(8) Plastic Behavior of Solids:

8.1 True Stress – True Strain Curve in Simple Tension:

The bulk of present day analysis in plasticity is predicted upon materials displaying idealized stress – strain curves as in the following figures,

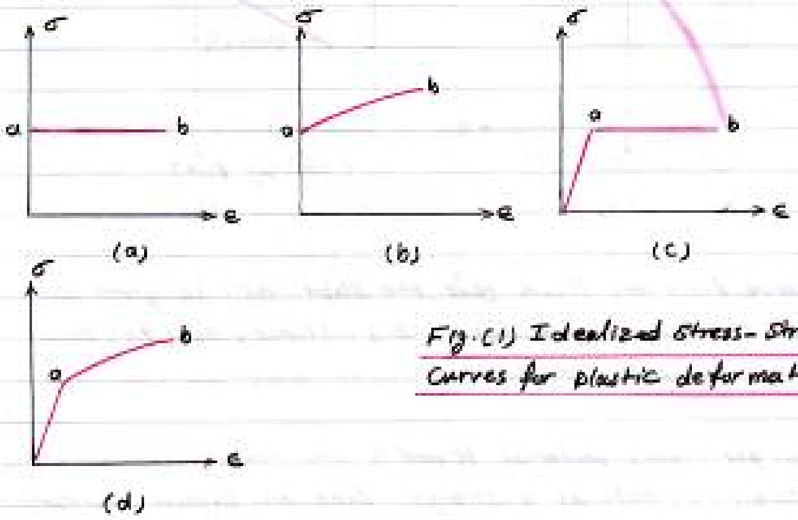


Fig. (1) Idealized Stress – Strain Curves for plastic deformation.

A general true stress – true strain curve may be represented by the empirical expression,

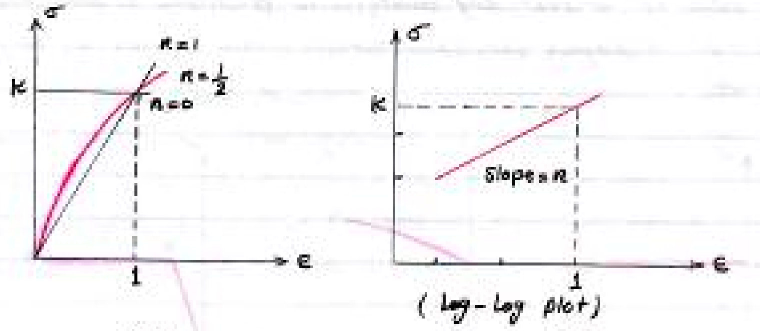
$$\sigma = \sigma_y + K\epsilon^n \qquad (8.1)$$

where n and K are termed the strain-hardening index and the strength coefficient. For $0 \leq n \leq 1$, the above form describes the material idealized in Fig. (1) b. For $\sigma_y = E\epsilon$, Eqn (8.1) represents an elastic – plastic stress as shown in the stress diagram of Fig. (1) d. Clearly, for $K = 0$, the above expression is represented by Fig. (1) a.

When no yielding occurs prior to plastic deformation (i.e. $\sigma_y = 0$), the true stress and the true strain are connected by a parabola,

$$\sigma = K\epsilon^{n} \qquad (8.2)$$

and the curves are as in the following figure,



(a) (b)

We observe from the figure that the slope $d\sigma / d\epsilon$ grows without limit as ϵ approach zero for $n \neq 1$. Hence, Equ (8.2) should not be used for small strains.

For a particular material K and n are readily evaluated inasmuch as Equ (8.2) plots as a straight line on logarithmic coordinates. We can thus rewrite Equ (8.2) in the form,

$$\log \sigma = \log K + n \log \epsilon \qquad (a)$$

Here n is the slope of the line and K the true stress associated with the true strain at 1.0 on the log-log plot. The strain-hardening coefficient n for commercially used materials falls between 0.2 and 0.5.

At the ultimate stress in a tensile test, an unstable flow results from the effects of strain hardening and the decreasing cross-sectional area of the specimen. These tend to weaken the material. When the rate of former effect is less than the latter, an instability occurs. This point corresponds to the maximum tensile load and is defined,

$$dP = 0 \qquad (b)$$

Since axial load P is a function of both the true stress and the area (i.e., $P = \sigma A$), the above is rewritten,

$$\sigma\, dA + A\, d\sigma = 0 \qquad (c)$$

The condition of incompressibility, $A_0 L_0 = AL$, also yields

$$L\, dA + A\, dL = 0 \qquad (d)$$

as the original volume $A_0 L_0$ is constant. Expressions (c) and (d) result in,

$$\frac{d\sigma}{\sigma} = \frac{dL}{L} = d\epsilon \qquad (e)$$

we thus obtain the relationships,

$$\frac{d\sigma}{d\epsilon} = \sigma \quad \text{or} \quad \frac{d\sigma}{d\epsilon_0} = \frac{\sigma}{1+\epsilon_0} = \sigma_0 \qquad (8.3)$$

for the instability of tensile member. Here the subscript o, denotes the engineering strain and stress.

Introduction of Equ (8.2) into Equ (8.3) results in,

$$\sigma = K\epsilon^n = \frac{d}{d\epsilon}(K\epsilon^n) = nK\epsilon^{n-1}$$

or,

$$\epsilon = n \qquad (8.4)$$

That is the instant of instability of flow in tension, the true strain ϵ has the same numerical value as the strain-hardening index. The state of true stress and the true strain under uniaxial tension are therefore,

$$\sigma_1 = K n^n \ , \quad \sigma_2 = \sigma_3 = 0$$

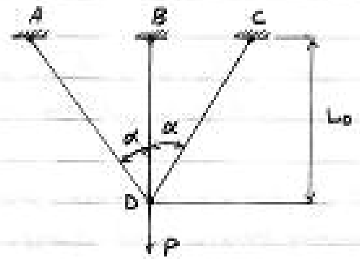
$$\epsilon_1 = n \ , \quad \epsilon_2 = \epsilon_3 = -\frac{n}{2} \qquad (f)$$

Example (8.1):

Determine the maximum allowable plastic stress and strain in the frame sustaining a vertical load P, as shown below. Assume that $\alpha = 45^{\circ}$ and that each element is constructed of an aluminum alloy with the following properties,

$$\sigma_y = 350 \text{ MPa} \ , \quad K = 840 \text{ MPa} \ , \quad n = 0.2 \ , \quad L_o = 3 \text{ m}$$

$$A_{AD} = A_{CD} = 10 \times 10^{-5} \text{ m}^2 \ , \quad A_{BD} = 15 \times 10^{-5} \text{ m}^2$$



Solution:

The frame is elastically statically indeterminate and the solution may be obtained as,

$$P = \sigma_y A_{BD} + 2\sigma_y A_{AD} \cos\alpha$$

$$= 350 \times 10^{6} \left[15 + 2 \times 10 \cos 45^{\circ} \right] \times 10^{-5}$$

$$= 101{,}990 \text{ N}$$

on Applying Eqns (f), the maximum allowable stress,

$$\sigma = K n^{n} = 840 \times 10^{6} (0.2)^{0.2} = 608.9 \text{ MPa}$$

occurs at the following axial and transvers strains,

$$\epsilon_1 = n = 0.2 \;, \quad \epsilon_2 = \epsilon_3 = -0.1$$

For example, the total elongation for instability of the central bar is thus,

$$3(0.2) = 0.6 \text{ m}$$

Example (8.2):

A tube of original mean diameter d_0 and thickness t_0 is subjected to axial tensile loading. Assume a true stress – engineering strain relation of the form $\sigma = K_1 \epsilon_0^{n}$ and derive expressions for thickness and diameter at the instant of instability. Let $n = 0.3$.

Solution:

Differentiating the given expression for stress,

$$\frac{d\sigma}{d\epsilon_0} = n K_1 \epsilon_0^{n-1} = \frac{n\sigma}{\epsilon_0}$$

This result and Equ(8.3) yield the engineering axial strain at instability,

$$\epsilon_0 = \frac{n}{1-n}$$

The transvers strains are $-\epsilon_0/2$, and hence the decrease of wall thickness equals $n t_0 / 2(1-n)$. The thickness at instability is thus,

$$t = t_0 - \frac{n t_0}{2(1-n)} = \frac{2-3n}{2(1-n)} t_0 \qquad (g)$$

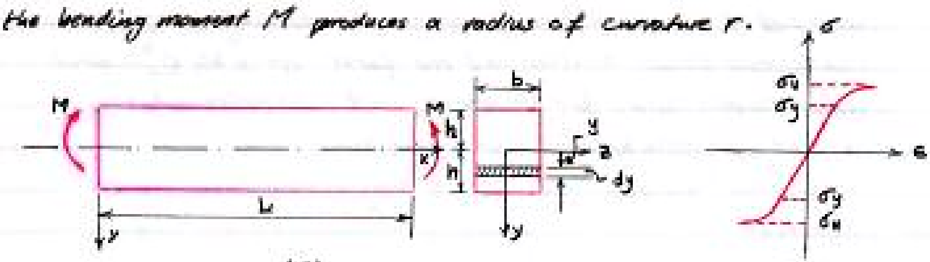
Similarly, the diameter at instability is,

$$d = d_0 - \frac{n d_0}{2(1-n)} = \frac{2-3n}{2(1-n)} d_0 \qquad (h)$$

From eqns (g) and (h), with $n = 0.3$, we have $t = 0.79 t_0$ and $d = 0.79 d_0$. Thus, for the tube under axial tension, the diameter and thickness decreases approximately 21% at the instant of instability.

8.2 Theory of Plastic Bending:

Consider a beam of rectangular section, as shown below, wherein the bending moment M produces a radius of curvature r.



(a) (b)

The longitudinal strain of any fiber located a distance y from the neutral surface, is given by,

$$\epsilon = \frac{y}{r} \qquad (8.5)$$

Owing to the distribution of stress shown, the longitudinal tensile and compressive forces cancel, and the equilibrium of axial forces is satisfied. The following describes the equilibrium of moments about the z-axis,

$$\int_A y\sigma \, dA = b\int_{-h}^{h} y\sigma \, dy = M \qquad (a)$$

Consider the true stress – true strain relationship of the form,

$$\sigma = K\epsilon^n$$

Introducing the above, together with Equ (8.5), into Equ(a), we obtain,

$$M = b\int_{-h}^{h} \frac{1}{r^n} K y^{n+1}\, dy = \frac{1}{r^n} K\, In \qquad (b)$$

where,

$$In = b\int_{-h}^{h} y^{n+1}\, dy \qquad (8.6)$$

From Equ (8.5), and $\sigma = K\epsilon^n$, and equ (b) the following is derived,

$$\frac{\sigma}{y^n} = \frac{K}{r^n} = \frac{M}{In} \qquad (c)$$

In addition, on the basis of elementary beam theory, we have, from previous analysis,

$$\frac{1}{r} = \frac{d^2v}{dx^2} \qquad (d)$$

Upon substitution Equ(c) into Equ(d), we obtain the following equation for a rigid plastic beam,

$$\frac{d^2v}{dx^2} = \left(\frac{M}{KIn}\right)^{1/n} \qquad (8.7)$$

It is noted that when $n = 1$ and, hence $K = E$, this expression reduces to that of an elastic beam.

Example (8.3):

Determine the deflection of a rigid plastic simply supported beam subjected to a downward concentrated force P at its midlength. The beam has a rectangular cross section of depth 2h and width b. The span length is L.

Solution:

The bending moment at any section is given by,

$$M = -\frac{1}{2}Px \qquad (e)$$

where the minus sign is due to sign convention.

Substituting Eqn(e) into Eqn(8.7) and integrating, one achieves

$$\frac{dv}{dx} = \frac{\lambda x^{(1/n)+1}}{(\frac{1}{n})+1} + C_1$$

$$v = \frac{\lambda x^{(\frac{1}{n})+2}}{[(\frac{1}{n})+1][(\frac{1}{n})+2]} + C_1 x + C_2 \qquad (f)$$

where,

$$\lambda = \left(\frac{P}{2KI_n}\right)^{\frac{1}{n}} \qquad (g)$$

The constants of equilibrium C_1 and C_2 depend upon the boundary conditions $v(0) = dv/dx(L/2) = 0$,

$$C_2 = 0, \quad C_1 = \frac{\lambda (L/2)^{(\frac{1}{n})+1}}{(\frac{1}{n})+1}$$

Upon introduction of C_1 and C_2 into Eqn (f), the beam deflection is found to be,

$$\boxed{v = \frac{\lambda}{(\frac{1}{n})+1}\left[\frac{x^{(\frac{1}{n})+2}}{(\frac{1}{n})+2} - \left(\frac{L}{2}\right)^{(\frac{1}{n})+1} x\right]} \qquad (8.8)$$

Interestingly, in the case of an elastic beam, the above expression, becomes:

$$v = \frac{P}{4EI}\left(\frac{x^3}{3} - \frac{L^2}{4}x\right)$$

For $x = L/2$, the familiar result is,

$$v = v_{max} = \frac{PL^3}{48EI}$$

The foregoing procedure is applicable to the determination of the deflection of beams subjected to a variety of end conditions and load configurations.

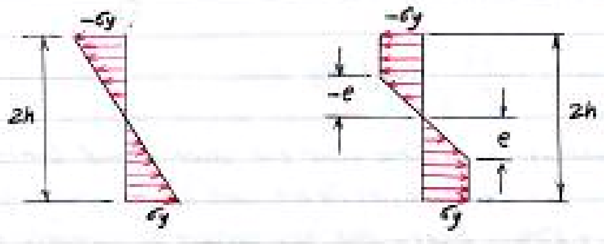
8.3 Analysis of Perfectly Plastic Beams:

In this section, our attension will be focus on the analysis of a perfectly plastic straight beam of rectangular section subjected to pure bending.

The bending moment at which plastic deformation impends, M_y, may be found directly from the flexure relationship,

$$M_y = \frac{\sigma_y I}{h} = \frac{2}{3} b h^2 \sigma_y \qquad (8.9)$$

Here σ_y represents the stress at which yielding begins.



(a) (b)

The stress distribution corresponding to M_y, assuming identical material properties in tension and compression, is shown in Fig.(a). As the bending moment is increased, the region of the beam which yielded progresses in toward the neutral surface, is shown in Fig.(b). The distance from the neutral surface to the point at which yielding begins denoted by the symbol e as shown.

It is clear, from Fig.(b), that the normal stress varies in accordance with the relations,

$$\sigma_x = \frac{\sigma_y y}{e} \qquad (-e \leq y \leq e) \qquad (a)$$

and,

$$\sigma_x = \sigma_y \qquad (e \le y \le h)$$
$$\sigma_x = -\sigma_y \qquad (-e \ge y \ge -h) \qquad (b)$$

It will be useful to determine the manner in which the bending moment M relates to the distance e. To do this, we begin with a statement of the x equilibrium of forces,

$$\int_{-h}^{-e} -\sigma_y \, b \, dy + \int_{-e}^{e} \sigma_x \, b \, dy + \int_{e}^{h} \sigma_y \, b \, dy = 0$$

Canceling the first and third integrals and combining the remaining integral with Equ(A), we have,

$$\frac{\sigma_y}{e} \int_{-e}^{e} y \, b \, dy = 0$$

The above expression indicates that the neutral and centroidal axes of the cross section coincide, as in the case of an entirely elastic distribution of stress. Next, the equilibrium of moments about the neutral axis provides the following relation,

$$\int_{-h}^{-e} -\sigma_y \, y \, b \, dy + \int_{-e}^{e} \sigma_x \, y \, b \, dy + \int_{e}^{h} \sigma_y \, y \, b \, dy = M$$

substituting σ_x from equ (A) into the above gives, after integration,

$$M = b \sigma_y \left(h^2 - \frac{e^2}{3} \right) \qquad (8.10)$$

For the case in which $e = h$. Equ (8.10) reduces to Equ (8.9) and $M = M_y$. For $e = 0$ which appyes to a totally plastic beam, Equ (8.10), becomes,

$$M_u = b h^2 \sigma_y \qquad (8.11)$$

where M_u is the ultimate moment.

In general, for any cross section the plastic or ultimate resisting moment for a beam is,

$$M_u = \sigma_y Z \qquad (8.12)$$

Where Z is the plastic section modulus. Clearly for rectangular beam analyzed above, $Z = bh^2$.

To express the beam curvature in terms of the yield stress we begin by noting that the longitudinal strain given by Eqn (8.5), combined with Hooke's Law, leads to,

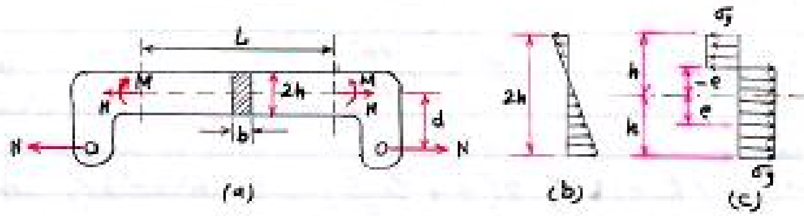
$$\sigma_y = E \epsilon_y = E \frac{e}{r}$$

The curvature and yield stress are thus related by,

$$\frac{1}{r} = \frac{\sigma_y}{Ee} \qquad (8.13)$$

Example (8.4):

A link of rectangular cross section is subjected to a load N, as shown below. Derive general relationships involving N and M which govern first the case of initial yielding, and then fully plastic deformation, for the straight part of the link of length L...



Solution:

Suppose N and M are such that the state of stress is as shown in Fig.(c) above at any straight beam section. The maximum stress in the beam is then, by superposition of the axial and bending stresses,

$$\sigma_y = \frac{N_1}{2hb} + \frac{3}{2}\frac{M_1}{bh^2} \qquad (c)$$

The upper limits on N ($M=0$) and M ($N=0$), corresponding to the condition of yielding are,

$$N_y = 2hb\sigma_y \quad , \quad M_y = \frac{2}{3}bh^2\sigma_y \qquad (d)$$

Substituting $2hb$ and I/h from the above into eq.(c) and rearrange terms, we have,

$$\boxed{\frac{N_1}{N_y} + \frac{M_1}{M_y} = 1} \qquad (8.14)$$

If N_1 is zero, then M_1 must achieve its maximum value M_y for yielding to impend. Similarly, for $M_1 = 0$, it is necessary for N_1 to equal N_y to initiate yielding.

For fully plastic case (Fig.(c)), we shall denote the state of loading by N_2 and M_2. It is apparent that stress acting within the range $-e<y<e$ contribute pure axial load only. The stresses within the range $e<y<h$ and $-e>y>-h$ form a couple, however. For the total load system described, we may write,

$$N_2 = 2eb\sigma_y \quad , \quad e = \frac{N_2}{2b\sigma_y} \qquad (e)$$

$$M_2 = (h-e)\,b\,\sigma_y \cdot 2\left(e + \frac{h-e}{2}\right) = b(h^2 - e^2)\sigma_y \qquad (f)$$

Introducing Eqns (8.11) and (e) into the above expression, one has,

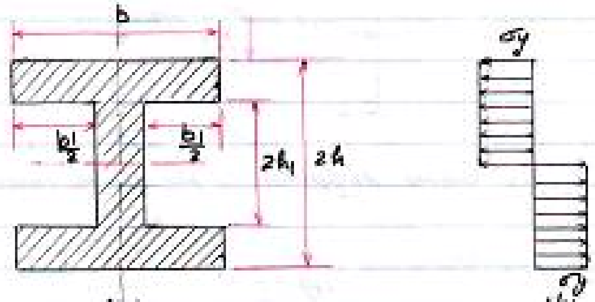
$$M_z = M_u - \frac{N_z^2}{4b\sigma_y}$$

Finally, divided by M_u and noting that $M_u = \frac{3}{2} M_y = bh^2\sigma_y$, $N_y = 2bh\sigma_y$, we obtain,

$$\boxed{\frac{2}{3}\frac{M_z}{M_y} + \left(\frac{N_z}{N_y}\right)^2 = 1} \qquad (8.15)$$

Example (8.6):

An I-beam, shown below, is subjected to pure bending resulting from end couples. Determine the moment causing initial yielding and that resulting in complete plastic deformation.



(a) (b)

Solution:

The moment corresponding to σ_y is, from Eqn (8.4),

$$M_y = \frac{I}{h}\sigma_y = \frac{1}{h}\left(\frac{2}{3}bh^3 - \frac{2}{3}b_1h_1^3\right)\sigma_y \qquad (g)$$

Refer now to the completely plastic stress distribution of Fig. (b). The moments of force owing to σ_y, taken about the neutral axis, provide,

$$M_u = (bh^2 - b_1h_1^2)\sigma_y \qquad (h)$$

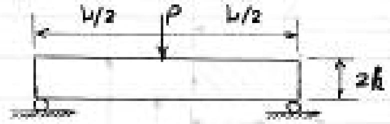
Combining Eqn (g) and (h), we have,

$$\frac{M_u}{M_y} = \frac{3}{2}\,\frac{1-(b_1h_1^2/bh^2)}{1-(b_1h_1^3/bh^3)}$$

From the above expression, it is seen that $M_u/M_y < \frac{3}{2}$, while it is $\frac{3}{2}$ for a beam of rectangular section ($h_1 = 0$). We conclude therefore that if a rectangular beam and an I-beam are designed plastically, the former will be more resistant to complete plastic failure.

Example (8.6):

Determine the maximum deflection due to an applied force P acting on the simply supported rectangular beam, as shown below.



Solution:

The centre deflection in the elastic range is given by,

$$v_{max} = \frac{PL}{48EI} \qquad (i)$$

At the start of yielding,

$$P = P_y\ ,\quad M_{max} = M_y = \frac{P_y L}{4} \qquad (j)$$

Expression (i), together with Eq. (j) and Eq. (8.9), leads to

$$v_{max} = \frac{M_y L^2}{12EI} = \frac{1}{12}\left(\frac{L^2}{Eh}\right)\sigma_y \qquad (k)$$

In a like manner, we obtain,

$$v_{max} = \frac{M_u L^2}{12EI} = \frac{1}{8}\left(\frac{L^2}{Eh}\right)\sigma_y \qquad (l)$$

for the centre deflection at the instant of plastic collapse.

8.4 Elastic-Plastic Stresses in Rotating Disks:

This section treats the stresses in a flat disk fabricated of a perfectly plastic material, rotating at constant angular velocity. The maximum elastic stresses for this geometry are,

For the solid disk at $r = 0$,

$$\sigma_\theta = \sigma_r = \frac{\rho\omega^2(3+\nu)b^2}{8} \qquad (a)$$

For the annular disk, at $r = a$,

$$\sigma_\theta = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right) \qquad (b)$$

Here a and b represent the inner and outer radii, respectively, ρ the mass density, and ω the angular speed.

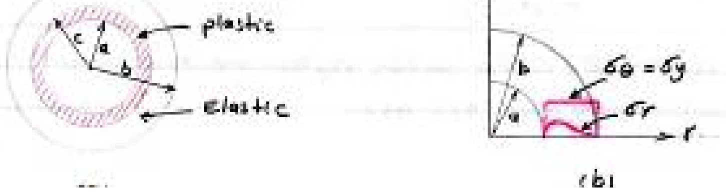
Initial Yielding:

According to the Tresca yield condition, yielding impends when the maximum stress is equal to the yield stress. Denoting the critical speed ω_c and using $\nu = \frac{1}{3}$, we have, from Eq. (b),

$$\omega_c = \left(\frac{6}{5b^2+a^2}\cdot\frac{\sigma_y}{\rho}\right)^{1/2} \qquad (8.16)$$

Partial Yielding:

For angular speeds in excess of ω_c, but lower than speeds resulting in total plasticity, the disk contains both an elastic and plastic region, as shown in the figure below,



(b)

In the plastic range, the equation of radial equilibrium, with σ_y replacing the maximum stress σ_θ, becomes,

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_y + \rho w^2 r^2 = 0 \qquad (8.17)$$

or

$$\frac{d}{dr}(r\sigma_r) - \sigma_y + \rho w^2 r^3 = 0$$

The solution of the above is given by,

$$r\sigma_r - \sigma_y r + \frac{\rho w^2 r^3}{3} + C_1 = 0 \qquad (c)$$

By satisfying the boundary condition $\sigma_r = 0$ at $r = a$, Equ (c) provides an expression for the constant C_1, which when introduced above results in,

$$r\sigma_r - \sigma_y (r-a) + \frac{\rho w^2}{3}(r^3 - a^3) = 0 \qquad (d)$$

The stress within the plastic region is now determined by letting $r = c$ in equ (d),

$$\sigma_c = \frac{\rho w^2}{3}\cdot\frac{a^3 - c^3}{c} + \frac{c-a}{c}\sigma_y \qquad (8.18)$$

Reffering to the elastic region, the distribution of stress is determind from $\sigma_r = \sigma_c$ at $r = c$, and $\sigma_r = 0$ at $r = b$. Applying these conditions, we obtain,

$$C_1 = -\frac{c^2(1-\nu)\sigma_c}{E(b^2-c^2)} + \frac{\rho w^2(1-\nu)(3+\nu)}{8E}(b^2+c^2)$$

$$(e)$$

$$C_2 = -\frac{b^2c^2(1+\nu)\sigma_c}{E(b^2-c^2)} + \frac{\rho w^2(1+\nu)(3+\nu)\,b^2c^2}{8E}$$

The stresses in the outer region are then obtained by substituting Equs (e), as,

$$\sigma_r = \frac{c^2}{b^2-c^2}\left(-1+\frac{b^2}{r^2}\right)\sigma_c + \frac{\rho\omega^2}{8}(3+\nu)\left(b^2+c^2-\frac{b^2c^2}{r^2}-r^2\right) \quad \text{—(8.19)a}$$

$$\sigma_\theta = -\frac{c^2}{b^2-c^2}\left(1+\frac{b^2}{r^2}\right)\sigma_c + \frac{\rho\omega^2}{8}(3+\nu)\left(b^2+c^2-\frac{1+3\nu}{3+\nu}r^2+\frac{b^2c^2}{r^2}\right) \quad \text{—(8.19)b}$$

In order to determine that value of ω which causes yielding up to radius C, one need only substitute σ_θ for σ_y above, and introduce σ_c as given by Eqn (8.18).

Complete Yielding:

We turn finally to a determination of the speed ω_1 at which the disk becomes fully plastic. First Eqn(c) is rewritten,

$$\sigma_r - \sigma_y + \frac{\rho\omega^2r^2}{3} + \frac{C_1}{r} = 0 \qquad (f)$$

Applying the boundary conditions, $\sigma_r = 0$ at $r=a$ and $r=b$ in Eqn (f), we have,

$$C_1 = a\sigma_y - \frac{\rho^2\omega^2a^3}{3} \qquad (g)$$

and the critical speed ($\omega = \omega_1$) is given by

$$\omega_1 = \left(\frac{3\sigma_y}{\rho} - \frac{b-a}{b^3-a^3}\right)^{1/2} \qquad (8.20)$$

Substituting of Eqn (g) and (8.20) into eqn (f) provides the radial stress in a fully plastic disk,

$$\sigma_r = \left(1-\frac{a}{r}-\frac{1-a^2/b^2}{1-a^3/b^3}-\frac{r^2}{b^2}\;\frac{1-a/b}{1-a^3/b^3}\right)\sigma_y \qquad (8.21)$$

The distribution of radial and tangential stress are plotted in Fig. (b) above.

8.5 Plastic Stress-Strain Relations:

Consider an element subjected to true stresses $\sigma_1, \sigma_2, \sigma_3$ with corresponding true straining. The true strain, which is plastic, is denoted $\epsilon_1, \epsilon_2, \epsilon_3$. A simple way to derive expressions relating true stress and strain is to replace the elastic constants E and ν by E_s and $\frac{1}{2}$, respectively. In so doing, we obtain equations of the total strain theory or the deformational theory, also known as Hencky's plastic stress-strain relations,

$$\epsilon_1 = \frac{1}{E_s}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] \qquad (8.22)\,a$$

$$\epsilon_2 = \frac{1}{E_s}\left[\sigma_2 - \frac{1}{2}(\sigma_1+\sigma_3)\right] \qquad (8.22)\,b$$

$$\epsilon_3 = \frac{1}{E_s}\left[\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)\right] \qquad (8.22)\,c$$

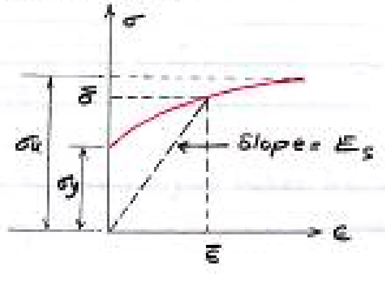
The forgoing may be restated,

$$\frac{\epsilon_1}{\sigma_1-\frac{1}{2}(\sigma_2+\sigma_3)} = \frac{\epsilon_2}{\sigma_2-\frac{1}{2}(\sigma_1+\sigma_3)} = \frac{\epsilon_3}{\sigma_3-\frac{1}{2}(\sigma_1+\sigma_2)} = \frac{1}{E_s} \qquad (8.23)$$

Here E_s, a function of the state of plastic stress, is termed the modulus of plasticity or secant-modulus. It is defined by,

$$E_s = \frac{\bar{\sigma}}{\bar{\epsilon}} \qquad (8.24)$$

in which the quantities $\bar{\sigma}$ and $\bar{\epsilon}$ are the effective stress and the effective strain, respectively.



While other yield theories may be employed to determine $\bar{\sigma}$, the maximum energy of distortion or Mises theory is most suitable. According to the Mises theory the following relationship connects the uniaxial yield stress to the general state of stress at a point,

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \qquad (8.25)$$

The above equation represents the logical extension of the yield condition to describe plastic deformation after the yield stress is exceeded. Collecting terms of Eqns (8.22) and (8.23), we have,

$$\left[\frac{2}{3} (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) \right]^{1/2} = \frac{1}{\sqrt{2}\, E_s} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \quad (8.26)$$

The foregoing, together with Eqn (8.24) and (8.25), leads to definition,

$$\bar{\epsilon} = \left[\frac{2}{3} (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) \right]^{1/2} \qquad (8.27)a$$

or (on the basis of $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$) in different form,

$$\bar{\epsilon} = \frac{\sqrt{2}}{3} \left[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2 \right]^{1/2} \qquad (8.27)b$$

relating the effective plastic strain and the true strain components.

Hencky's equations as they appear in Eqn (8.22) and Eqn (8.23) have little utility. In order to give these expressions generality and convert them to a more convenient form, it is useful to employ the empirical relationship, given before,

$$\bar{\sigma} = K (\bar{\epsilon})^n$$

from which,

$$\frac{\bar{\epsilon}}{\bar{\sigma}} = \frac{(\bar{\sigma})^{\frac{1}{n} - 1}}{K^{\frac{1}{n}}} \qquad (a)$$

The true stress-strain relations, upon substitution of Eqs (8.26) and (a) into (8.28), then assume the following more useful form,

$$\epsilon_1 = \left(\frac{\sigma_1}{k}\right)^{\frac{1}{n}} \left[\alpha^2 - \beta^2 - \alpha\beta - \alpha - \beta + 1\right]^{\frac{1-n}{2n}} \left(1 - \frac{\alpha}{2} - \frac{\beta}{2}\right) \qquad (8.28)a$$

$$\epsilon_2 = \left(\frac{\sigma_1}{k}\right)^{\frac{1}{n}} \left[\alpha^2 - \beta^2 - \alpha\beta - \alpha - \beta + 1\right]^{\frac{1-n}{2n}} \left(\alpha - \frac{\beta}{2} - \frac{1}{2}\right) \qquad (8.28)b$$

$$\epsilon_3 = \left(\frac{\sigma_1}{k}\right)^{\frac{1}{n}} \left[\alpha^2 - \beta^2 - \alpha\beta - \alpha - \beta + 1\right]^{\frac{1-n}{2n}} \left(\beta - \frac{\alpha}{2} - 1\right) \qquad (8.28)c$$

Where $\alpha = \sigma_2/\sigma_1$ and $\beta = \sigma_3/\sigma_1$.

Example (8.7):

A thin-walled cylinder of initial radius r_0 is subjected to internal pressure P. Assume that the values of r_0, P, and the material properties (k and n) are given. Apply Hencky's relations to determine (a) the maximum allowable stress and (b) the initial thickness t_0 for the cylinder to become unstable at internal pressure P.

Solution:

The current radius, thickness, and the length are denoted by r, t, and L, respectively. In the plastic range, the hoop, axial, and radial stresses are,

$$\sigma_\theta = \sigma_1 = \frac{Pr}{t}, \quad \sigma_z = \sigma_2 = \frac{Pr}{2t}, \quad \sigma_r = \sigma_3 = 0 \qquad (b)$$

We thus have $\alpha = \sigma_2/\sigma_1$ and $\beta = 0$ in Eqns (8.28).

Corresponding to the above stresses, the components of true strain, are from,

$$\epsilon_\theta = \epsilon_1 = \ln \frac{r}{r_0}$$

$$\epsilon_z = \epsilon_2 = \ln \frac{h}{h_0} \qquad (c)$$

$$\epsilon_t = \epsilon_3 = \ln \frac{t}{t_0}$$

Based upon the constancy of volume, we have,

$$\epsilon_3 = -(\epsilon_1 + \epsilon_2) = \ln \frac{t}{t_0}$$

or

$$t = t_0 e^{-\epsilon_1 - \epsilon_2} = t_0 e^{\epsilon_3} \qquad (d)$$

The first of Eqns (c) gives

$$r = r_0 e^{\epsilon_1} \qquad (e)$$

The tangential stress, the first of Eqns (b), is therefore,

$$\sigma_1 = P r_0 e^{\epsilon_1} \frac{1}{t_0 e^{-\epsilon_1 - \epsilon_2}}$$

from which,

$$P = \sigma_1 \left(\frac{t_0}{r_0}\right) e^{-\epsilon_1 \left(2 + \frac{\epsilon_2}{\epsilon_1}\right)} \qquad (f)$$

Simultaneous solution of Eqns (8-28) leads readily to:

$$\frac{\epsilon_1}{\epsilon_2} = \frac{2-\alpha}{2\alpha - 1} \qquad (g)$$

Equation (f) then appears as,

$$P = \sigma_1 \left(\frac{t_0}{r_0}\right) e^{-3\epsilon_1/(2-\alpha)} \qquad (h)$$

For material instability,

$$dP = \frac{\partial P}{\partial \sigma_1} d\sigma_1 + \frac{\partial P}{\partial \epsilon_1} d\epsilon_1 = 0$$

Which upon substitution of $\partial P/\partial \sigma_1$ and $\partial P/\partial \epsilon_1$ derived from eqs (h), becomes,

$$dP = \left(\frac{t_0}{r_0}\right) e^{-3\epsilon_1/(2-\alpha)} d\sigma_1 + \sigma_1 \left(\frac{t_0}{r_0}\right) e^{-3\epsilon_1/(2-\alpha)} \left(-\frac{3}{2-\alpha}\right) d\epsilon_1 = 0$$

or,

$$\frac{d\sigma_1}{d\epsilon_1} = \sigma_1 \left(\frac{3}{2-\alpha}\right) \qquad (i)$$

In eqns (8.28) it is observed that σ_1 depends upon α and ϵ_1^n. That is,

$$\sigma = f(\alpha)\, \epsilon_1^n \qquad (j)$$

Deferentiating the above, we have,

$$\frac{d\sigma_1}{d\epsilon_1} = n f(\alpha)\, \epsilon_1^{n-1} \qquad (k)$$

Expressions (i) and (j) and (k) lead to the instability condition,

$$\boxed{\epsilon_1 = \left(\frac{2-\alpha}{3}\right) n} \qquad (8.29)$$

(a) Equating expressions (8.29) and (8.28)a, we obtain

$$\left(\frac{2-\alpha}{3}\right) n = \left(\frac{\sigma_1}{K}\right)^{\frac{1}{n}} (\alpha^2-\alpha+1)^{\frac{(1-n)}{2n}} \left(\frac{2-\alpha}{2}\right)$$

and the true tangential stress is thus,

$$\boxed{\sigma_1 = K \left(\frac{2n}{3}\right)^n \left(\frac{1}{\alpha^2-\alpha+1}\right)^{\frac{(1-n)}{2}}} \qquad (8.30)$$

(b) on the other hand Equs (b), (d), (e), and (8.28) yields,

$$\sigma_1 = \frac{P r_0}{t_0} \frac{e^{(\frac{\sigma_1}{K})^{\frac{1}{n}}(\alpha^2-\alpha+1)^{(1-n)/2n}\,[(2-\alpha)/2]}}{e^{(\frac{\sigma_1}{K})^{\frac{1}{n}}(\alpha^2-\alpha+1)^{(1-n)/2n}\,(-1-\alpha)}}$$

From which, the required original thickness is found to be,

$$\boxed{t_0 = \frac{P r_0}{\sigma_1} e^{\frac{\alpha-2}{2(1+\alpha)}}} \qquad (8.31)$$

wherein σ_1 is given by Equ (8.30).

In the case under consideration, $\alpha = \frac{1}{2}$ and Equs (8.29), (8.30), and (8.31) thus became,

$$\epsilon_1 = \frac{n}{2} \qquad (8.32)a$$

$$\sigma_1 = K\left(\frac{2n}{3}\right)^n \left(\frac{4}{3}\right)^{(1-n)/2} = \frac{2K}{\sqrt{3}}\left(\frac{n}{\sqrt{3}}\right)^n \qquad (8.32)b$$

$$t_0 = \frac{0.606\, P r_0}{\left(\frac{2K}{\sqrt{3}}\right)\left(\frac{n}{\sqrt{3}}\right)^n} \qquad (8.32)c$$

For a thin walled spherical shell under internal pressure, the two principal stresses are equal and hence $\alpha = 1$. Equs (8.29), (8.30), and (8.31) then reduce to,

$$\boxed{\epsilon_1 = \frac{n}{3}, \quad \sigma_1 = K\left(\frac{2n}{3}\right)^n, \quad t_0 = \frac{0.717\, P r_0}{K\left(\frac{2n}{3}\right)^n}} \qquad (8.33)$$

It is significant that for loading situation in which components of stress do not increase continuously, Hencky's equations provide results which are somewhat in error and the incremental theory must be used.

8.6 Plastic Stress–Strain Increment Relations:

The incremental theory offers another approach, treating not the total strain associated with a state of stress, but rather the increment of strain.

Suppose now that the true stresses at a point experience very small changes in magnitude $d\sigma_1, d\sigma_2, d\sigma_3$. As a consequence of these increments, the effective stress $\bar{\sigma}$ will be altered by $d\bar{\sigma}$ and the effective strain $\bar{\epsilon}$ by $d\bar{\epsilon}$. The plastic strains thus suffer increments $d\epsilon_1, d\epsilon_2, d\epsilon_3$.

The following modification of Hencky's equations, due to Lévy and Mises, describe the foregoing giving good results in metals,

$$d\epsilon_1 = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] \qquad (8.34)\text{a}$$

$$d\epsilon_2 = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_2 - \frac{1}{2}(\sigma_1+\sigma_3)\right] \qquad (8.34)\text{b}$$

$$d\epsilon_3 = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)\right] \qquad (8.34)\text{c}$$

An alternate form of the above is,

$$\frac{d\epsilon_1}{\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)} = \frac{d\epsilon_2}{\sigma_2 - \frac{1}{2}(\sigma_1+\sigma_3)} = \frac{d\epsilon_3}{\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)} = \frac{d\bar{\epsilon}}{\bar{\sigma}} \qquad (8.34)\text{d}$$

The plastic strain is, as before, assumed to occur at constant volume,

$$d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$$

The effective strain increment, reffering to Eqn (8.27)b may be, written,

$$d\bar{\epsilon} = \frac{\sqrt{2}}{3}\left[(d\epsilon_1-d\epsilon_2)^2+(d\epsilon_2-d\epsilon_3)^2+(d\epsilon_3-d\epsilon_1)^2\right]^{\frac{1}{2}} \qquad (8.35)$$

In a particular case of straining of sheet metal under biaxial tension the Levy-Mises equations (8.34) become,

$$\frac{d\epsilon_1}{2-\alpha} = \frac{d\epsilon_2}{2\alpha-1} = -\frac{d\epsilon_3}{1+\alpha} \qquad (8.36)$$

where $\alpha = \sigma_2/\sigma_1$ and σ_3 taken as zero. The effective stress and strain increment Eqns (8.26) and (8.35), is now written,

$$\bar{\sigma} = \sigma_1 (1-\alpha+\alpha^2)^{1/2} \qquad (8.37)$$

$$d\bar{\epsilon} = d\epsilon_1 \left(\frac{2}{2-\alpha}\right)(1-\alpha+\alpha^2)^{1/2} \qquad (8.38)$$

Combining Eqns (8.36) and (8.38) and integrating yields,

$$\frac{\bar{\epsilon}}{2(1-\alpha+\alpha^2)^{1/2}} = \frac{\epsilon_1}{2-\alpha} = \frac{\epsilon_2}{2\alpha-1} = \frac{\epsilon_3}{1+\alpha} \qquad (8.39)$$

To generalize the above result, it is useful to employ Eqn (8.2), $\bar{\sigma} = K(\bar{\epsilon})^n$, to include strain hardening characteristics. Differentiating this expression, we have,

$$\frac{d\bar{\sigma}}{d\bar{\epsilon}} = \frac{n\bar{\sigma}}{\bar{\epsilon}} \qquad (8.40)$$

Note that for simple tension, $n = \bar{\epsilon} = \epsilon$ and Eqn (8.40) reduces to Eqn (8.3).

In closing, we note that the total (elastic-plastic) strains are determined by adding the elastic strains to the plastic strains. The elastic-plastic strain relations together with the equations of equilibrium and appropriate boundary conditions, completely describe a given situation.

Example (8.8):

Redo Example (8.7) employing the Lévy-Mises Stress-Strain increment relations.

Solution:

For the thin-walled cylinder under internal pressure, the plastic stresses are,

$$\sigma_\theta = \sigma_1 = \frac{Pr}{t} \quad , \sigma_z = \sigma_2 = \frac{Pr}{2t} \quad , \sigma_r = \sigma_3 = 0 \qquad (a)$$

Where r and t are the current radius and the thickness. At instability,

$$dP = \frac{\partial P}{\partial \sigma_z} d\sigma_z + \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial t} dt = 0 \qquad (\text{Eq } 1,2) \qquad (b)$$

As $\alpha = \frac{1}{2}$, introduction of Eqs(a) into the above, provides,

$$\boxed{\frac{d\sigma_1}{\sigma_1} = \frac{d\sigma_2}{\sigma_2} = \frac{dr}{r} - \frac{dt}{t} = d\epsilon_2 - d\epsilon_3} \qquad (8.41)$$

clearly, dr/r is the hoop strain increment $d\epsilon_2$, and dt/t is the incremental thickness strain or radial strain increment $d\epsilon_3$. Eqs(8.41) is the condition of instability for the cylinder material.
Upon application of Eqs (8.37) and (8.38), the effective stress and the effective strains are found to be,

$$\bar{\sigma} = \frac{\sqrt{3}}{2}\sigma_1$$

$$d\bar{\epsilon} = \frac{2}{\sqrt{3}} d\epsilon_1 = -\frac{2}{\sqrt{3}} d\epsilon_3 \quad , d\epsilon_2 = 0 \qquad (c)$$

It is observed that axial strain does not occur and the situation is one of plane strain. The first of Eqs(c) leads to

$$d\bar{\sigma} = \left(\frac{\sqrt{3}}{2}\right) d\sigma_1$$

and condition (8.41) gives,

$$d\bar{\sigma} = \frac{\sqrt{3}}{2}\sigma_1(d\epsilon_1 - d\epsilon_3) = \sqrt{3}\,\bar{\sigma}\,d\bar{\epsilon}$$

from which,

$$\frac{d\bar{\sigma}}{d\bar{\epsilon}} = \sqrt{3}\,\bar{\sigma}$$

A comparison of this result with Equ (8.40) shows that,

$$\boxed{\bar{\epsilon} = \frac{n}{\sqrt{3}}} \qquad (8.42)$$

For a sperical shell subjected to internal pressure $\sigma_1 = \sigma_2 = \frac{pr}{2t}$, $\alpha = 1$ and $d\epsilon_1 = d\epsilon_2 = -d\epsilon_3/2$. At stability $dp = 0$, we now have,

$$\frac{d\sigma_1}{\sigma_1} = \frac{dr}{r} - \frac{dt}{t} = d\epsilon_1 - d\epsilon_3 \qquad (d)$$

Equations (8.37) and (8.38) result in,

$$\bar{\sigma} = \sigma_1 \quad , \quad d\bar{\epsilon} = 2d\epsilon_1 = 2d\epsilon_2 = -d\epsilon_3 \qquad (e)$$

Equations (d) and (e) are combined to yield,

$$\frac{d\bar{\sigma}}{d\bar{\epsilon}} = \frac{2}{3}\bar{\sigma} \qquad (f)$$

From Equs (f) and (8.40) it is concluded that,

$$\boxed{\bar{\epsilon} = \frac{2}{3}n} \qquad (8.43)$$

True stress and true strain are easily obtained, and are the same as the values obtained by a different method.

Example (8-9) :

Determine the plastic stress distribution in a long thick-walled cylinder, subjected only to internal pressure P. Assume the cylinder to be fabricated of a rigid plastic material of yield strength σ_y and ultimate strength σ_u.

Solution :

The elastic stress distribution in the cylinder is described before. As the pressure is increased, it's clear that the entire cylinder will eventually yield. The limiting pressure as well as the stress distribution corresponding to this pressure is determined by application of the Lévy - Mises relations. In polar coordinates the axial strain increment is,

$$\boxed{d\epsilon_z = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_z - \frac{1}{2}(\sigma_r + \sigma_\theta)\right]} \qquad (8.44)$$

If the ends of the cylinder are restrained so that the axial displacement $\omega = 0$, the problem may be regarded as a case of plane strain, for which $\epsilon_z = 0$. It follows that $d\epsilon_z = 0$ and eq (8.44) gives,

$$\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta) \qquad (g)$$

The equation of equilibrium is,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (h)$$

Subjected to the following boundary conditions,

$$(\sigma_r)_{r=a} = -P, \quad (\sigma_r)_{r=b} = 0 \qquad (i)$$

Setting $\sigma_1 = \sigma_r$, $\sigma_2 = \sigma_\theta$, and σ_z as given in eq (g), the

Mises yield criterion results in $(\bar{\sigma})^2 = \frac{3}{4}(\sigma_r - \sigma_\theta)^2$. From this expression,

$$\sigma_r - \sigma_\theta = \pm \frac{2}{\sqrt{3}} \bar{\sigma} \qquad (j)$$

Introducing Equ (j) into Equ (h), we obtain,

$$\frac{d\sigma_r}{dr} = \pm \frac{2\bar{\sigma}}{\sqrt{3}\, r}$$

which has the solution,

$$\sigma_r = \pm \frac{2}{\sqrt{3}} \bar{\sigma} \ln r + C_1 \qquad (k)$$

The constant of integration is determined by applying the second of conditions (i),

$$C_1 = \pm \frac{2}{\sqrt{3}} \bar{\sigma} \ln b$$

Equation (k) is thus,

$$\sigma_r = \pm \frac{2}{\sqrt{3}} \bar{\sigma} \ln \frac{b}{r} \qquad (l)$$

The first of conditions (i) now leads to,

$$\boxed{P = \frac{2}{\sqrt{3}} \bar{\sigma} \ln\left(\frac{b}{a}\right)} \quad (r = a) \qquad (8.45)$$

The pressure causes the initial plastic yielding when $\bar{\sigma} = \sigma_y$, and the collapse of the cylinder when $\bar{\sigma} = \sigma_u$.
An expression for σ_θ can now be found by substituting Equ (l) into Equ (j). Consequently, Equ (g) provides σ_z. The complete plastic stress distribution, for a specified value of $\bar{\sigma}$, is thus found to be,

$$\boxed{\sigma_r = -\frac{2}{\sqrt{3}} \bar{\sigma} \ln \frac{b}{r}} \qquad (8.46)a$$

$$\boxed{\begin{aligned} \sigma_\theta &= -\frac{2}{\sqrt{3}}\bar{\sigma}\left(1-\ln\frac{b}{r}\right) \quad (8.46)b \\ \sigma_z &= -\frac{2}{\sqrt{3}}\bar{\sigma}\left(\frac{1}{2}-\ln\frac{b}{r}\right) \quad (8.46)c \end{aligned}}$$

Following procedure similar to that outlined above, a number of problems of particular importance, involving spherical and cylinderical symmetry, can be solved.

Problems :

(8.1) A solid circular cylinder of 0.1 m diameter is subjected to a bending moment $M = 3.375$ KN.m, an axial tensile force $P = 90$ KN, and a twisting end couple $M_t = 4.5$ KN.m. Determine the stress deviator tensor.

(8.2) Determine the deflection of a uniformly loaded rigid plastic cantilever beam of length L. Locate the origin of coordinates at the fixed end, and denote the loading by P.

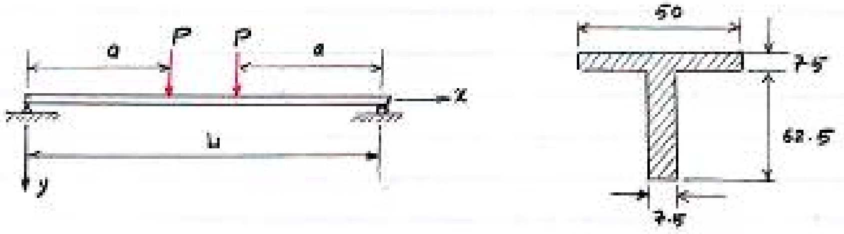
(8.3) Consider a beam of rectangular section, subjected to end moments. Assuming that the relationship for tensile and compressive stress for the material is approximated by $\sigma = K\epsilon^{1/4}$, determine the maximum stress.

(8.4) Consider a perfectly plastic cantilever beam of rectangular cross section and length L subjected to a concentrated downward force P at the free end. Determine the maximum deflection at the beginning of inelastic action and at the instant of collapse.

(8.5) A simply supported rigid plastic beam is described below. Compute the maximum deflection. Reduce the result to the case of a linearly elastic material,

$$EI\,v_{max} = \frac{1}{8} PaL^2 - \frac{1}{6} Pa^3$$

Let $P = 8\ kN$, $E = 200\ GPa$, $L = 1.2\ m$ and $a = 0.45\ m$. Cross-sectional dimensions shown are in millimeters.



(8.6) Consider a thin-walled cylinder of original radius r_0, subjected to internal pressure P. Determine the value of the required original thickness at instability employing Hencky's relations. Use the following,

$K = 900\ MPa$, $n = 0.2$

$r_0 = 0.5\ m$, $P = 14\ MPa$