented straight-sided triangular elements can be determined using a formula

$$A = \frac{1}{2}\left|\det[J]\right|$$

where [J] is a square matrix called the *Jacobian*, given by

$$[J] = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

The *determinant* of the Jacobian [J] is

$$\det[J] = x_{13}y_{23} - x_{23}y_{13}$$

Note: “I I” represents the *“magnitude of”*. Most computer software use counter-clockwise order of *local* node numbering, and use *det*[J] for computing the area of the triangular element.

8-8 Strain-Displacement Matrix

The strains within the triangular element are related to the components of the nodal displacement by a relation

$$\{\varepsilon\} = [B]\{q\}$$

where $[B]$ is a (3 x 6) rectangular matrix called the *strain-displacement* matrix, given by

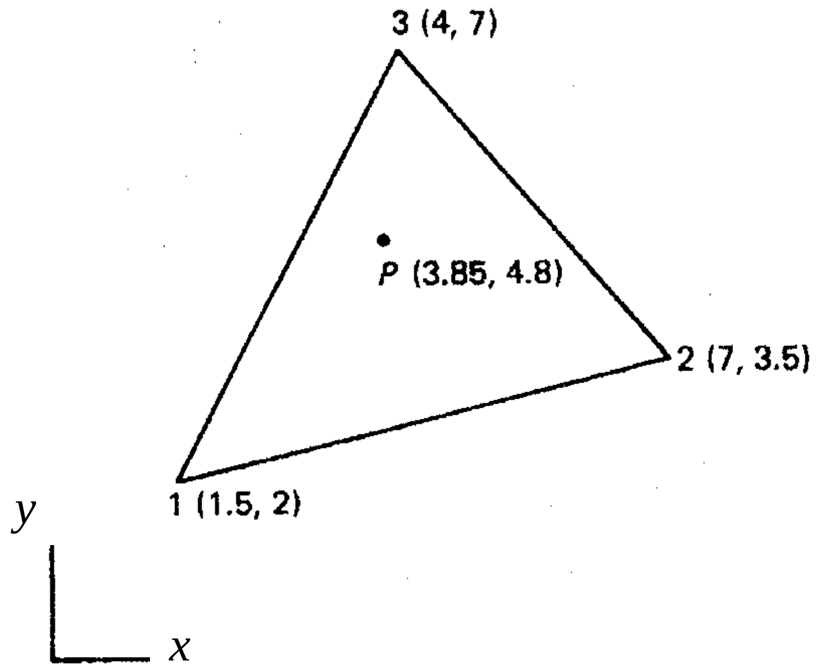
$$[B] = \frac{1}{\det[J]}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Note: For the given magnitude of $\{q\}$, the strains within the element depend only on $[B]$ matrix, which in turns depends on the nodal coordinates, which are constant.

Hence the strains are the same everywhere within the element, thus the name *constant-strain triangle* (CST).

Exercise 8-2

Consider a triangular element in Exercise 8-1. a) Write the *Jacobian* matrix; b) Find the *determinant* of the Jacobian matrix; c) Compute the area of the triangular element; d) Establish the *strain-displacement* matrix for the element.

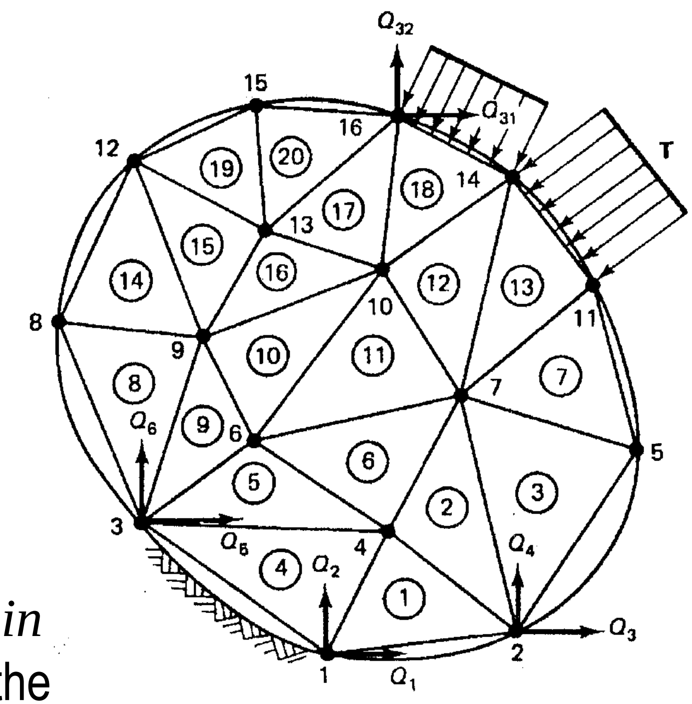


The triangular element for solution.

7-9 Potential Energy Approach

The *total potential energy* of a 2-D body, discretized using triangular elements, is given by

$$\Pi = \sum_e \frac{1}{2}\int_e \{\varepsilon\}^T [D]\{\varepsilon\} tdA - \sum_e \int_e \{u\}^T \{f\} tdA - \int_L \{u\}^T \{T\} tdL - \sum_i \{u\}_i^T \{P\}_i$$



The **first** term represents the sum of *internal strain energy* of all elements, U_e. For a **single** element, the internal strain energy is

$$U_e = \frac{1}{2}\int_e \{\varepsilon\}^T [D]\{\varepsilon\} t \cdot dA$$

8-10 Element Stiffness Matrix

We will derive the *stiffness matrix* of a triangular element using the *potential energy* approach. Recall, the internal strain energy of an element, U_e is given by

$$U_e = \frac{1}{2}\int_e \{\varepsilon\}^T [D]\{\varepsilon\} t \cdot dA \qquad \text{(i)}$$

The strains $\{\varepsilon\}$ are related to nodal displacements $\{q\}$ by,

$$\{\varepsilon\} = [B]\{q\} \qquad \text{(ii)}$$

Substituting Eq.(ii) into Eq.(i), we get

$$U_e = \frac{1}{2}\int_e \{q\}^T [B]^T [D][B]\{q\} t \cdot dA \qquad \text{(iii)}$$

Taking all constants in Eq.(iii) out of the integral we obtain,

$$U_e = \frac{1}{2}\{q\}^T [B]^T [D][B] t \left(\int_e dA\right)\{q\} \qquad \text{(iv)}$$

Note that, $\int_e dA = A_e$, i.e. the area of the triangular element.

Substituting this into eq.(iv) and further simplifying, we get,

$$U_e = \frac{1}{2}\{q\}^T t_e A_e [B]^T [D][B]\{q\} \qquad \text{(v)}$$

The internal strain energy of the element can now be written as

$$U_e = \frac{1}{2}\{q\}^T [k]^e \{q\} \qquad \text{(vi)}$$

From eq.(vi) we identify the stiffness matrix $[k]^e$ of the triangular (CST) element as,

$$[k]^e = t_e A_e [B]^T [D] \cdot [B]$$

Note: Since there are 6 DOFs for a given element, $[k]^e$ will be a (6 x 6) rectangular symmetric matrix.

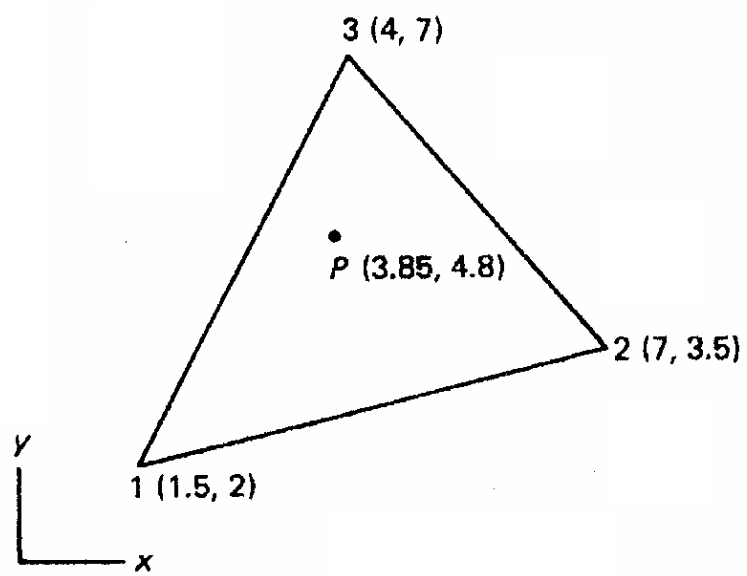
Exercise 8-3

Determine the *stiffness matrix* for the straight-sided triangular element of thickness t = 1 mm, as shown. Use E = 70 GPa, ν = 0.3 and assume a *plane stress* condition.

Solution

Element stiffness matrix is given by

$$[k]^e = t_e A_e [B]^T [D] \cdot [B] \qquad \text{(i)}$$



(Dimension is in mm)

where,

$$t_e = 1 \text{ mm}$$

$$A_e = \frac{1}{2}\left|\det[J]\right| = \frac{1}{2}\left|x_{13}y_{23} - x_{23}y_{13}\right|$$

$$= \frac{1}{2}\left|23.75\right|$$

$$A_e = 11.875 \ \text{ mm}^2$$

The *strain-displacement matrix*, $[B]$ is given by

$$[B] = \frac{1}{\det[J]}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{23.75}\begin{bmatrix} 3.5-7 & 0 & 7-2 & 0 & 2-3.5 & 0 \\ 0 & 4-7 & 0 & 1.5-4 & 0 & 7-1.5 \\ 4-7 & 3.5-7 & 1.5-4 & 7-2 & 7-1.5 & 2-3.5 \end{bmatrix}$$

$$[B] = \frac{1}{23.75}\begin{bmatrix} -3.5 & 0 & 5 & 0 & -1.5 & 0 \\ 0 & -3 & 0 & -2.5 & 0 & 5.5 \\ -3 & -3.5 & -2.5 & 5 & -5.5 & -1.5 \end{bmatrix}$$

The transpose of [B] matrix is,

$$[B]^T = \frac{1}{23.75}\begin{bmatrix} -3.5 & 0 & -3 \\ 0 & -3 & -3.5 \\ 5 & 0 & -2.5 \\ 0 & -2.5 & 5 \\ -1.5 & 0 & 5.5 \\ 0 & 5.5 & -1.5 \end{bmatrix}$$

For a *plane stress* condition, the *material's matrix* [D] is given by

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = \frac{70\times10^3}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-0.3) \end{bmatrix}$$

Substituting all the terms into eq.(i) we have,

$$[k]^e = 1\times 11.875\times\frac{1}{23.75}\begin{bmatrix} -3.5 & 0 & -3 \\ 0 & -3 & -3.5 \\ 5 & 0 & -2.5 \\ 0 & -2.5 & 5 \\ -1.5 & 0 & 5.5 \\ 0 & 5.5 & -1.5 \end{bmatrix}\times\frac{70\times 10^3}{\left(1-0.3^2\right)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\times\frac{1}{23.75}\begin{bmatrix} -3.5 & 0 & 5 & 0 & -1.5 & 0 \\ 0 & -3 & 0 & -2.5 & 0 & 5.5 \\ -3 & -3.5 & -2.5 & 5 & -5.5 & -1.5 \end{bmatrix}$$

Multiplying and simplifying, we obtain

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad q_1 & \quad q_2 & \quad q_3 & \quad q_4 & \quad q_5 & \quad q_6 \end{array} \\
[k]^e = 10^4 \cdot & \begin{bmatrix} 2.494 & 1.105 & -2.409 & -0.425 & -0.085 & -0.68 \\ & 2.152 & -0.233 & 0.223 & -0.873 & -2.374 \\ & & 4.403 & -1.316 & -1.994 & 1.549 \\ & & & 2.429 & 1.741 & -2.652 \\ & & & & 2.079 & -0.868 \\ & symmetry & & & & 5.026 \end{bmatrix}
\end{array}
$$

Note: Connectivity with the **local** DOFs is shown.

8-11 Element Force Vector

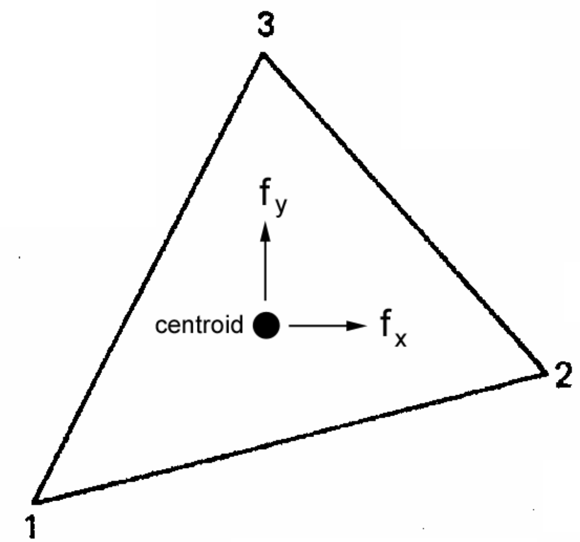
We will derive the force vector for a **single** element, which is contributed by a) *body force*, f and b) *traction force*, T.

We need to convert both f and T into the *equivalent nodal forces*.

Note: The *concentrated forces* can be included directly into the *global* load vector, appropriate DOF direction.

a) Body Force

Suppose body force components, f_x and f_y, act at the *centroid* of a triangular element.



The *potential energy* due to these forces is given by,

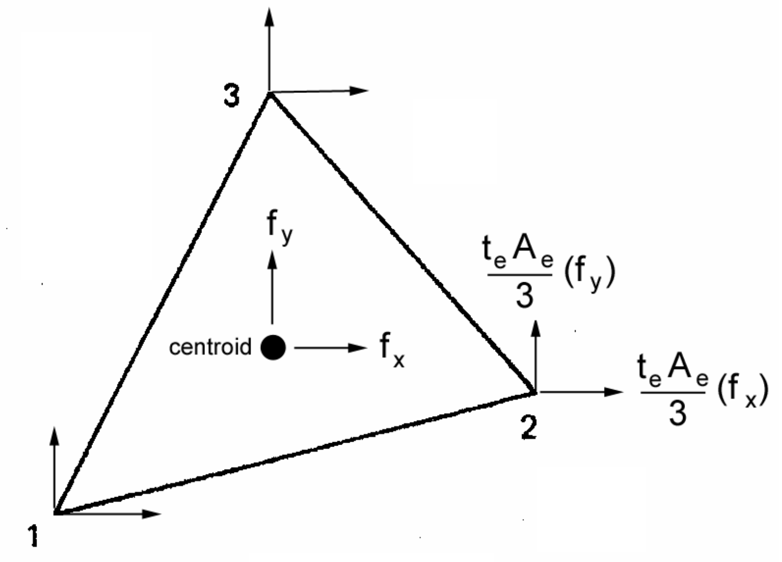
$$\int_e \{u\}^T \{f\} t \cdot dA = t_e \int_e \left(u f_x + v f_y \right) dA \qquad \text{(i)}$$

Recall,
$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

Also,
$$\int_e N_i \, dA = \frac{1}{3} A_e$$

Substituting the above into eq.(i), we get

$$\int_e \{u\}^T \{f\} t \cdot dA = \{q\}^T \{f\}^e$$



where $\{f\}^e$ is the element body force vector, given by

$$\{f\}^e = \frac{t_e A_e}{3} \begin{bmatrix} f_x, & f_y, & f_x, & f_y, & f_x, & f_y \end{bmatrix}^T$$

Note: Physical representation of force vector $\{f\}^e$ is shown.

8-12 Concentrated Force

The concentrated force term can be easily considered by having a node at the point of application of the force.

If concentrated load components P_x and P_y are applied at a point i, then

$$\{u\}_i^T \{P\}_i = Q_{2i-1}P_x + Q_{2i}P_y$$

Thus, P_x and P_y, i.e. the x and y components of $\{P\}_i$ get added to the $(2_i - 1)$th component and (2_i)th components of the global force vector, $\{F\}$.

Note: The contribution of the body, traction and concentrated forces to the global force vector, $\{F\}$ is represented by,

$$\{F\} \leftarrow \sum_e \left(\{f\}^e + \{T\}^e \right) + \{P\}$$

8-13 Strains and Stress Calculations

a) Strains

The strains in a triangular element are,

$$\{\varepsilon\}^e = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \left(\frac{du}{dy} + \frac{dv}{dx}\right) \end{Bmatrix}$$

$$\{\varepsilon\}^e = [B]\{q\}$$

Note: We observed that $\{\varepsilon\}^e$ depends on the $[B]$ matrix, which in turn depends only on nodal coordinates (x_i, y_i), which are constant. Therefore, for a given nodal displacements $\{q\}$, the strains $\{\varepsilon\}^e$ within the element are constant.

Hence the triangular element is called a *constant-strain triangle*.

b) Stresses

The stresses in a triangular element can be determined using the stress-strain relation,

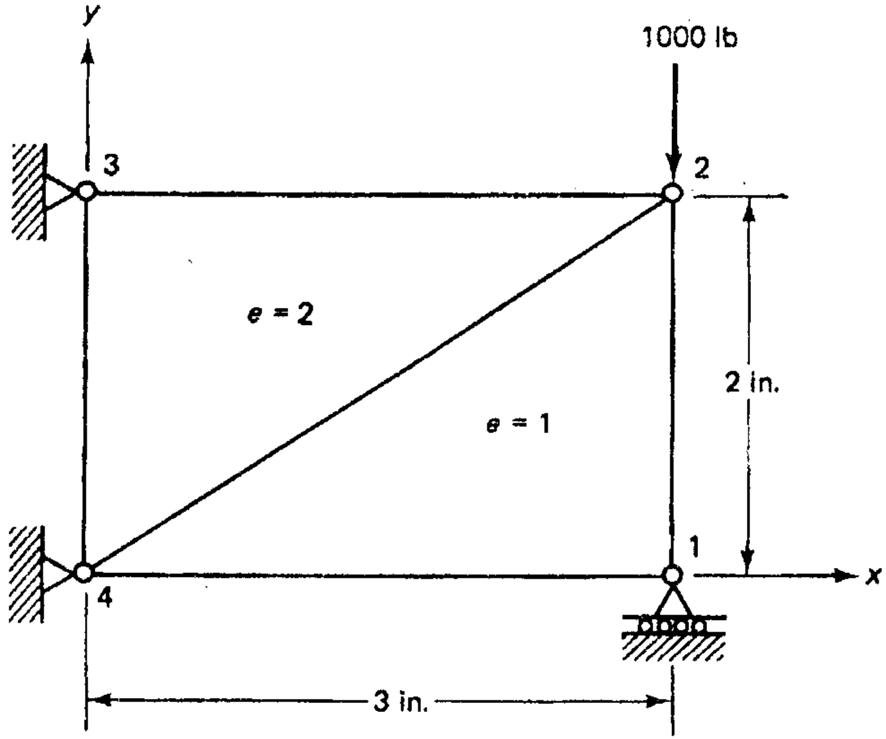
$$\{\sigma\}^e = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\{\varepsilon\}^e = [D][B]\{q\}^e$$

Note:

1. Since the strains $\{\varepsilon\}^e$ are constant within the element, the stresses are also the same at any point in the element.
2. Stresses for plane stress problem differ from those for plane strain problem by the material's matrix $[D]$.
3. For interpolation purposes, the calculated stresses may be used as the values at the *centroid* of the element.
4. Principal stresses and their directions are calculated using the *Mohr circle*.

Example 8-1

Consider a thin plate having thickness t = 0.5 in. being modeled using two CST elements, as shown. Assuming plane stress condition, (a) determine the displacements of nodes 1 and 2, and (b) estimate the stresses in both elements.



Thickness t = 0.5 in., E = 30 X 10^6 psi, ν = 0.25

Solution

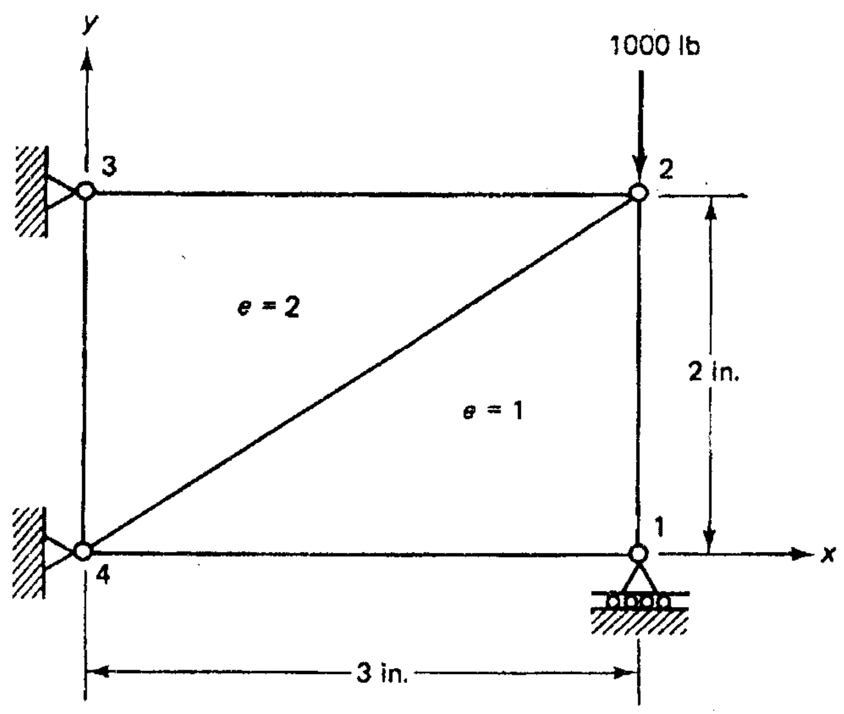
Element connectivity

	Local Nodes		
Element No	**1**	**2**	**3**
1	1	2	4
2	3	4	2

For **plane stress** problem, the *materials matrix* is given by

$$[D]=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}$$

$$[D]=32\times10^6\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}$$



Thickness t = 0.5 in., E = 30 X 10^6 psi, ν = 0.25

Element 1

Area of element, $A_1 = \frac{1}{2}\left|\det[J]\right| = \frac{1}{2}(6) = 3\ \text{in}^2$

The strain-displacement matrix,

$$[B] = \frac{1}{\det[J]}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 0 & -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

Multiplying matrices $[D][B]$ we get,

$$[D][B]^{(1)} = 10^7 \cdot \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}$$

The *stiffness matrix* is given by,

$$[k]^{(1)} = t_1 A_1 [B]_1^T [D][B]_1$$

Substitute all parameters and multiplying the matrices, yields

$$[k]^{(1)} = 10^7 \begin{array}{c} \begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_7 & Q_8 \end{array} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$

Note: Connectivity with global DOFs are shown.

Element 2

Area of element, $$A_2 = \frac{1}{2}\left|\det[J]\right| = \frac{1}{2}(6) = 3\ \text{in}^2$$

The *strain-displacement* matrix is

$$[B] = \frac{1}{\det[J]}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

Multiplying matrices [D][B] we get,

$$[D][B]^{(2)} = 10^7 \cdot \begin{bmatrix} -1.067 & 0.4 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.6 & 0 & -1.6 & 0.267 & 0 \\ 0.6 & -0.4 & -0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

The *stiffness matrix* is given by,

$$[k]^{(2)} = t_2 A_2 [B]_2^T [D][B]_2$$

Substituting all parameters and multiplying the matrices yield

$$[k]^{(2)} = 10^7 \begin{array}{c} \begin{array}{cccccc} Q_5 & Q_6 & Q_7 & Q_8 & Q_3 & Q_4 \end{array} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$

Note: Connectivity with global DOFs are shown.

Write the global system of linear equations, $[K]\{Q\} = \{F\}$, and then apply the *boundary conditions:* Q_2, Q_5, Q_6, Q_7, and $Q_8 = 0$.

The reduced *system of linear equations* are,

$$10^7 \cdot \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}$$

Solving the reduced SLEs simultaneously yields,

$$\begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 1.913 \\ 0.875 \\ -7.436 \end{Bmatrix} \cdot 10^{-5} \quad \text{in.}$$

Stresses in element 1

For element 1, the element nodal displacement vector is

$$\{q\}^{(1)} = 10^{-5}\left[1.913, \quad 0, \quad 0.875, \quad -7.436, \quad 0 \quad 0\right]^T$$

The element stresses, $\{\sigma\}^{(1)}$ are calculated from $[D][B]^{(1)}\{q\}$ as

$$\{\sigma\}^{(1)} = \left[-93.3, \quad -1138.7, \quad -62.3\right]^T \quad \text{psi}$$

Stresses in element 2

For element 2, the element nodal displacement vector is

$$\{q\}^{(2)} = 10^{-5}\left[0, \quad 0, \quad 0 \quad 0, \quad 0.875, \quad -7.436\right]^T$$

The element stresses, $\{\sigma\}^{(2)}$ are calculated from $[D][B]^{(2)}\{q\}$ as

$$\{\sigma\}^{(2)} = \left[93.4, \quad 23.4, \quad -297.4\right]^T \quad \text{psi}$$