CHAPTER 3

LINEAR STRESS–STRAIN–TEMPERATURE RELATIONS

In Chapter 2, we presented separate theories for stress and strain. These theories are based on the concept of a general continuum. Consequently, they are applicable to all continua. In particular, the theory of stress is based solely on the concept of force and the associated concept of force per unit area. Similarly, the theory of strain is based on geometrical concepts of infinitesimal line extensions and rotations between two infinitesimal lines. However, to relate the stress at a point in a material to the corresponding strain at that point, knowledge of material properties is required. These properties enter into the stress–strain–temperature relations as material coefficients. The theoretical basis for these relations is the first law of thermodynamics, but the material properties themselves must be determined experimentally.

In this chapter, we employ the first law of thermodynamics to derive linear stress–strain–temperature relations. In addition, certain concepts, such as complementary strain energy, that have application to nonlinear problems are introduced. These relations and concepts are utilized in many applications presented in subsequent chapters of this book.

3.1 FIRST LAW OF THERMODYNAMICS, INTERNAL-ENERGY DENSITY, AND COMPLEMENTARY INTERNAL-ENERGY DENSITY

The derivation of load–stress and load–deflection relations requires stress–strain relations that relate the components of the strain tensor to components of the stress tensor. The form of the stress–strain relations depends on material behavior. In this book, we treat mainly materials that are isotropic; that is, at any point they have the same properties in all directions. Stress–strain relations for linearly elastic isotropic materials are well known and are presented in Section 3.4.

Stress–strain relations may be derived with the first law of thermodynamics, a precise statement of the law of conservation of energy. The total amount of internal energy in a system is generally indeterminate. Hence, only changes of internal energy are measurable. If electromagnetic effects are disregarded, this law is described as follows:

The work performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase in internal energy plus the increase in kinetic energy.

Symbolically, the first law of thermodynamics is expressed by the equation

$$\delta W + \delta H = \delta U + \delta K \tag{3.1}$$

where δW is the work performed on the system by *external forces*, δH is the heat that flows into the system, δU is the increase in internal energy, and δK is the increase in kinetic energy.

To apply the first law of thermodynamics, we consider a loaded member in equilibrium. The deflections are assumed to be known. They are specified by known displacement components (u, v, w) for each point in the deflected member. We allow each point to undergo infinitesimal increments (variations) in the displacement components (u, v, w) indicated by $(\delta u, \delta v, \delta w)$. The stress components at every point of the member are considered to be unchanged under variations of the displacements. These displacement variations are arbitrary, except that two or more particles cannot occupy the same point in space, nor can a single particle occupy more than one position (the member does not tear). In addition, displacements of certain points in the member may be specified (e.g., at a fixed support); such specified displacements are referred to as forced boundary conditions (Langhaar, 1989). By Eq. 2.81, the variations of the strain components resulting from variations $(\delta u, \delta v, \delta w)$ are

$$\begin{aligned}
\delta\epsilon_{xx} &= \frac{\partial \delta u}{\partial x}, & \delta\epsilon_{xy} &= \frac{1}{2}\left[\frac{\partial(\delta v)}{\partial x} + \frac{\partial(\delta u)}{\partial y}\right] \\
\delta\epsilon_{yy} &= \frac{\partial \delta v}{\partial y}, & \delta\epsilon_{yz} &= \frac{1}{2}\left[\frac{\partial(\delta w)}{\partial y} + \frac{\partial(\delta v)}{\partial z}\right] \\
\delta\epsilon_{zz} &= \frac{\partial \delta w}{\partial z}, & \delta\epsilon_{zx} &= \frac{1}{2}\left[\frac{\partial(\delta w)}{\partial x} + \frac{\partial(\delta u)}{\partial z}\right]
\end{aligned} \tag{3.2}$$

To introduce force quantities, consider an arbitrary volume V of the deformed member enclosed by a closed surface S. We assume that the member is in static equilibrium following the displacement variations $(\delta u, \delta v, \delta w)$. Therefore, the part of the member considered in volume V is in equilibrium under the action of surface forces (represented by stress distributions on surface S) and body forces (represented by distributions of body forces per unit volume B_x, B_y, and B_z in volume V).

For adiabatic conditions (no net heat flow into V, $\delta H = 0$) and static equilibrium ($\delta K = 0$), the first law of thermodynamics states that, during the displacement variations $(\delta u, \delta v, \delta w)$, the variation in work of the external forces δW is equal to the variation of internal energy δU for each volume element. Hence, for V, we have

$$\delta W = \delta U \tag{3.1a}$$

It is convenient to divide δW into two parts: the work of the surface forces δW_S and the work of the body forces δW_B. At point P of surface S, consider an increment of area dS. The stress vector $\boldsymbol{\sigma}_P$ acting on dS has components σ_{Px}, σ_{Py}, and σ_{Pz} defined by Eqs. 2.10. The surface force is equal to the product of these stress components and dS. The work δW_S is equal to the sum of the work of these forces over the surface S. Thus,

$$\begin{aligned}
\delta W_s &= \int_S \sigma_{Px}\delta u dS + \int_S \sigma_{Py}\delta v dS + \int_S \sigma_{Pz}\delta w dS \\
&= \int_S [(\sigma_{xx}l + \sigma_{yx}m + \sigma_{zx}n)\delta u + (\sigma_{xy}l + \sigma_{yy}m + \sigma_{zy}n)\delta v \\
&\quad + (\sigma_{xz}l + \sigma_{yz}m + \sigma_{zz}n)\delta w]dS
\end{aligned} \tag{3.3}$$

For a volume element dV in volume V, the body forces are given by products of dV and the body force components per unit volume (B_x, B_y, B_z). The work δW_B of the body forces that act throughout V is

$$\delta W_B = \int_V (B_x \delta u + B_y \delta v + B_z \delta w)\, dV \tag{3.4}$$

The variation of work δW of the external forces that act on volume V with surface S is equal to the sum of δW_S and δW_B. The surface integral in Eq. 3.3 may be converted into a volume integral by use of the divergence theorem (Boresi and Chong, 2000). Thus,

$$\begin{aligned}\delta W = \delta W_S + \delta W_B = \int_V \Big[& \frac{\partial}{\partial x}(\sigma_{xx}\delta u + \sigma_{xy}\delta v + \sigma_{xz}\delta w) \\ & + \frac{\partial}{\partial y}(\sigma_{yx}\delta u + \sigma_{yy}\delta v + \sigma_{yz}\delta w) \\ & + \frac{\partial}{\partial z}(\sigma_{zx}\delta u + \sigma_{zy}\delta v + \sigma_{zz}\delta w) \\ & + B_x \delta u + B_y \delta v + B_z \delta w \Big] dV\end{aligned} \tag{3.5}$$

With Eqs. 3.2 and 2.45, Eq. 3.5 reduces to

$$\begin{aligned}\delta W = \int_V & (\sigma_{xx}\delta\epsilon_{xx} + \sigma_{yy}\delta\epsilon_{yy} + \sigma_{zz}\delta\epsilon_{zz} + 2\sigma_{xy}\delta\epsilon_{xy} \\ & + 2\sigma_{yz}\delta\epsilon_{yz} + 2\sigma_{zx}\delta\epsilon_{zx})\, dV\end{aligned} \tag{3.6}$$

The internal energy U for volume V is expressed in terms of the internal energy per unit volume, that is, in terms of the *internal-energy density* U_0. Thus,

$$U = \int_V U_0 \, dV$$

and the variation of internal energy becomes

$$\delta U = \int_V \delta U_0 \, dV \tag{3.7}$$

Substitution of Eqs. 3.6 and 3.7 into Eq. 3.1a gives the variation of the internal-energy density δU_0 in terms of the stress components and the variation in strain components. Thus,

$$\delta U_0 = \sigma_{xx}\,\delta\epsilon_{xx} + \sigma_{yy}\,\delta\epsilon_{yy} + \sigma_{zz}\,\delta\epsilon_{zz} + 2\sigma_{xy}\,\delta\epsilon_{xy} + 2\sigma_{xz}\,\delta\epsilon_{xz} + 2\sigma_{yz}\,\delta\epsilon_{yz} \tag{3.8}$$

This equation is used later in the derivation of expressions that relate the stress components to the strain-energy density U_0 (see Eqs. 3.11).

3.1.1 Elasticity and Internal-Energy Density

The strain-energy density U_0 is a function of certain variables; we need to determine these variables. For elastic material behavior, the total internal energy U in a loaded member is equal to the potential energy of the internal forces (called the *elastic strain energy*). Each stress component is related to the strain components; therefore, the internal-energy density U_0 at a given point in the member can be expressed in terms of the six components of the strain tensor. If the material is nonhomogeneous (has different properties at different points in the member), the function U_0 depends on location (x, y, z) in the member as well. The strain-energy density U_0 also depends on the temperature T (see Section 3.4).

Since the strain-energy density function U_0 generally depends on the strain components, the coordinates, and the temperature, we may express it as function of these variables. Thus,

$$U_0 = U_0(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}, x, y, z, T) \tag{3.9}$$

Then, if the displacements (u, v, w) undergo a variation $(\delta u, \delta v, \delta w)$, the strain components take variations $\delta\epsilon_{xx}$, $\delta\epsilon_{yy}$, $\delta\epsilon_{zz}$, $\delta\epsilon_{xy}$, $\delta\epsilon_{xz}$, and $\delta\epsilon_{yz}$, and the function U_0 takes on the variation

$$\delta U_0 = \frac{\partial U_0}{\partial \epsilon_{xx}}\delta\epsilon_{xx} + \frac{\partial U_0}{\partial \epsilon_{yy}}\delta\epsilon_{yy} + \frac{\partial U_0}{\partial \epsilon_{zz}}\delta\epsilon_{zz} + \frac{\partial U_0}{\partial \epsilon_{xy}}\delta\epsilon_{xy} + \frac{\partial U_0}{\partial \epsilon_{xz}}\delta\epsilon_{xz} + \frac{\partial U_0}{\partial \epsilon_{yz}}\delta\epsilon_{yz} \tag{3.10}$$

Therefore, since Eqs. 3.8 and 3.10 are valid for arbitrary variations $(\delta u, \delta v, \delta w)$, comparison yields for rectangular coordinate axes (x, y, z)

$$\begin{aligned} \sigma_{xx} &= \frac{\partial U_0}{\partial \epsilon_{xx}}, & \sigma_{yy} &= \frac{\partial U_0}{\partial \epsilon_{yy}}, & \sigma_{zz} &= \frac{\partial U_0}{\partial \epsilon_{zz}}, \\ \sigma_{xy} &= \frac{1}{2}\frac{\partial U_0}{\partial \epsilon_{xy}}, & \sigma_{xz} &= \frac{1}{2}\frac{\partial U_0}{\partial \epsilon_{xz}}, & \sigma_{yz} &= \frac{1}{2}\frac{\partial U_0}{\partial \epsilon_{yz}} \end{aligned} \tag{3.11}$$

3.1.2 Elasticity and Complementary Internal-Energy Density

In many members of engineering structures, there may be one dominant component of the stress tensor; call it σ. This situation may arise in axially loaded members, simple columns, beams, or torsional members. Then the strain-energy density U_0 (Eq. 3.9) depends mainly on the associated strain component ϵ; consequently, for a given temperature T, σ depends mainly on ϵ.

By Eq. 3.11, $\sigma = dU_0/d\epsilon$ and, therefore, $U_0 = \int \sigma\, d\epsilon$. It follows that U_0 is represented by the area under the stress–strain diagram (Figure 3.1). The rectangular area (0, 0), $(0, \epsilon)$, (σ, ϵ), $(\sigma, 0)$ is represented by the product $\sigma\epsilon$. Hence, this area is given by

$$\sigma\epsilon = U_0 + C_0 \tag{3.12}$$

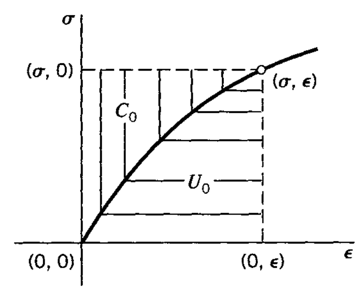

FIGURE 3.1 Strain-energy densities.

where C_0 is called the *complementary internal-energy density* or *complementary strain-energy density*. C_0 is represented by the area above the stress–strain curve and below the horizontal line from $(\sigma, 0)$ to (σ, ϵ). Hence, by Figure 3.1,

$$C_0 = \int \epsilon \, d\sigma \tag{3.13}$$

or

$$\epsilon = \frac{dC_0}{d\sigma} \tag{3.14}$$

This graphical interpretation of the complementary strain energy is applicable only for the case of a single nonzero component of stress. However, it can be generalized for several nonzero components of stress as follows. We assume that Eqs. 3.11 may be integrated to obtain the strain components as functions of the stress components. Thus, we obtain

$$\begin{aligned} \epsilon_{xx} &= f_1(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}) \\ \epsilon_{yy} &= f_2(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}) \\ &\vdots \\ \epsilon_{yz} &= f_6(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}) \end{aligned} \tag{3.15}$$

where $f_1, f_2, \ldots, f_6$ denote functions of the stress components. Substitution of Eqs. 3.15 into Eqs. 3.9 yields U_0 as a function of the six stress components. Then direct extension of Eq. 3.12 yields

$$C_0 = -U_0 + \sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + 2\sigma_{xy}\epsilon_{xy} + 2\sigma_{xz}\epsilon_{xz} + 2\sigma_{yz}\epsilon_{yz} \tag{3.16}$$

By Eqs. 3.15 and 3.16, the complementary energy density C_0 may be expressed in terms of the six stress components. Hence, differentiating Eq. 3.16 with respect to σ_{xx}, noting by the chain rule of differentiation that

$$\begin{aligned} \frac{\partial U_0}{\partial \sigma_{xx}} &= \frac{\partial U_0}{\partial \epsilon_{xx}}\frac{\partial \epsilon_{xx}}{\partial \sigma_{xx}} + \frac{\partial U_0}{\partial \epsilon_{yy}}\frac{\partial \epsilon_{yy}}{\partial \sigma_{xx}} + \frac{\partial U_0}{\partial \epsilon_{zz}}\frac{\partial \epsilon_{zz}}{\partial \sigma_{xx}} \\ &\quad + \frac{\partial U_0}{\partial \epsilon_{xy}}\frac{\partial \epsilon_{xy}}{\partial \sigma_{xx}} + \frac{\partial U_0}{\partial \epsilon_{xz}}\frac{\partial \epsilon_{xz}}{\partial \sigma_{xx}} + \frac{\partial U_0}{\partial \epsilon_{yz}}\frac{\partial \epsilon_{yz}}{\partial \sigma_{xx}} \end{aligned} \tag{3.17}$$

and employing Eq. 3.11, we find

$$\epsilon_{xx} = \frac{\partial C_0}{\partial \sigma_{xx}} \tag{3.18}$$

Similarly, taking derivatives of Eq. 3.16 with respect to the other stress components (σ_{yy}, σ_{zz}, σ_{xy}, σ_{xz}, σ_{yz}), we obtain the generalization of Eq. 3.14:

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial C_0}{\partial \sigma_{xx}}, & \epsilon_{yy} &= \frac{\partial C_0}{\partial \sigma_{yy}}, & \epsilon_{zz} &= \frac{\partial C_0}{\partial \sigma_{zz}} \\ \epsilon_{xy} &= \frac{1}{2}\frac{\partial C_0}{\partial \sigma_{xy}}, & \epsilon_{xz} &= \frac{1}{2}\frac{\partial C_0}{\partial \sigma_{xz}}, & \epsilon_{yz} &= \frac{1}{2}\frac{\partial C_0}{\partial \sigma_{yz}} \end{aligned} \tag{3.19}$$

Because of their relationship to Eqs. 3.11, Eqs. 3.19 are said to be conjugate to Eqs. 3.11. Equations 3.19 are known also as the *Legendre transform* of Eqs. 3.11 (Boresi and Chong, 2000).

3.2 HOOKE'S LAW: ANISOTROPIC ELASTICITY

In the one-dimensional case, for a linear elastic material the stress σ is proportional to the strain ϵ; that is, $\sigma = E\epsilon$, where the proportionality factor E is called the *modulus of elasticity*. The modulus of elasticity is a property of the material. Thus, for the one-dimensional case, only one material property is required to relate stress and strain for linear elastic behavior. The relation $\sigma = E\epsilon$ is known as *Hooke's law*. More generally, in the three-dimensional case, Hooke's law asserts that each of the stress components is a linear function of the components of the strain tensor; that is (with γ_{xy}, γ_{xz}, γ_{yz}; see Eq. 2.73),

$$
\begin{aligned}
\sigma_{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} + C_{14}\gamma_{xy} + C_{15}\gamma_{xz} + C_{16}\gamma_{yz} \\
\sigma_{yy} &= C_{21}\epsilon_{xx} + C_{22}\epsilon_{yy} + C_{23}\epsilon_{zz} + C_{24}\gamma_{xy} + C_{25}\gamma_{xz} + C_{26}\gamma_{yz} \\
\sigma_{zz} &= C_{31}\epsilon_{xx} + C_{32}\epsilon_{yy} + C_{33}\epsilon_{zz} + C_{34}\gamma_{xy} + C_{35}\gamma_{xz} + C_{36}\gamma_{yz} \\
\sigma_{xy} &= C_{41}\epsilon_{xx} + C_{42}\epsilon_{yy} + C_{43}\epsilon_{zz} + C_{44}\gamma_{xy} + C_{45}\gamma_{xz} + C_{46}\gamma_{yz} \\
\sigma_{xz} &= C_{51}\epsilon_{xx} + C_{52}\epsilon_{yy} + C_{53}\epsilon_{zz} + C_{54}\gamma_{xy} + C_{55}\gamma_{xz} + C_{56}\gamma_{yz} \\
\sigma_{yz} &= C_{61}\epsilon_{xx} + C_{62}\epsilon_{yy} + C_{63}\epsilon_{zz} + C_{64}\gamma_{xy} + C_{65}\gamma_{xz} + C_{66}\gamma_{yz}
\end{aligned}
\tag{3.20}
$$

where the 36 coefficients, C_{11}, ..., C_{66}, are called elastic coefficients. Materials that exhibit such stress–strain relations involving a number of independent elastic coefficients are said to be *anisotropic*. (See also Section 3.5.)

In reality, Eq. 3.20 is not a law but merely an assumption that is reasonably accurate for many materials subjected to small strains. For a given temperature, time, and location in the body, the coefficients C_{ij} are constants that are characteristics of the material.

Equations 3.11 and 3.20 yield

$$
\begin{aligned}
\frac{\partial U_0}{\partial \epsilon_{xx}} &= \sigma_{xx} = C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} + C_{14}\gamma_{xy} + C_{15}\gamma_{xz} + C_{16}\gamma_{yz} \\
&\ \ \vdots \\
\frac{\partial U_0}{\partial \gamma_{yz}} &= \sigma_{yz} = C_{61}\epsilon_{xx} + C_{62}\epsilon_{yy} + C_{63}\epsilon_{zz} + C_{64}\gamma_{xy} + C_{65}\gamma_{xz} + C_{66}\gamma_{yz}
\end{aligned}
\tag{3.21}
$$

Hence, the appropriate differentiations of Eqs. 3.21 yield

$$
\begin{aligned}
&\frac{\partial^2 U_0}{\partial \epsilon_{xx}\,\partial \epsilon_{yy}} = C_{12} = C_{21} \\
&\frac{\partial^2 U_0}{\partial \epsilon_{xx}\,\partial \epsilon_{zz}} = C_{13} = C_{31}, \ldots, \frac{\partial^2 U_0}{\partial \gamma_{yz}\,\partial \gamma_{xy}} = C_{46} = C_{64} \\
&\frac{\partial^2 U_0}{\partial \gamma_{yz}\,\partial \gamma_{xz}} = C_{56} = C_{65}
\end{aligned}
\tag{3.22}
$$

These equations show that the elastic coefficients $C_{ij} = C_{ji}$ are symmetrical in the subscripts i, j. Therefore, there are only 21 *distinct* C's. In other words, the general anisotropic linear elastic material has 21 elastic coefficients. In view of the preceding relation, the strain-energy density of a general anisotropic material is (by integration of Eqs. 3.21; see Boresi and Chong, 2000)

$$\begin{aligned} U_0 = {} & \frac{1}{2}C_{11}\epsilon_{xx}^2 + \frac{1}{2}C_{12}\epsilon_{xx}\epsilon_{yy} + \cdots + \frac{1}{2}C_{16}\epsilon_{xx}\gamma_{yz} \\ & + \frac{1}{2}C_{12}\epsilon_{xx}\epsilon_{yy} + \frac{1}{2}C_{22}\epsilon_{yy}^2 + \cdots + \frac{1}{2}C_{26}\epsilon_{yy}\gamma_{yz} \\ & + \frac{1}{2}C_{13}\epsilon_{xx}\epsilon_{zz} + \frac{1}{2}C_{23}\epsilon_{yy}\epsilon_{zz} + \cdots + \frac{1}{2}C_{36}\epsilon_{zz}\gamma_{yz} \\ & + \cdots + \frac{1}{2}C_{16}\epsilon_{xx}\gamma_{yz} + \frac{1}{2}C_{26}\epsilon_{yy}\gamma_{yz} + \cdots + \frac{1}{2}C_{66}\gamma_{yz}^2 \end{aligned} \tag{3.23}$$

3.3 HOOKE'S LAW: ISOTROPIC ELASTICITY

3.3.1 Isotropic and Homogeneous Materials

If the constituents of the material of a solid member are distributed sufficiently randomly, any part of the member will display essentially the same material properties in all directions. If a solid member is composed of such randomly oriented constituents, it is said to be *isotropic*. Accordingly, if a material is isotropic, its physical properties at a point are invariant under a rotation of axes. A material is said to be *elastically isotropic* if its characteristic elastic coefficients are invariant under any rotation of coordinates.

If the material properties are identical for every point in a member, the member is said to be *homogeneous*. In other words, homogeneity implies that the physical properties of a member are invariant under a translation. Alternatively, a member whose material properties change from point to point is said to be nonhomogeneous.

If an elastic member is composed of isotropic materials, the strain-energy density depends only on the principal strains, since for isotropic materials the elastic coefficients are invariant under arbitrary rotations (see Eq. 3.25).

3.3.2 Strain-Energy Density of Isotropic Elastic Materials

The strain-energy density of an elastic isotropic material depends only on the principal strains (ϵ_1, ϵ_2, ϵ_3). Accordingly, if the elasticity is linear, Eq. 3.23 yields

$$\begin{aligned} U_0 = {} & \frac{1}{2}C_{11}\epsilon_1^2 + \frac{1}{2}C_{12}\epsilon_1\epsilon_2 + \frac{1}{2}C_{13}\epsilon_1\epsilon_3 + \frac{1}{2}C_{12}\epsilon_1\epsilon_2 + \frac{1}{2}C_{22}\epsilon_2^2 + \frac{1}{2}C_{23}\epsilon_2\epsilon_3 \\ & + \frac{1}{2}C_{13}\epsilon_1\epsilon_3 + \frac{1}{2}C_{23}\epsilon_2\epsilon_3 + \frac{1}{2}C_{33}\epsilon_3^2 \end{aligned} \tag{3.24}$$

By symmetry, the naming of the principal axes is arbitrary. Hence, $C_{11} = C_{22} = C_{33} = C_1$, and $C_{12} = C_{23} = C_{13} = C_2$. Consequently, Eq. 3.24 contains only two distinct coefficients. For linear elastic isotropic materials, the strain-energy density may be expressed in the form

$$U_0 = \frac{1}{2}\lambda\left(\epsilon_1 + \epsilon_2 + \epsilon_3\right)^2 + G\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2\right) \tag{3.25}$$

where $\lambda = C_2$ and $G = (C_1 - C_2)/2$ are elastic coefficients called Lamé's elastic coefficients. If the material is homogeneous and temperature is constant everywhere, λ and G are constants at all points. In terms of the strain invariants (see Eq. 2.78), Eq. 3.25 may be written in the following form:

$$U_0 = \left(\frac{1}{2}\lambda + G\right)\bar{I}_1^2 - 2G\bar{I}_2 \tag{3.26}$$

Returning to orthogonal curvilinear coordinates (x, y, z) and introducing the general definitions of $\bar{I}_1$ and $\bar{I}_2$ from Eq. 2.78, we obtain

$$U_0 = \frac{1}{2}\lambda\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right)^2 + G\left(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2 + 2\epsilon_{xy}^2 + 2\epsilon_{xz}^2 + 2\epsilon_{yz}^2\right) \tag{3.27}$$

where $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ are strain components relative to orthogonal coordinates (x, y, z); see Eqs. 2.84. Equations 3.11 and 3.27 now yield Hooke's law for a linear elastic isotropic material in the form (for orthogonal curvilinear coordinates x, y, z)

$$\begin{aligned} &\sigma_{xx} = \lambda e + 2G\epsilon_{xx}, \quad &\sigma_{yy} = \lambda e + 2G\epsilon_{yy}, \quad &\sigma_{zz} = \lambda e + 2G\epsilon_{zz} \\ &\sigma_{xy} = 2G\epsilon_{xy}, &\sigma_{xz} = 2G\epsilon_{xz}, &\sigma_{yz} = 2G\epsilon_{yz} \end{aligned} \tag{3.28}$$

where $e \approx \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \bar{I}_1$ is the classical small-displacement volumetric strain (also called cubical strain; see Boresi and Chong, 2000). Thus, we have shown that for isotropic linear elastic materials, the stress–strain relations involve only two elastic constants. An analytic proof of the fact that no further reduction is possible on a theoretical basis can be constructed (Jeffreys, 1957).

By means of Eqs. 3.28, we find (with Eqs. 2.21 and 2.78)

$$\begin{aligned} I_1 &= (3\lambda + 2G)\bar{I}_1 \\ I_2 &= \lambda(3\lambda + 4G)\bar{I}_1^2 + 4G^2\bar{I}_2 \\ I_3 &= \lambda^2(\lambda + 2G)\bar{I}_1^3 + 4\lambda G^2\bar{I}_1\bar{I}_2 + 8G^3\bar{I}_3 \end{aligned} \tag{3.29}$$

which relate the stress invariants I_1, I_2, I_3 to the strain invariants $\bar{I}_1, \bar{I}_2, \bar{I}_3$.

Inverting Eqs. 3.28, we obtain

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) \\ \epsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}) \\ \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}) \\ \epsilon_{xy} &= \frac{1}{2G}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} \\ \epsilon_{xz} &= \frac{1}{2G}\sigma_{xz} = \frac{1+\nu}{E}\sigma_{xz} \\ \epsilon_{yz} &= \frac{1}{2G}\sigma_{yz} = \frac{1+\nu}{E}\sigma_{yz} \end{aligned} \tag{3.30}$$

where

$$E = \frac{G(3\lambda + 2G)}{\lambda + G}, \quad \nu = \frac{\lambda}{2(\lambda + G)} \tag{3.31a}$$

are elastic coefficients called Young's modulus and Poisson's ratio, respectively. Also, inverting Eqs. 3.31a, we obtain the Lamé coefficients λ and G in terms of E and ν as (see also Example 3.2)

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{3\nu K}{1+\nu}, \quad G = \frac{E}{2(1+\nu)} \tag{3.31b}$$

where

$$K = \frac{E}{3(1-2\nu)} \tag{3.31c}$$

is the *bulk modulus*. The bulk modulus relates the mean stress $\sigma_m = I_1/3$ to the volumetric strain e by $\sigma_m = Ke$.

Alternatively, Eqs. 3.28 may be written in terms of E and ν as follows:

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{xx} + \nu(\epsilon_{yy} + \epsilon_{zz})] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{yy} + \nu(\epsilon_{xx} + \epsilon_{zz})] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})] \\
\sigma_{xy} &= \frac{E}{1+\nu}\epsilon_{xy}, \quad \sigma_{xz} = \frac{E}{1+\nu}\epsilon_{xz}, \quad \sigma_{yz} = \frac{E}{1+\nu}\epsilon_{yz}
\end{aligned} \tag{3.32}$$

For the case of plane stress, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, Eqs. 3.32 reduce to

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) \\
\sigma_{yy} &= \frac{E}{1-\nu^2}(\nu\epsilon_{xx} + \epsilon_{yy}) \\
\sigma_{xy} &= \frac{E}{1+\nu}\epsilon_{xy}
\end{aligned} \tag{3.32a}$$

For the case of plane strain, $\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$, Eqs. 3.32 reduce to

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{xx} + \nu\epsilon_{yy}] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_{xx} + (1-\nu)\epsilon_{yy}] \\
\sigma_{zz} &= \frac{\nu E}{(1+\nu)(1-2\nu)}(\epsilon_{xx} + \epsilon_{yy}) \\
\sigma_{xy} &= \frac{E}{1+\nu}\epsilon_{xy}, \quad \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \tag{3.32b}$$

Substitution of Eqs. 3.30 into Eq. 3.27 yields the strain-energy density U_0 in terms of stress quantities. Thus, we obtain

$$\begin{aligned} U_0 &= \frac{1}{2E}[\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}) \\ &\quad + 2(1+\nu)(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)] \\ &= \frac{1}{2E}[I_1^2 - 2(1+\nu)I_2] \end{aligned} \tag{3.33}$$

If the (x, y, z) axes are directed along the principal axes of strain, then $\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$. Hence, by Eq. 3.32, $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$. Therefore, the (x, y, z) axes must also lie along the principal axes of stress. Consequently, for an isotropic material, the principal axes of stress are coincident with the principal axes of strain. *When we deal with isotropic materials, no distinction need be made between principal axes of stress and principal axes of strain. Such axes are called simply principal axes.*

EXAMPLE 3.1
Flat Plate Bent Around a Circular Cylinder

A flat rectangular plate lies in the (x, y) plane (Figure E3.1*a*). The plate, of uniform thickness $h = 2.00$ mm, is bent around a circular cylinder (Figure E3.1*b*) with the y axis parallel to the axis of the cylinder. The plate is made of an isotropic aluminum alloy ($E = 72.0$ GPa and $\nu = 0.33$). The radius of the cylinder is 600 mm.

(a) Assuming that plane sections for the undeformed plate remain plane after deformation, determine the maximum circumferential stress $\sigma_{\theta\theta(\max)}$ in the plate for linearly elastic behavior.

(b) The reciprocal of the radius of curvature R for a beam subject to pure bending is the curvature $\kappa = 1/R = M/EI$. For the plate, derive a formula for the curvature $\kappa = 1/R$ in terms of the applied moment M per unit width.

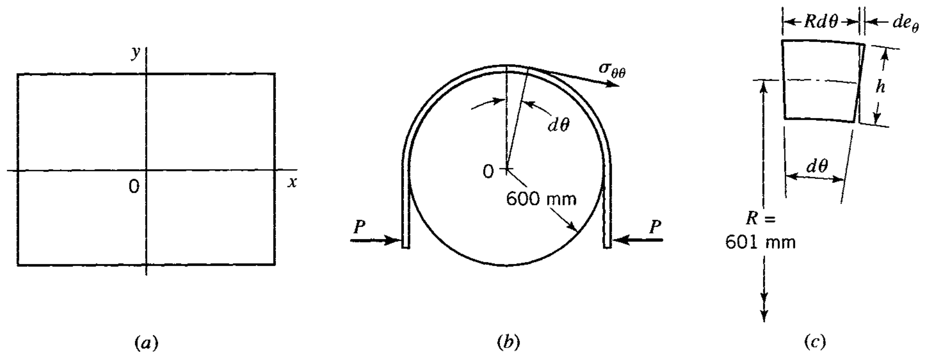

FIGURE E3.1

Solution

(a) We assume that the middle surface of the plate remains unstressed and that the stress through the thickness is negligible. Hence, the flexure formula is valid for the bending of the plate. Therefore, $\sigma_{\theta\theta} = \sigma_{xx} = 0$ for the middle surface and $\sigma_{rr} = 0$ throughout the plate thickness h. Equations 3.30 yield the results $\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{xx} = 0$ in the middle surface of the plate. Since the length of the plate in the y direction is large compared to the thickness h, the plate deforms approximately under conditions of plane strain; that is, $\epsilon_{yy} \approx 0$ throughout the plate thickness. Equations 3.30 give

$$\epsilon_{yy} = 0 = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{\theta\theta}$$

throughout the plate thickness. Thus, for plane strain relative to the (r, θ) plane

$$\sigma_{yy} = \nu\sigma_{\theta\theta} \tag{a}$$

With Eqs. 3.30, Eq. (a) yields

$$\epsilon_{\theta\theta} = \frac{1}{E}\sigma_{\theta\theta} - \frac{\nu}{E}\sigma_{yy} = \frac{(1-\nu^2)}{E}\sigma_{\theta\theta} \tag{b}$$

The relation between the radius of curvature R of the deformed plate and $\epsilon_{\theta\theta}$ may be determined by the geometry of deformation of a plate segment (Figure E3.1*c*). By similar triangles, we find from Figure E3.1*c* that

$$\frac{R\,d\theta}{R} = \frac{2de_\theta}{h} = \frac{2\epsilon_{\theta\theta(\max)}R\,d\theta}{h}$$

or

$$\epsilon_{\theta\theta(\max)} = \frac{h}{2R} \tag{c}$$

Equations (b) and (c) yield the result

$$\sigma_{\theta\theta(\max)} = \frac{Eh}{2(1-\nu^2)R} = \frac{72.0\times 10^3(2)}{2(1-0.33^2)(601)} = 134 \text{ MPa} \tag{d}$$

(b) In plate problems, it is convenient to consider a unit width of the plate (in the y direction) and let M be the moment per unit width. The moment of inertia for this unit width is $I = bh^3/12 = h^3/12$. Since $\sigma_{\theta\theta(\max)} = M(h/2)/I$, this relation may be used with Eq. (d) to give

$$\frac{1}{R} = \frac{\sigma_{\theta\theta(\max)}(2)(1-\nu^2)}{Eh} = \frac{Mh(12)}{2h^3}\frac{2(1-\nu^2)}{Eh} = \frac{M}{D} \tag{e}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{f}$$

is called the *flexure rigidity* of the plate. (See Chapter 13.)

EXAMPLE 3.2 *The Simple Tension Test*

In Section 1.3, the axial tension test and its role in the determination of material properties were discussed. The axial tension test in the linear elastic range of stress–strain may be used to interpret the Lamé coefficients λ and G. For example, consider a prismatic bar subjected to the following state of stress relative to the (x, y, z) axes, with the z axis directed along the longitudinal axis of the bar:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0, \quad \sigma_{zz} = \sigma = \text{constant} \tag{a}$$

For this state of stress to exist, the stresses on the lateral surface of the bar must be zero. On the ends of the bar, the normal stress is σ and the shear stress is zero. In other words, the state of stress in the bar is one of simple tension.

Equations 3.28 yield $\lambda e + 2G\epsilon_{xx} = \lambda e + 2G\epsilon_{yy} = \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$. Solving these equations for the strain components, we obtain

$$\epsilon_{xx} = \epsilon_{yy} = \frac{\lambda\sigma}{[2G(3\lambda+2G)]}, \quad \epsilon_{zz} = \frac{(\lambda+G)\sigma}{[G(3\lambda+2G)]} \tag{b}$$

It follows from Eqs. (b) that

$$-\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{\epsilon_{yy}}{\epsilon_{zz}} = \frac{\lambda}{[2(\lambda+G)]} = \nu, \quad \epsilon_{zz} = \frac{\sigma}{E} \tag{c}$$

where the quantities

$$E = \frac{G(3\lambda+2G)}{(\lambda+G)}, \quad \nu = \frac{\lambda}{[2(\lambda+G)]} \tag{d}$$

are *Young's modulus of elasticity* and *Poisson's ratio*, respectively. In terms of E and ν, Eq. (b) becomes

$$\epsilon_{xx} = \epsilon_{yy} = -\frac{\nu\sigma}{E}, \quad \epsilon_{zz} = \frac{\sigma}{E} \tag{e}$$

Solving Eqs. (d) for the Lamé coefficients λ and G, in terms of E and ν, we obtain

$$\lambda = \frac{\nu E}{[(1+\nu)(1-2\nu)]}, \quad G = \frac{E}{[2(1+\nu)]} \tag{f}$$

The Lamé coefficient G is also called the *shear modulus of elasticity*. It may be given a direct physical interpretation (see Example 3.3). The Lamé coefficient λ has no direct physical interpretation. However, if the first of Eqs. 3.32 is written in the form

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\epsilon_{yy} + \epsilon_{zz}) \tag{g}$$

the coefficient $E(1-\nu)/[(1+\nu)(1-2\nu)]$ can be called the *axial modulus*, since it relates the axial strain component ϵ_{xx} to its associated axial stress σ_{xx}.

Similarly, the Lamé coefficient $\lambda = \nu E/[(1+\nu)(1-2\nu)]$ may be called the *transverse modulus*, since it relates the strain components ϵ_{yy} and ϵ_{zz} (which act transversely to σ_{xx}) to the axial stress σ_{xx}. The second and third equations of Eqs. 3.32 may be written in a form similar to Eq. (g), with the same interpretation.

EXAMPLE 3.3 The Pure Shear Test and the Shear Modulus

The pure shear test may be characterized by the stress state $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = 0$ and $\sigma_{yz} = \tau =$ constant. For this state of stress, Eqs. 3.28 yield the strain components

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \gamma_{xy} = \gamma_{xz} = 0, \quad \gamma_{yz} = \frac{\tau}{G} \tag{a}$$

where γ is used to represent engineering shear strain because of its convenient geometric interpretation (see Eq. 2.73). These formulas show that a rectangular parallelepiped $ABCD$ (Figure E3.3) whose faces are parallel to the coordinate planes is sheared in the yz plane so that the right angle between the edges of the parallelepiped parallel to the y and z axes decreases by the amount γ_{yz}. For this reason, the coefficient G is called the *shear modulus of elasticity*. A pure shear state of stress can be obtained quite accurately by the torsion of a hollow circular cylinder with thin walls (see Chapter 6).

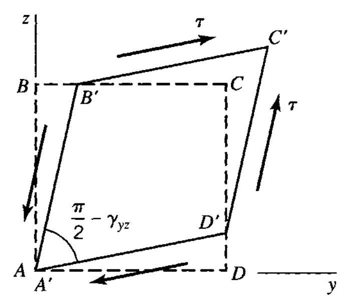

FIGURE E3.3

EXAMPLE 3.4
Elimination of Friction Effect in the Uniaxial Compression Test

In a uniaxial compression test, the effect of friction between the test specimen and the testing machine platens restrains the ends of the specimen from expanding freely in the lateral directions. This restraint may lead to erroneous measurement of the specimen strain. One way to eliminate this effect is to design the specimen and machine platens so that 1. the specimen and the end of the platens in contact with the specimen have the same cross sections and 2. a certain relation exists between the material properties of the specimen and the platens.

To illustrate this point, let quantities associated with the specimen be denoted by subscript s and those associated with the platens be denoted by subscript p. Let P be the load applied to the specimen through the end platens. Because the cross-sectional shapes of the specimen and the platens are the same, we denote the areas by A. Let coordinate z be taken along the longitudinal axis of the specimen and coordinate x be perpendicular to axis z. Then, under a machine load P, the longitudinal strains in the specimen and platens are, respectively,

$$(\epsilon_{zz})_{\mathrm{s}} = \frac{P}{E_{\mathrm{s}}A}, \quad (\epsilon_{zz})_{\mathrm{p}} = \frac{P}{E_{\mathrm{p}}A} \tag{a}$$

The associated lateral strains are

$$(\epsilon_{xx})_{\mathrm{s}} = -\nu_{\mathrm{s}}(\epsilon_{zz})_{\mathrm{s}} = -\frac{\nu_{\mathrm{s}}P}{E_{\mathrm{s}}A}, \quad (\epsilon_{xx})_{\mathrm{p}} = -\nu_{\mathrm{p}}(\epsilon_{zz})_{\mathrm{p}} = -\frac{\nu_{\mathrm{p}}P}{E_{\mathrm{p}}A} \tag{b}$$

If the lateral strains in the specimen and platens are equal, they will expand laterally the same amount, thus eliminating friction that might be induced by the tendency of the specimen to move laterally relative to the platens. By Eq. (b), the requirement for friction to be nonexistent is that $(\epsilon_{xx})_{\mathrm{s}} = (\epsilon_{xx})_{\mathrm{p}}$, or

$$\frac{\nu_{\mathrm{s}}}{E_{\mathrm{s}}} = \frac{\nu_{\mathrm{p}}}{E_{\mathrm{p}}} \tag{c}$$

In addition to identical cross sections of specimen and platens, the moduli of elasticity and Poisson's ratios must satisfy Eq. (c). To reduce or eliminate the effect of friction on the tests results, it is essential to select the material properties of the platens to satisfy Eq. (c) as closely as possible.

3.4 EQUATIONS OF THERMOELASTICITY FOR ISOTROPIC MATERIALS

Consider an unconstrained member made of an isotropic elastic material in an arbitrary zero configuration. Let the uniform temperature of the member be increased by a small amount ΔT. Experimental observation has shown that, for a homogeneous and isotropic material, all infinitesimal line elements in the volume undergo equal expansions. Furthermore, all line elements maintain their initial directions. Therefore, the strain components resulting from the temperature change ΔT are, with respect to rectangular Cartesian coordinates (x, y, z),

$$\epsilon'_{xx} = \epsilon'_{yy} = \epsilon'_{zz} = \alpha\,\Delta T, \quad \epsilon'_{xy} = \epsilon'_{xz} = \epsilon'_{zy} = 0 \tag{3.34}$$

where α denotes the coefficient of thermal expansion of the material.

Now let the member be subjected to forces that induce stresses $\sigma_{xx}, \sigma_{yy}, \ldots, \sigma_{yz}$ at point 0 in the member. Accordingly, if $\epsilon_{xx}, \epsilon_{yy}, \ldots, \epsilon_{yz}$ denote the strain components at point 0 after the application of the forces, the change in strain produced by the forces is represented by the equations

$$\begin{aligned} \epsilon''_{xx} &= \epsilon_{xx} - \alpha\,\Delta T, & \epsilon''_{yy} &= \epsilon_{yy} - \alpha\,\Delta T, & \epsilon''_{zz} &= \epsilon_{zz} - \alpha\,\Delta T \\ \epsilon''_{xy} &= \epsilon_{xy}, & \epsilon''_{xz} &= \epsilon_{xz}, & \epsilon''_{yz} &= \epsilon_{yz} \end{aligned} \tag{3.35}$$

In general, ΔT may depend on the location of point 0 and time t. Hence $\Delta T = \Delta T(x, y, z, t)$. Substitution of Eq. 3.35 into Eqs. 3.28 yields

$$\begin{aligned} &\sigma_{xx} = \lambda e + 2G\epsilon_{xx} - c\Delta T, \quad \sigma_{yy} = \lambda e + 2G\epsilon_{yy} - c\Delta T \\ &\sigma_{zz} = \lambda e + 2G\epsilon_{zz} - c\Delta T \\ &\sigma_{xy} = 2G\epsilon_{xy}, \quad \sigma_{xz} = 2G\epsilon_{xz}, \quad \sigma_{yz} = 2G\epsilon_{yz} \end{aligned} \tag{3.36}$$

where

$$c = (3\lambda + 2G)\alpha = \frac{E\alpha}{(1-2\nu)} \tag{3.37}$$

Similarly, substitution of Eqs. 3.36 into Eqs. 3.30 yields

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \alpha\Delta T \\ \epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] + \alpha\Delta T \\ \epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] + \alpha\Delta T \\ \epsilon_{xy} &= \frac{(1+\nu)}{E}\sigma_{xy}, \quad \epsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz}, \quad \epsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \end{aligned} \tag{3.38}$$

Finally, substituting Eqs. 3.38 into Eqs. 3.26 or 3.27, we find that

$$U_0 = \left(\frac{1}{2}\lambda + G\right)\bar{I}_1^2 - 2G\bar{I}_2 - c\bar{I}_1\,\Delta T + \frac{3}{2}c\alpha(\Delta T)^2 \tag{3.39}$$

In terms of the strain components (see Eqs. 2.78), we obtain

$$\begin{aligned} U_0 = {} & \frac{1}{2}\lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})^2 + G(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2 + 2\epsilon_{xy}^2 + 2\epsilon_{xz}^2 + 2\epsilon_{yz}^2) \\ & - c(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})\Delta T + \frac{3}{2}c\alpha(\Delta T)^2 \end{aligned} \tag{3.40}$$

Equations 3.36 and 3.38 are the basic stress–strain relations of classical thermoelasticity for isotropic materials. For temperature changes ΔT, the strain-energy density is modified by a temperature-dependent term that is proportional to the volumetric strain $e = \bar{I}_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ and by a term proportional to $(\Delta T)^2$ (Eqs. 3.39 and 3.40).

We find by Eqs. 3.38 and 3.40

$$U_0 = \frac{1}{2E}\left[I_1^2 - 2(1+\nu)I_2\right] \tag{3.41}$$

and

$$\begin{aligned} U_0 = {} & \frac{1}{2E}\Big[\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}) \\ & + 2(1+\nu)(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\Big] \end{aligned} \tag{3.42}$$

in terms of stress components. Equation 3.42 does not contain ΔT explicitly. However, the temperature distribution may affect the stresses. Note that Eqs. 3.41 and 3.42 are identical to the results in Eq. 3.33.

3.5 HOOKE'S LAW: ORTHOTROPIC MATERIALS

An important class of materials, called *orthotropic* materials, is discussed in this section. Materials such as wood, laminated plastics, cold rolled steels, reinforced concrete, various composite materials, and even forgings can be treated as orthotropic. Orthotropic materials possess three orthogonal planes of material symmetry and three corresponding orthogonal axes called the orthotropic axes. In some materials, for example, forged materials, these axes may vary from point to point. In other materials, for example, fiber-reinforced plastics and concrete reinforced with steel bars, the orthotropic directions remain constant as long as the fibers and steel reinforcing bars maintain constant directions. In any case, for an elastic orthotropic material, the elastic coefficients C_{ij} (Eq. 3.20) remain unchanged at a point under a rotation of 180° about any of the orthotropic axes.

Let the (x, y, z) axes denote the orthotropic axes for an orthotropic material and let the (x, y) plane be a plane of material symmetry. Then, under the coordinate transformation $x \to x$, $y \to y$, and $z \to -z$, called a reflection with respect to the (x, y) plane, the elastic coefficients C_{ij} remain invariant. The direction cosines for this transformation (see Table 2.2) are defined by

$$l_1 = m_2 = 1, \quad n_3 = -1, \quad l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = 0 \tag{3.43}$$

Substitution of Eqs. 3.43 into Eqs. 2.15, 2.17, and 2.76 reveals that, for a reflection with respect to the (x, y) plane,

$$\sigma_{XX} = \sigma_{xx}, \quad \sigma_{YY} = \sigma_{yy}, \quad \sigma_{ZZ} = \sigma_{zz}, \quad \sigma_{XY} = \sigma_{xy}, \quad \sigma_{XZ} = -\sigma_{xz}, \quad \sigma_{YZ} = -\sigma_{yz} \tag{3.44}$$

and

$$\epsilon_{XX} = \epsilon_{xx}, \epsilon_{YY} = \epsilon_{yy}, \epsilon_{ZZ} = \epsilon_{zz}, \gamma_{XY} = \gamma_{xy}, \gamma_{XZ} = -\gamma_{xz}, \gamma_{YZ} = -\gamma_{yz} \tag{3.45}$$

Since the C_{ij} are constant under the transformation of Eq. 3.43, the first of Eqs. 3.20 yields

$$\sigma_{XX} = C_{11}\epsilon_{XX} + C_{12}\epsilon_{YY} + C_{13}\epsilon_{ZZ} + C_{14}\gamma_{XY} + C_{15}\gamma_{XZ} + C_{16}\gamma_{YZ} \tag{3.46}$$

Substitution of Eqs. 3.44 and 3.45 into Eq. 3.46 yields

$$\sigma_{xx} = \sigma_{XX} = C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} + C_{14}\gamma_{xy} - C_{15}\gamma_{xz} - C_{16}\gamma_{yz} \tag{3.47}$$

Comparison of the first of Eqs. 3.20 with Eq. 3.47 yields the conditions $C_{15} = -C_{15}$ and $C_{16} = -C_{16}$, or $C_{15} = C_{16} = 0$. Similarly, considering σ_{YY}, σ_{ZZ}, σ_{XY}, σ_{XZ}, and σ_{YZ}, we find that $C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0$. Thus, the coefficients for a material whose elastic properties are invariant under a reflection with respect to the (x, y) plane (i.e., for a material that possesses a plane of elasticity symmetry) are summarized by the matrix

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \tag{3.48}$$

A general orthotropic material has two additional planes of elastic material symmetry, in this case, the (x, z) and (y, z) planes. Consider the (x, z) plane. Let $x \to x$, $y \to -y$, $z \to z$. Then, proceeding as before, noting that $l_1 = n_3 = 1$, $m_2 = -1$, and $l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = 0$, we find $C_{14} = C_{24} = C_{34} = C_{56} = 0$. Then, the matrix of Eq. 3.48 reduces to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{3.49}$$

A reflection with respect to the (y, z) plane does not result in further reduction in the number of elastic coefficients C_{ij}.

The matrix of coefficients in Eq. 3.49 contains nine elastic coefficients. Consequently, the stress–strain relations for the most general orthotropic material contain nine independent elastic coefficients relative to the orthotropic axes (x, y, z). Equations 3.20 are simplified accordingly. It should be noted, however, that this simplification occurs only when the orthotropic axes are used as the coordinate axes for which the C_{ij} are defined. The resulting equations are

$$\begin{aligned} \sigma_{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} \\ \sigma_{yy} &= C_{12}\epsilon_{xx} + C_{22}\epsilon_{yy} + C_{23}\epsilon_{zz} \\ \sigma_{zz} &= C_{13}\epsilon_{xx} + C_{23}\epsilon_{yy} + C_{33}\epsilon_{zz} \\ \sigma_{xy} &= C_{44}\gamma_{xy} \\ \sigma_{xz} &= C_{55}\gamma_{xz} \\ \sigma_{yz} &= C_{66}\gamma_{yz} \end{aligned} \tag{3.50}$$

The stress–strain relations for orthotropic materials in terms of orthotropic moduli of elasticity and orthotropic Poisson's ratios may be written in the form

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E_x}\sigma_{xx} - \frac{\nu_{yx}}{E_y}\sigma_{yy} - \frac{\nu_{zx}}{E_z}\sigma_{zz} \\ \epsilon_{yy} &= -\frac{\nu_{xy}}{E_x}\sigma_{xx} + \frac{1}{E_y}\sigma_{yy} - \frac{\nu_{zy}}{E_z}\sigma_{zz} \\ \epsilon_{zz} &= -\frac{\nu_{xz}}{E_x}\sigma_{xx} - \frac{\nu_{yz}}{E_y}\sigma_{yy} + \frac{1}{E_z}\sigma_{zz} \\ \gamma_{xy} &= \frac{1}{G_{xy}}\sigma_{xy} \\ \gamma_{xz} &= \frac{1}{G_{xz}}\sigma_{xz} \\ \gamma_{yz} &= \frac{1}{G_{yz}}\sigma_{yz} \end{aligned} \tag{3.51}$$

where E_x, E_y, E_z denote the orthotropic moduli of elasticity and G_{xy}, G_{xz}, G_{yz} denote the orthotropic shear moduli for shear deformation in the x–y, x–z, and y–z planes, respectively. The term ν_{xy} is a Poisson ratio that characterizes the strain in the y direction produced by the stress in the x direction, with similar interpretations for the other Poisson ratios, ν_{yx}, ν_{zx}, ν_{xz}, ν_{yz}, and ν_{zy}. For example, by Eq. 3.51, for a tension specimen of orthotropic material subjected to a uniaxial stress $\sigma_{zz} = \sigma$, the axial strain is $\epsilon_{zz} = \sigma/E_z$ and the laterial strains are $\epsilon_{xx} = -\nu_{zx}\sigma/E_z$ and $\epsilon_{yy} = -\nu_{zy}\sigma/E_z$. (See Example 3.2 for the analogous isotropic tension test.)

Because of the symmetry of the coefficients in the stress–strain relations, we have by Eqs. 3.51 the identities

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}, \quad \frac{\nu_{xz}}{E_x} = \frac{\nu_{zx}}{E_z}, \quad \frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z} \tag{3.52}$$

EXAMPLE 3.5
Stress–Strain Relations for Orthotropic Materials: The Plane Stress Case

A wood panel with orthotropic axes (x, y, z) is subjected to a plane stress state relative to its face in the (x, y) plane. Let a rectangular region in the body be subjected to extensional stress σ_{xx} (Figure E3.5*a*). By Eq. 3.51, the strain components are

$$\begin{aligned}
\epsilon_{xx} &= \frac{\sigma_{xx}}{E_x} \\
\epsilon_{yy} &= -\nu_{xy}\epsilon_{xx} = -\frac{\nu_{xy}\sigma_{xx}}{E_x} \\
\epsilon_{zz} &= -\nu_{xz}\epsilon_{xx} = -\frac{\nu_{xz}\sigma_{xx}}{E_x}
\end{aligned} \tag{a}$$

where ν_{xy} and ν_{xz} are orthotropic Poisson ratios.

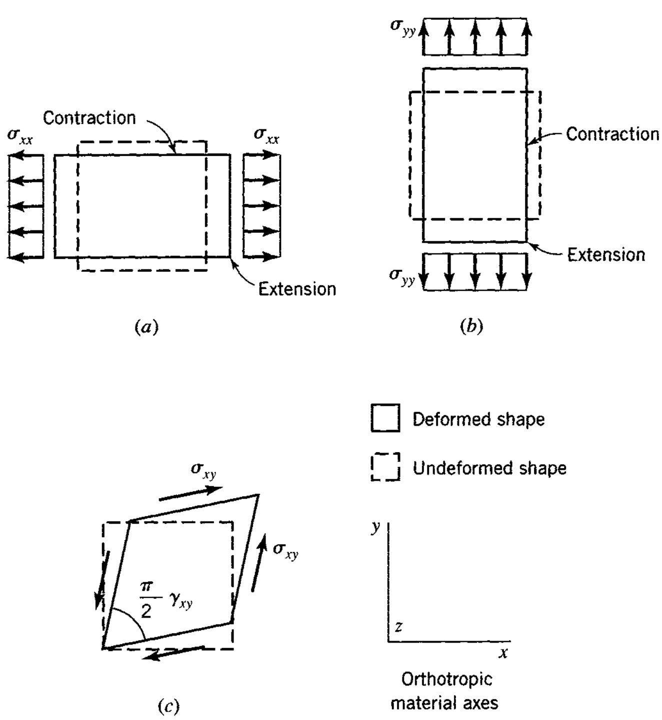

FIGURE E3.5 Orthotropic material. (*a*) Applied stress σ_{xx}. (*b*) Applied stress σ_{yy}. (*c*) Applied stress σ_{xy}.

Consider next the case where the rectangular region is subjected to an extensional stress σ_{yy} (Figure E3.5*b*). By Eqs. 3.51, the strain components are

$$\begin{aligned}\epsilon_{yy} &= \frac{\sigma_{yy}}{E_y}\\ \epsilon_{xx} &= -\nu_{yx}\epsilon_{yy} = -\frac{\nu_{yx}\sigma_{yy}}{E_y}\\ \epsilon_{zz} &= -\nu_{yz}\epsilon_{yy} = -\frac{\nu_{yz}\sigma_{yy}}{E_y}\end{aligned} \tag{b}$$

where ν_{yx} and ν_{yz} are orthotropic Poisson ratios.

For a combination of stresses $(\sigma_{xx}, \sigma_{yy})$, the addition of Eqs. (a) and (b) yields

$$\begin{aligned}\epsilon_{xx} &= \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}\sigma_{yy}}{E_y}\\ \epsilon_{yy} &= -\frac{\nu_{xy}\sigma_{xx}}{E_x} + \frac{\sigma_{yy}}{E_y}\\ \epsilon_{zz} &= -\frac{\nu_{xz}\sigma_{xx}}{E_x} - \frac{\nu_{yz}\sigma_{yy}}{E_y}\end{aligned} \tag{c}$$

Solving the first two of Eqs. (c) for $(\sigma_{xx}, \sigma_{yy})$ in terms of the in-plane strains $(\epsilon_{xx}, \epsilon_{yy})$, we obtain

$$\begin{aligned}\sigma_{xx} &= \frac{E_x}{1-\nu_{xy}\nu_{yx}}(\epsilon_{xx} + \nu_{yx}\epsilon_{yy})\\ \sigma_{yy} &= \frac{E_y}{1-\nu_{xy}\nu_{yx}}(\nu_{xy}\epsilon_{xx} + \epsilon_{yy})\end{aligned} \tag{d}$$

Finally, consider the element subjected to shear stress σ_{xy} (Figure E3.5*c*). By Eqs. 3.51, we have

$$\sigma_{xy} = G_{xy}\gamma_{xy} \tag{e}$$

where G_{xy} is the orthotropic shear modulus in the (x, y) plane and γ_{xy} is the engineering shear strain in the (x, y) plane. Thus, for the orthotropic material in a state of plane stress, we have the stress–strain relations [by Eqs. (d) and (e)]

$$\begin{aligned}\sigma_{xx} &= \frac{E_x}{1-\nu_{xy}\nu_{yx}}(\epsilon_{xx} + \nu_{yx}\epsilon_{y})\\ \sigma_{yy} &= \frac{E_y}{1-\nu_{xy}\nu_{yx}}(\nu_{xy}\epsilon_{xx} + \epsilon_{yy})\\ \sigma_{xy} &= G_{xy}\gamma_{xy}\end{aligned} \tag{f}$$

With these stress–strain relations, the theory for plane stress orthotropic problems of wood panels follows in the same manner as for plane stress problems for isotropic materials.

EXAMPLE 3.6 *Stress–Strain Relations of a Fiber–Resin Lamina*

A lamina (a thin plate, sheet, or layer of material) of a section of an airplane wing is composed of unidirectional fibers and a resin matrix that bonds the fibers. Let the volume fraction (the proportion of fiber volume to the total volume of the composite) be f. Determine the effective linear stress–strain relations of the lamina.

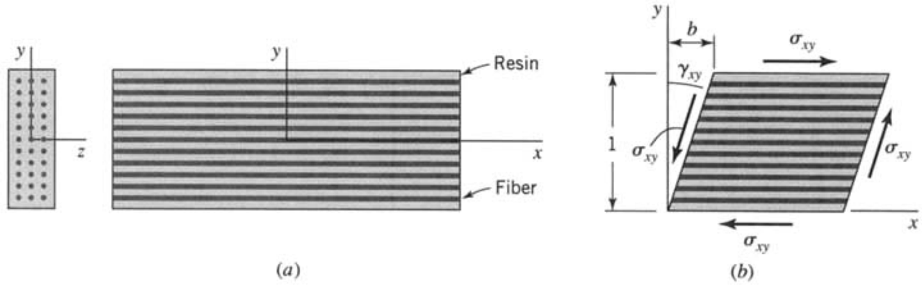

FIGURE E3.6 Lamina: fiber volume fraction = f, resin volume fraction = $1 - f$.

Solution

Let the modulus of elasticity and the Poisson ratio of the fibers be denoted E_F and ν_F, respectively, and the modulus of elasticity and the Poisson ratio of the resin be E_R and ν_R. Since the lamina is thin, the effective state of stress in the lamina is approximately one of plane stress in the x–y plane of the lamina (see Figure E3.6*a*). Hence, the stress–strain relations for the fibers and the resin are

$$\begin{aligned}
\epsilon_{xxF} &= \frac{1}{E_F}(\sigma_{xxF} - \nu_F\sigma_{yyF}) \\
\epsilon_{yyF} &= \frac{1}{E_F}(\sigma_{yyF} - \nu_F\sigma_{xxF}) \\
\epsilon_{xxR} &= \frac{1}{E_R}(\sigma_{xxR} - \nu_R\sigma_{yyR}) \\
\epsilon_{yyR} &= \frac{1}{E_R}(\sigma_{yyR} - \nu_R\sigma_{xxR})
\end{aligned} \tag{a}$$

where $(\sigma_{xxF}, \sigma_{yyF})$, $(\sigma_{xxR}, \sigma_{yyR})$, $(\epsilon_{xxF}, \epsilon_{yyF})$, and $(\epsilon_{xxR}, \epsilon_{yyR})$ denote stress and strain components in the fiber (F) and resin (R), respectively.

Since the fibers and resin are bonded, the effective lamina strain ϵ_{xx} (Figure E3.6*a*) is the same as that in the fibers and in the resin; that is, in the x direction,

$$\epsilon_{xx} = \epsilon_{xxF} = \epsilon_{xxR} \tag{b}$$

In the y direction, the effective lamina strain ϵ_{yy} is proportional to the amount of fiber per unit length in the y direction and the amount of resin per unit length in the y direction. Hence,

$$\epsilon_{yy} = f\epsilon_{yyF} + (1 - f)\epsilon_{yyR} \tag{c}$$

Also, by equilibrium of the lamina in the x direction, the effective lamina stress σ_{xx} is

$$\sigma_{xx} = f\sigma_{xxF} + (1 - f)\sigma_{xxR} \tag{d}$$

In the y direction, the effective lamina stress σ_{yy} is the same as in the fibers and in the resin; that is,

$$\sigma_{yy} = \sigma_{yyF} = \sigma_{yyR} \tag{e}$$

Solving Eqs. (a) through (e) for ϵ_{xx} and ϵ_{yy} in terms of σ_{xx} and σ_{yy}, we obtain the effective stress–strain relations for the lamina as

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\
\epsilon_{yy} &= \frac{1}{E}(\beta\sigma_{yy} - \nu\sigma_{xx})
\end{aligned} \tag{f}$$

where

$$
\begin{aligned}
E &= fE_{\mathrm{F}} + (1-f)E_{\mathrm{R}} \\
v &= fv_{\mathrm{F}} + (1-f)v_{\mathrm{R}} \\
\beta &= f(1-f)\left[(1-v_R^2)\frac{E_{\mathrm{F}}}{E_{\mathrm{R}}} + (1-v_F^2)\frac{E_{\mathrm{R}}}{E_{\mathrm{F}}} + 2v_{\mathrm{F}}v_{\mathrm{R}} + \frac{1-f}{f} + \frac{f}{1-f}\right]
\end{aligned}
\tag{g}
$$

To determine the shear stress–strain relation, we apply a shear stress σ_{xy} to a rectangular element of the lamina (Figure E3.6*b*), and we calculate the angle change γ_{xy} of the rectangle. By Figure E3.6*b*, the relative displacement b of the top of the element is

$$b = f\gamma_{\mathrm{F}} + (1-f)\gamma_{\mathrm{R}} \tag{h}$$

where γ_{F} and γ_{R} are the angle changes attributed to the fibers and the resin, respectively; that is,

$$\gamma_{\mathrm{F}} = \frac{\sigma_{xy}}{G_{\mathrm{F}}}, \quad \gamma_{\mathrm{R}} = \frac{\sigma_{xy}}{G_{\mathrm{R}}} \tag{i}$$

and G_{F} and G_{R} are the shear moduli of elasticity of the fiber and resin, respectively. Hence, the change γ_{xy} in angle of the element (the shear strain) is, with Eqs. (h) and (i),

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{b}{1} = \left[\frac{fG_{\mathrm{R}} + (1-f)G_{\mathrm{F}}}{G_{\mathrm{F}}G_{\mathrm{R}}}\right]\sigma_{xy} \tag{j}$$

By Eq. (j), the shear stress–strain relation is

$$\sigma_{xy} = G\gamma_{xy} = 2G\epsilon_{xy} \tag{k}$$

where

$$G = \frac{G_{\mathrm{F}}G_{\mathrm{R}}}{fG_{\mathrm{R}} + (1-f)G_{\mathrm{F}}} \tag{l}$$

Thus, by Eqs. (f), (g), (k), and (l), we obtain the stress–strain relations of the lamina, in the form of Eqs. 3.50, as

$$
\begin{aligned}
\sigma_{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} \\
\sigma_{yy} &= C_{12}\epsilon_{xx} + C_{22}\epsilon_{yy} \\
\sigma_{xy} &= C_{33}\gamma_{xy}
\end{aligned}
\tag{m}
$$

where

$$
\begin{aligned}
C_{11} &= \frac{\beta E}{\beta - v^2}, \quad C_{12} = \frac{vE}{\beta - v^2} \\
C_{22} &= \frac{E}{\beta - v^2}, \quad C_{33} = G
\end{aligned}
\tag{n}
$$

EXAMPLE 3.7
Composite Thin-Wall Cylinder Subjected to Pressure and Temperature Increase

Consider a composite cylinder of length L formed from an inner cylinder of aluminum with outer radius R and thickness t_{A} and an outer cylinder of steel with inner radius R and thickness t_{S} (Figure E3.7*a*); $t_{\mathrm{A}} << R$ and $t_{\mathrm{S}} << R$. The composite cylinder is supported snugly in an upright, unstressed state between rigid supports. An inner pressure p is applied to the cylinder (Figure E3.7*b*), and the entire assembly is subjected to a uniform temperature change ΔT. Determine the stresses in both the aluminum and the steel cylinders for the case $t_{\mathrm{A}} = t_{\mathrm{S}} = t = 0.02R$. For aluminum, $E_{\mathrm{A}} = 69$ GPa, $v_{\mathrm{A}} = 0.333$, and $\alpha_{\mathrm{A}} = 21.6 \times 10^{-6}$/°C. For steel, $E_{\mathrm{S}} = 207$ GPa, $v_{\mathrm{S}} = 0.280$, and $\alpha_{\mathrm{S}} = 10.8 \times 10^{-6}$/°C.

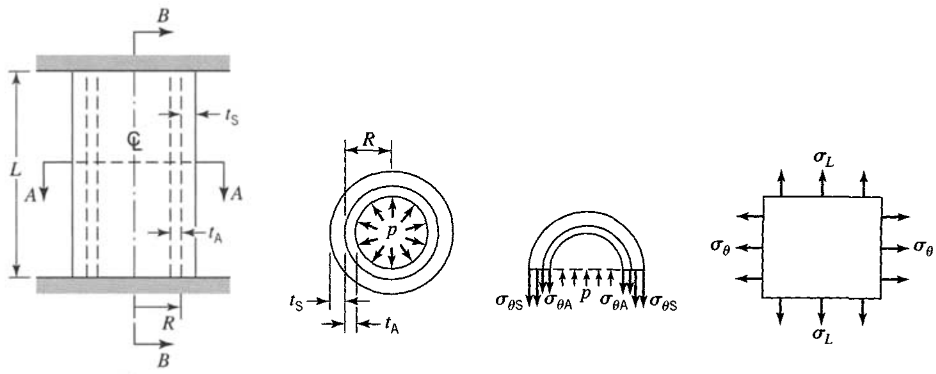

FIGURE E3.7 (*a*) Composite cylinder. (*b*) Cross section *A–A*. (*c*) Longitudinal section *B–B*. (*d*) Cylinder element.

Solution

Since both cylinders are thin, we may assume that the stresses in the tangential direction θ, $\sigma_{\theta A}$ and $\sigma_{\theta S}$ in the aluminum and steel, respectively, are constant through the thicknesses t_A and t_S (Figure E3.7*c*). Also, it is sufficiently accurate to use the approximation $R - t = R$. From the free-body diagram of Figure E3.7*c*, we have $\sum F = 2pRL - 2\sigma_{\theta S}tL - 2\sigma_{\theta A}tL = 0$. Hence,

$$\sigma_{\theta A} + \sigma_{\theta S} = \left(\frac{R}{t}\right)p = 50p \tag{a}$$

Since ordinarily the radial stress σ_r in the cylinder is very small (of the order p) compared with both the tangential stress σ_θ and the longitudinal stress σ_L, we assume that σ_r is negligible. Therefore, the cylinder is subjected approximately to a state of plane stress (σ_L, σ_θ) (Figure E3.7*d*). Hence, for plane stress, the stress–strain–temperature relations for each cylinder are

$$\begin{aligned} E\epsilon_L &= \sigma_L - \nu\sigma_\theta + E\alpha(\Delta T) \\ E\epsilon_\theta &= \sigma_\theta - \nu\sigma_L + E\alpha(\Delta T) \end{aligned} \tag{b}$$

Equations (b) hold for all points in the cylinder, provided that the ends are free to expand radially. The cylinder is restrained from expanding longitudinally, since the end walls are rigid. Then, $\epsilon_L = 0$. Also, at radial distance R (the interface between the aluminum and the steel sleeves), the radial displacement is u and the tangential strain is $\epsilon_\theta = [2\pi(R + u) - 2\pi R]/2\pi R = u/R$. Assuming that t is so small that this strain is the same throughout the aluminum and the steel sleeves, we have by Eqs. (b)

$$\begin{aligned} \epsilon_{LA} &= \frac{1}{E_A}(\sigma_{LA} - \nu_A\sigma_{\theta A}) + \alpha_A(\Delta T) = 0 \\ \epsilon_{LS} &= \frac{1}{E_S}(\sigma_{LS} - \nu_S\sigma_{\theta S}) + \alpha_S(\Delta T) = 0 \\ \epsilon_{\theta A} &= \epsilon_{\theta S} = \frac{1}{E_A}(\sigma_{\theta A} - \nu_A\sigma_{LA}) + \alpha_A(\Delta T) = \frac{1}{E_S}(\sigma_{\theta S} - \nu_S\sigma_{LS}) + \alpha_S(\Delta T) \end{aligned} \tag{c}$$

Also, from the given data, $3E_A = E_S$ and $\alpha_A = 2\alpha_S$. Therefore, with Eqs. (a) and (c), we may write

$$\begin{aligned} &\sigma_{LA} + \frac{1}{3}\sigma_{\theta S} - \frac{50}{3}p + \frac{2}{3}E_S\alpha_S(\Delta T) = 0 \\ &\sigma_{LS} - 0.28\sigma_{\theta S} + E_S\alpha_S(\Delta T) = 0 \\ &3(50p - \sigma_{\theta S}) - \sigma_{LA} + 2E_S\alpha_S(\Delta T) = \sigma_{\theta S} - 0.28\sigma_{LS} + E_S\alpha_S(\Delta T) = 0 \end{aligned} \tag{d}$$

By the first two of Eqs. (d) and with $E_S\alpha_S = 2.236$ MPa/°C, we find that

$$\begin{aligned}\sigma_{LA} &= \frac{50}{3}p - 1.491(\Delta T) - \frac{1}{3}\sigma_{\theta S}\\ \sigma_{LS} &= 0.28\sigma_{\theta S} - 2.236(\Delta T)\end{aligned} \tag{e}$$

Substitution of Eqs. (e) into the last of Eqs. (d) yields for the tangential stress in the steel cylinder

$$\sigma_{\theta S} = 37.16p + 0.8639(\Delta T) \tag{f}$$

By Eqs. (a) and (f), we find the tangential stress in the aluminum cylinder to be

$$\sigma_{\theta A} = 50p - 37.16p - 0.8639(\Delta T) = 12.84p - 0.8639(\Delta T)$$

and by Eqs. (e) and (f), we find the longitudinal stresses in the aluminum and steel cylinders, respectively,

$$\sigma_{LA} = 4.28p - 1.779(\Delta T)$$
$$\sigma_{LS} = 10.40p - 1.994(\Delta T)$$

Thus, for $p = 689.4$ kPa and $\Delta T = 100$°C

$$\sigma_{\theta A} = -77.4 \text{ MPa}, \qquad \sigma_{LA} = -175 \text{ MPa}$$
$$\sigma_{\theta S} = 112 \text{ MPa}, \qquad \sigma_{LS} = -192 \text{ MPa}$$

EXAMPLE 3.8 Douglas Fir Stress–Strain Relations

Wood is generally considered to be an orthotropic material. For example, the elastic constants for Douglas fir (FPS, 1999), relative to material axes (x, y, z), are

$$\begin{gathered} E_x = 14{,}700 \text{ MPa}, \quad E_y = 1000 \text{ MPa}, \quad E_z = 735 \text{ MPa}\\ G_{xy} = 941 \text{ MPa}, \quad G_{xz} = 1147 \text{ MPa}, \quad G_{yz} = 103 \text{ MPa}\\ \nu_{xy} = 0.292, \quad \nu_{xz} = 0.449, \quad \nu_{yz} = 0.390 \end{gathered} \tag{a}$$

where the x axis is longitudinal (parallel to the grain), the y axis is radial (across the grain), and the z axis is tangent to the growth rings (across the grain).

At a point in a Douglas fir timber, the nonzero components of stress are

$$\sigma_{xx} = 7 \text{ MPa}, \quad \sigma_{yy} = 2.1 \text{ MPa}, \quad \sigma_{zz} = -2.8 \text{ MPa}, \quad \sigma_{xy} = 1.4 \text{ MPa} \tag{b}$$

(a) Determine the orientation of the principal axes of stress.

(b) Determine the strain components.

(c) Determine the orientation of the principal axes of strain.

Solution

(a) Since $\sigma_{xz} = \sigma_{yz} = 0$, the z axis is a principal axis of stress and $\sigma_{zz} = -2.8$ MPa is a principal stress. Therefore, the orientation of the principal axes in the (x, y) plane is given by Eq. 2.36, which is

$$\tan 2\theta = \frac{2\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})} = 0.5714 \tag{c}$$

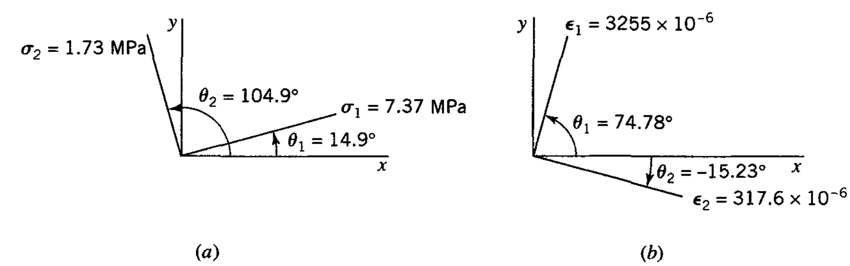

FIGURE E3.8 (*a*) Principal stress axes. (*b*) Principal strain axes.

Equation (c) yields $\theta = 14.9°$ or $\theta = 104.9°$. The maximum principal stress is $\sigma_1 = 7.37$ MPa and occurs in the direction $\theta = 14.9°$, and the intermediate principal stress $\sigma_2 = 1.73$ MPa occurs in the direction $\theta = 104.9°$ (see Figure E3.8*a*). As mentioned above, the minimum principal stress is $\sigma_3 = \sigma_{zz} = -2.8$ MPa.

(b) With the material constants in Eq. (a), we can write the stress–strain relations using Eqs. 3.51 and 3.52:

$$
\begin{aligned}
10^6 \times \epsilon_{xx} &= 68.0\sigma_{xx} - 19.9\sigma_{yy} - 30.6\sigma_{zz} \\
10^6 \times \epsilon_{yy} &= -19.9\sigma_{xx} + 1000\sigma_{yy} - 390\sigma_{zz} \\
10^6 \times \epsilon_{zz} &= -30.6\sigma_{xx} - 390\sigma_{yy} + 1361\sigma_{zz} \\
10^6 \times \gamma_{xy} &= 1063\sigma_{xy} \\
10^6 \times \gamma_{xz} &= 872\sigma_{xz} \\
10^6 \times \gamma_{yz} &= 9710\sigma_{yz}
\end{aligned}
\tag{d}
$$

Now with the stresses in Eq. (b), we can find the strains from Eq. (d) as

$$
\begin{aligned}
&\epsilon_{xx} = 520 \times 10^{-6}, \quad &&\epsilon_{yy} = 3053 \times 10^{-6}, \quad &&\epsilon_{zz} = -4844 \times 10^{-6} \\
&\gamma_{xy} = 1488 \times 10^{-6}, &&\gamma_{xz} = 0, &&\gamma_{yz} = 0
\end{aligned}
\tag{e}
$$

(c) Since $\gamma_{xz} = \gamma_{yz} = 0$, the z axis is a principal axis of strain. The orientation of the principal axes of strain in the (x, y) plane is given by

$$
\tan 2\theta = \frac{\gamma_{xy}}{(\epsilon_{xx} - \epsilon_{yy})} = -0.5875
\tag{f}
$$

Hence, $\theta = -15.22°$ or $\theta = 74.78°$. The maximum principal strain is $\epsilon_1 = 3255 \times 10^{-6}$ and occurs in the direction $\theta = 74.78°$, and the intermediate principal strain $\epsilon_2 = 317.6 \times 10^{-6}$ occurs in the direction $\theta = -15.22°$ (see Figure E3.8*b*). The minimum principal strain is $\epsilon_3 = -4844 \times 10^{-6}$, which is oriented along the z axis. Thus, the principal axes of stress and strain do not coincide, as they do for an isotropic material.

PROBLEMS

The problems for Chapter 3 generally require the use of stress–strain relations to determine principal stress, principal strains, maximum shear stresses, and directional strains. These quantities play important roles in failure theories and design specifications.

Sections 3.1–3.4

3.1. Table P3.1 lists principal strains that have been calculated for several points in a test of a machine part made of AISI–3140 steel (see Table A.1). Determine the corresponding principal stresses.

TABLE P3.1

Strain	Point 1	Point 2	Point 3	Point 4	Point 5
ϵ_1	0.008	0.006	−0.007	0.004	0.009
ϵ_2	−0.002	−0.003	−0.008	−0.005	0.002
ϵ_3	0	0	0	0	0

3.2. A wing of an airplane is subjected to a test in bending, and the principal strains are measured at several points on the wing surface (see Table P3.2). The wing material is aluminum alloy 7075 T6 (see Table A.1). Determine the corresponding principal stresses and the third principal strain.

TABLE P3.2

Strain	Point 1	Point 2	Point 3	Point 4	Point 5
ϵ_1	−0.004	0.008	0.006	−0.005	0.002
ϵ_2	−0.006	0.002	0.002	−0.008	−0.002

3.3. A square plate in the side of a ship with 800-mm sides parallel to the x and y axes has a uniform thickness h = 10 mm and is made of an isotropic steel (E = 200 GPa and ν = 0.29). The plate is subjected to a uniform state of stress. If $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$ (plane stress), $\sigma_{xx} = \sigma_1$ = 500 MPa, and $\epsilon_{yy} = 0$ for the plate, determine $\sigma_{yy} = \sigma_2$ and the final dimensions of the plate, assuming linearly elastic conditions.

3.4. The ship's plate in Problem 3.3 is subjected to plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$). If $\sigma_{xx} = \sigma_1$ = 500 MPa and $\epsilon_{xx} = 2\epsilon_{yy}$, determine the magnitude of $\sigma_{yy} = \sigma_2$ and $\sigma_{zz} = \sigma_3$, assuming linearly elastic conditions.

3.5. For an isotropic elastic medium subjected to a hydrostatic state of stress, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$, where p denotes pressure [FL^{-2}]. Show that for this state of stress $p = -Ke$, where $K = E/[3(1 - 2\nu)]$ is the bulk modulus and $e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ is the classical small-displacement cubical strain (also called the volumetric strain).

3.6. A triaxial state of principal stress acts on the faces of a unit cube of soil. Show that these stresses will not produce a volume change if ν = 0.5. Assume soil is a linearly elastic isotropic material. If $\nu \neq 0.5$, show that the condition necessary for the volume to remain unchanged is for $\sigma_1 + \sigma_2 + \sigma_3 = 0$.

3.7. An airplane wing is made of an isotropic linearly elastic aluminum alloy (E = 72.0 GPa and ν = 0.33). Consider a point in the free surface of the wing that is tangent to the (x, y) plane. If σ_{xx} = 250 MPa, σ_{yy} = –50 MPa, and σ_{xy} = –150 MPa, determine the directions for strain gages at that point to measure two of the principal strains. What are the magnitudes of these principal strains?

3.8. A bearing made of isotropic bronze (E = 82.6 GPa and ν = 0.35) is subjected to a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$). Determine σ_{zz}, ϵ_{xx}, ϵ_{yy}, and γ_{xy}, if σ_{xx} = 90 MPa, σ_{yy} = –50 MPa, and σ_{xy} = 70 MPa.

3.9. Solve Problem 3.3 for the condition that $\epsilon_{xx} = 2\epsilon_{yy}$.

3.10. A rectangular rosette (Figure 2.20b), is cemented to the free surface of an airplane wing made of an aluminum alloy 7075 T6 (see Appendix A). Under load, the strain readings are $\epsilon_a = \epsilon_{xx} = 0.00250$, $\epsilon_b = 0.00140$, $\epsilon_c = \epsilon_{yy} = -0.00125$.

a. Determine the principal stresses. Note that the stress components on the free surface are zero.

b. Show the orientation of the volume element on which the principal stresses in the plane of the rosette act.

c. Determine the maximum shear stress τ_{max}.

d. Show the orientation of the volume element on which τ_{max} acts.

3.11. The nonzero stress components at a point in a steel plate (E = 200 GPa and ν = 0.29) are σ_{xx} = 80 MPa, σ_{yy} = 120 MPa, and σ_{xy} = 50 MPa. Determine the principal strains.

3.12. Determine the extensional strain in Problem 3.11 in a direction 30° clockwise from the x axis.

3.13. A steel plate (E = 200 GPa and ν = 0.29) is subjected to a state of plane stress (σ_{xx} = –80 MPa, σ_{yy} = 100 MPa, and σ_{xy} = 50 MPa). Determine the principal stresses and principal strains.

3.14. In Problem 3.13, determine the extensional strain in a direction 20° counterclockwise from the x axis.

3.15. An airplane wing spar (Figure P3.15) is made of an aluminum alloy (E = 72 GPa and ν = 0.33), and it has a square cross section perpendicular to the plane of the figure. Stress components σ_{xx} and σ_{yy} are uniformly distributed as shown.

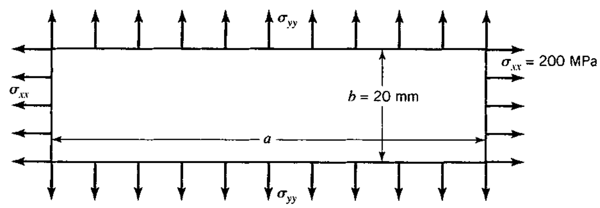

FIGURE P3.15

a. If σ_{xx} = 200 MPa, determine the magnitude of σ_{yy} so that the dimension b = 20 mm does not change under the load.

b. Determine the amount by which the dimension a changes.

c. Determine the change in the cross-sectional area of the spar.

3.16. Solve Example 3.7 for the case where p = 689.4 kPa is applied externally and ΔT = 100°C is a decrease in temperature. Discuss the results.

3.17. A rectangular rosette strain gage (Figure 2.20b) is cemented to the free surface of a machine part made of class 30, gray cast iron (Table A.1). In a test of the part, the following strains were recorded: ϵ_a = 0.00080, ϵ_b = 0.00010, and ϵ_c = 0.00040. Determine the stress components at the point on the surface with respect to the x–y axis shown.

3.18. The nonzero strain components at a critical point in an aluminum spar of an airplane (E = 72 GPa and ν = 0.33) are measured on a free surface as ϵ_{xx} = 0.0020, ϵ_{yy} = 0.0010, and ϵ_{xy} = 0.0010.

a. Determine the corresponding nonzero stress components.

b. A design criterion for the spar is that the maximum shear stress cannot exceed τ_{max} = 70 MPa. Is this condition satisfied for the measured strain state?

3.19. On the free surface of a bearing made of commercial bronze (half-hard; see Table A.1), the principal strains are determined to be ϵ_1 = 0.0015 and ϵ_2 = 0.0005. A design criterion for the bearing is that the maximum tensile stress not exceed 200 MPa. Is this criterion satisfied for the given strain state?

Section 3.5

3.20. The lamina of Example 3.6 is composed of glass fibers and an epoxy resin. The fibers have a modulus of elasticity $E_F = 72.4$ GPa, a shear modulus $G_F = 27.8$ GPa, and a Poisson ratio $\nu_F = 0.30$. The resin has a modulus of elasticity $E_R = 3.50$ GPa, a shear modulus $G_R = 1.35$ GPa, and a Poisson ratio $\nu_R = 0.30$. The volume fraction of fibers is $f = 0.70$.

a. Determine the coefficients C_{ij} of the lamina stress–strain relations [see Eqs. (m) and (n) of Example 3.6].

b. For a given load, the measured strain components were found to be

$$\epsilon_{xx} = 500\mu, \quad \epsilon_{yy} = 350\mu, \quad \gamma_{xy} = 1000\mu$$

Determine the principal stresses and the orientation of the principal axes of stress.

3.21. A member whose material properties remain unchanged (invariant) under rotations of 90° about axes (x, y, z) is called a *cubic material* relative to axes (x, y, z) and has three independent elastic coefficients (C_1, C_2, C_3). Its stress–strain relations relative to axes (x, y, z) are (a special case of Eq. 3.50)

$$\begin{aligned}
\sigma_{xx} &= C_1\epsilon_{xx} + C_2\epsilon_{yy} + C_2\epsilon_{zz}\\
\sigma_{yy} &= C_2\epsilon_{xx} + C_1\epsilon_{yy} + C_2\epsilon_{zz}\\
\sigma_{zz} &= C_2\epsilon_{xx} + C_2\epsilon_{yy} + C_1\epsilon_{zz}\\
\sigma_{xy} &= C_3\gamma_{xy}\\
\sigma_{xz} &= C_3\gamma_{xz}\\
\sigma_{yz} &= C_3\gamma_{yz}
\end{aligned}$$

Although in practice aluminum is often assumed to be an isotropic material ($E = 72$ GPa and $\nu = 0.33$), it is actually a cubic material with $C_1 = 103$ GPa, $C_2 = 55$ GPa, and $C_3 = 27.6$ GPa. At a point in an airplane wing, the strain components are $\epsilon_{xx} = 0.0003$, $\epsilon_{yy} = 0.0002$, $\epsilon_{zz} = 0.0001$, $\epsilon_{xy} = 0.00005$, and $\epsilon_{xz} = \epsilon_{yz} = 0$.

a. Determine the orientation of the principal axes of strain.

b. Determine the stress components.

c. Determine the orientation of the principal axes of stress.

d. Calculate the stress components and determine the orientation of the principal axes of strain and stress under the assumption that the aluminum is isotropic.

3.22. A birch wood log has the following elastic constants (FPS, 1999) relative to orthotropic axes (x, y, z):

$$\begin{aligned}
&E_x = 15{,}290 \text{ MPa}, \quad E_y = 1195 \text{ MPa}, \quad E_z = 765 \text{ MPa}\\
&G_{xy} = 1130 \text{ MPa}, \quad G_{xz} = 1040 \text{ MPa}, \quad G_{yz} = 260 \text{ MPa}\\
&\nu_{xy} = 0.426, \quad \nu_{xz} = 0.451, \quad \nu_{yz} = 0.697
\end{aligned}$$

where the x axis is longitudinal to the grain, the y axis is radial in the tree, and the z axis is tangent to the growth rings of the tree. The unit of stress is [MPa]. At a point in a birch log, the components of stress are $\sigma_{xx} = 7$ MPa, $\sigma_{yy} = 2.1$ MPa, $\sigma_{zz} = -2.8$ MPa, $\sigma_{xy} = 1.4$ MPa, and $\sigma_{xz} = \sigma_{yz} = 0$.

a. Determine the orientation of the principal axes of stress.

b. Determine the strain components.

c. Determine the orientation of the principal axes of strain.

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